A Complete Theory of Operator-Valued Hardy Spaces

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Outline

Non-commutative L_p-space

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $S_{\mathcal{M}}^{+}$ be the set of all positive $x \in \mathcal{M}$ such that

$$\tau(s(x)) < \infty,$$

where $s(x)$ denotes the support of x, that is, the least projection $e \in \mathcal{M}$ such that $exe = x$. Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^{+}$. For any $p \in (0, \infty)$, we define

$$\|x\|_p = (\tau|x|^p)^{1/p}, \quad x \in S_{\mathcal{M}},$$

where $|x| = (x^*x)^{1/2}$. The usual non-commutative L_p-space, $L_p(\mathcal{M})$, associated with $(\mathcal{M}, \tau)$, is the completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$. For convenience, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with operator norm $\|\cdot\|_{\mathcal{M}}$.

Column and row spaces

Let $(\Omega,\, d\mu)$ be a measurable space. We say that f is a $S_{\mathcal{M}}$-valued simple function on $(\Omega,\, d\mu)$ if

$$f = \sum_{i=1}^{n} m_i \cdot \chi_{A_i},$$

where each $m_i \in S_{\mathcal{M}}$ and A_i's are disjoint measurable subsets of Ω with $\mu(A_i) < \infty$. Let $p \in [1,\, \infty)$. For a $S_{\mathcal{M}}$-valued simple function f, we define

$$\|f\|_{L_p(\mathcal{M};\, L_2^c(\Omega))} := \left\| \left(\int_\Omega f^* f \right)^{1/2} \right\|_{L_p(\mathcal{M})}, \quad \|f\|_{L_p(\mathcal{M};\, L_2^r(\Omega))} := \left\| \left(\int_\Omega f f^* \right)^{1/2} \right\|_{L_p(\mathcal{M})}.$$

The *column space* $L_p(\mathcal{M};\, L_2^c(\Omega))$ (resp. row space $L_p(\mathcal{M};\, L_2^r(\Omega))$) is defined to be the completion of the space of all $S_{\mathcal{M}}$-valued simple functions under the $L_p(\mathcal{M};\, L_2^c(\Omega))$ (resp. $L_p(\mathcal{M};\, L_2^r(\Omega))$) norm.

Column and row spaces

Remark 1.1

When Ω is a countable set such as $\mathbb{Z}$, $\mathbb{N}$ and $\mathcal{D}$, equipped with a counting measure, $L_p(\mathcal{M}; L_2^c(\Omega))$ (resp. $L_p(\mathcal{M}; L_2^r(\Omega))$) will be denoted by $L_p(\mathcal{M}; \ell_2^c)$ (resp. $L_p(\mathcal{M}; \ell_2^r)$). Moreover, the space $L_p(\mathcal{M}; \ell_2^c)$ can be regarded as a space of sequences of elements of $L_p(\mathcal{M})$.

Operator-Valued Hardy Spaces

Let P be the Poisson kernel of $\mathbb{R}^n$: $P(x) = \widetilde{C}\frac{1}{(|x|^2+1)^{\frac{n+1}{2}}}$ with $\widetilde{C}$ being a normalizing constant. For $y > 0$, let

$$P_y(x) := \frac{1}{y^n}P(\frac{x}{y}) = \widetilde{C}\frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}.$$

For any function f on $\mathbb{R}^n$ with values in $\mathcal{M}$, its Poisson integral, whenever exists, will be denoted by $f(x, y)$: $f(x, y) := \int_{\mathbb{R}^n} P_y(x-t)f(t)\,dt, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}_+$. For any $S_{\mathcal{M}}$-valued simple function f, the Lusin area functions of f are defined by

$$S^c(f)(t) := \left(\int\int_\Gamma \left|\frac{\partial}{\partial y}f(x+t, y)\right|^2 \frac{dxdy}{y^{n-1}}\right)^{1/2} \text{ and } S^r(f)(t) := S^c(f^*)(t).$$

where $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}_+ : |x| < y\}$.

Operator-Valued Hardy Spaces

For $p \in [1, \infty)$, we set

$$\|f\|_{H_p^c(\mathbb{R}^n, \mathcal{M})} := \|S^c(f)\|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})} \quad \text{and} \quad \|f\|_{H_p^r(\mathbb{R}^n, \mathcal{M})} := \|f^*\|_{H_p^c(\mathbb{R}^n, \mathcal{M})}.$$

The *column Hardy space* $H_p^c(\mathbb{R}^n, \mathcal{M})$ (resp. *the row Hardy space* $H_p^r(\mathbb{R}^n, \mathcal{M})$) is defined to be the completion of the space of all $S_{\mathcal{M}}$-valued simple functions under the $H_p^c(\mathbb{R}^n, \mathcal{M})$ (resp. $H_p^r(\mathbb{R}^n, \mathcal{M})$) norm.

[1] T. Mei, Operator-valued Hardy spaces, Mem. Amer. Math. Soc., 188 (2007), 1-64.

Define the *mixture space* $H_p(\mathbb{R}^n,\, \mathcal{M})$ as follows: For $p \in [1,\, 2)$,

$$H_p(\mathbb{R}^n,\, \mathcal{M}) := H_p^c(\mathbb{R}^n,\, \mathcal{M}) + H_p^r(\mathbb{R}^n,\, \mathcal{M})$$

equipped with the sum norm

$$\begin{aligned}&\|f\|_{H_p(\mathbb{R}^n,\, \mathcal{M})}\\ &:= \inