Forward and reverse entropy power inequalities

Jiange Li

Institute for Advanced Study in Mathematics
Harbin Institute of Technology

2022.08.19

Contents

- 1: Forward EPI
- 2: Reverse EPI
- 3: Entropy jump
- 4: Minimum entropy

1. Forward EPI

What is entropy?

Let X be a random vector in $\mathbb{R}^d$ with density f (w.r.t the Lebesgue measure). The Shannon-Boltzmann entropy of X is defined as

$$h(X) = -\int_{\mathbb{R}^n} f(x) \log f(x) dx.$$

One can think of $h(X)$ as the logarithm of the volume of the effective support of X. This suggests an informal parallelism between entropy inequalities of random vectors on the one hand, and cardinality/volume inequalities of sets on the other hand.

Brunn-Minkowski Inequality (BMI)

Brunn-Minkowski Inequality: Let A and B be nonempty compact subsets in $\mathbb{R}^d$. We have

$$|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}.$$

Here, $|\cdot|$ and $+$ denote volume and vector sum, respectively. Equality holds precisely when A and B are homothetic (i.e., equal up to dilation and translation).

In other words, the Lebesgue measure on $\mathbb{R}^d$ is $1/d$ concave. This general BMI is due to Lusternik'35.

Remark

BMI was inspired by issues around the classical isoperimetric problem. The fundamental geometric content makes BMI a cornerstone of the Brunn-Minkowski theory.

BMI now consolidates its role as an analytical tool and a compelling picture has emerged of its relations to other analytical inequalities, including Prékopa-Leindler inequality, (reverse) Young's inequality, (reverse) Brascamp-Lieb inequality, entropy power inequality, etc.

Entropy power inequality (EPI)

Shannon'48: Let X and Y be independent random vectors in $\mathbb{R}^d$ such that the entropies of X, Y and $X+Y$ exist. We have

$$e^{\frac{2}{d}h(X+Y)} \geq e^{\frac{2}{d}h(X)} + e^{\frac{2}{d}h(Y)}.$$

Equality holds if and only if X and Y are Gaussian random vectors with proportional covariance matrices.

The first complete proof is due to Stam'59.

Remark

EPI was first used by Shannon to study the fundamental limits of communication channels. It is now recognized as an extremely useful inequality in probability theory, convex geometry, functional analysis, etc. Particularly, EPI implies the log-Sobolev inequality for Gaussian measures.

EPI for free entropy: Szarek-Voiculescu'96
Quantum EPI: König-Smith'14, Audenaert-Datta-Ozols'16

Rényi entropy

Let X be a random vector in $\mathbb{R}^d$ with density f (w.r.t the Lebesgue measure). For $p \in [0, \infty]$, the p-Rényi entropy of X is defined as

$$h_p(X) = \frac{1}{1-p} \log \int_{\mathbb{R}^n} f(x)^p dx.$$

For $p \in \{0, 1, \infty\}$, the definition is understood in the limiting sense:

- $h_1(X)$ is the Shannon-Boltzmann entropy
- $h_0(X) = \log |\mathrm{supp}(f)|$, $h_\infty(X) = -\log \|f\|_\infty$

EPI for p-Rényi entropy

Bobkov-Chistyakov'15: Let $p > 1$. There exists an absolute constant c_p such that for any independent random vectors $X_1, \cdots, X_n$ in $\mathbb{R}^d$,

$$e^{\frac{2}{d} h_p(X_1+\cdots+X_n)} \geq c_p \cdot \left(e^{\frac{2}{d} h_p(X_1)} + \cdots + e^{\frac{2}{d} h_p(X_n)} \right).$$

In particular, one can take $c_p = p^{\frac{1}{p-1}}/e$, which decreases from 1 to $1/e$.

EPI for p-Rényi entropy (cont'd)

L.'18: Let X and Y be independent random vectors in $\mathbb{R}^d$ such that X, Y and $X+Y$ have finite p-Rényi entropy for $p > 1$. We have

$$e^{\frac{\alpha_p}{d} h_p(X+Y)} \geq e^{\frac{\alpha_p}{d} h_p(X)} + e^{\frac{\alpha_p}{d} h_p(Y)},$$

$$\alpha_p = 2\left[1 + \frac{1}{\log 2}\left(\frac{p+1}{p-1}\log\frac{p+1}{2p} + \frac{\log p}{p-1}\right)\right]^{-1}.$$

It improves Bobkov-Marsiglietti'17. We have $\alpha_0 = 1$ and $\alpha_1 = 2$. For p large, α_p is asymptotically optimal up to a multiplicative constant. It is unknown whether this inequality is sharp.

Remark

L.-Marsiglietti-Melbourne'19: In general, there are no such EPIs for Rényi entropy of order $0 < p < 1$. One can take random vectors that are essentially truncations of some random vector with finite covariance matrix, but infinite p-Rényi entropy. However, we can establish analogs of Rényi EPIs of order $0 < p < 1$ if the distributions satisfy certain convexity/concavity.

2. Reverse EPI

Reverse BMI

V. Milman'86: There is an absolute constant c such that for any convex bodies A and B in $\mathbb{R}^d$ one can find volume preserving linear maps φ and ψ such that

$$|\varphi(A)+\psi(B)|^{1/d} \leq c \cdot (|A|^{1/d}+|B|^{1/d}).$$

Milman's reverse BMI has connections with high dimensional phenomena in convex geometry. Ball'86, Bourgain-Klartag-V. Milman'04: reverse BMI for convex bodies in isotropic position is equivalent to Bourgain's slicing problem.

Reverse EPI

Bobkov-Madiman'12: There is an absolute constant c such that for independent log-concave random vectors X and Y in $\mathbb{R}^d$ one can find volume preserving linear maps φ and ψ such that

$$e^{\frac{2}{d}h(\varphi(X)+\psi(Y))} \leq c \cdot \left(e^{\frac{2}{d}h(X)} + e^{\frac{2}{d}h(Y)}\right).$$

One can replace the Shannon-Boltzmann entropy by general p-Rényi entropies. The selection of φ and ψ is related to Bourgain's slicing problem.

Busemann's intersection body theorem

Busemann'49: The intersection body of a symmetric convex body is convex.

Intersection body was introduced by Lutwak'88. It plays a key role in the solution of the Busemann-Petty problem in convex geometry.

Conjecture (Gardner-Giannopoulos'99): The p-cross-section body of a symmetric convex body is convex.

Busemann's theorem corresponds to the $p = \infty$ case.

A reverse Rényi EPI

G-G conjecture (entropic version): Let $(X, Y) \in \mathbb{R}^2$ be a symmetric log-concave random vector, (X and Y are not necessarily independent). We have

$$e^{h_p(X+Y)} \leq e^{h_p(X)} + e^{h_p(Y)}.$$

Ball'88: the $p = \infty$ case.
Ball-Nayar-Tkocz'16: the $p = 1$ case with an exponent $1/5$.
L'18: the $p = 2$ case, symmetry is not required.

3. Entropy jump

Entropy jump

Carlen-Soffer'91: If X is not Gaussian, and X, Y are i.i.d, then the entropy of $(X+Y)/\sqrt{2}$ is strictly larger than that of X.

Ball-Barthe-Naor'03, Barron-Johnson'04: If X, Y satisfy the Poincare inequality with constant $c > 0$. Then

$$h\left(\frac{X+Y}{\sqrt{2}}\right) - h(X) \geq \frac{c}{2+2c}(h(G) - h(X)).$$

Ball-Nguyen'12: extention to log-concave random vectors under spectral condition.

Entropy jump yields the convergence rate of the entropic CLT. Ball'03 observed that entropy jump of isotropic log-concave random vectors implies Bourgain's slicing problem.

Entropy jump (cont'd)

Conjecture (folklore): Let X and Y be i.i.d log-concave random vectors in $\mathbb{R}^d$. The largest increment $h(X+Y)-h(X)$ is achieved by exponential random vectors (the numerical value is γd, where $\gamma \approx 0.57$ is Euler's constant).

A (trivial) uppber bound is $d \log 2$. Any improvement of this bound will find great applications to Hadwiger's covering problem.

L.-Marsiglietti-Melbourne: analogs hold for p-Rényi entropy of order $p \in \{0, 2, \infty\}$.

Entropy jump (cont'd)

Conjecture (folklore): Let X and Y be i.i.d log-concave random vectors in $\mathbb{R}^d$. The largest increment $h(X-Y)-h(X)$ is achieved by exponential random vectors (the numerical value is $d\log 2$).

Rogers-Shephard inequality: Let A be a convex body in $\mathbb{R}^d$. Then

$$|A-A| \leq \binom{2d}{d}|A|.$$

Equality holds if and only if A is a simplex.

Melbourne-Tkocz'21: analogs hold for Rényi entropy of order $p \geq 2$.

L.: a different proof which sheds more light on the optimality of exponential distribution.

4. Minimum entropy

Minimum entropy

It is a well-know fact that Gaussian distribution has the maximum Shannon-Boltzmann entropy within the class of distributions having the same covariance matrix.

Conjecture (folklore): Exponential distribution has the minimum Shannon-Boltzmann entropy within the class of log-concave distributions having the same covariance matrix.

An affirmative answer implies Bourgain's slicing problem.

Białobrzeski-Nayar'21: analogs hold for Rényi etropy of order $p \geq 2$ in one-dimension.

L.: a much simpler proof.

Thanks for your attention!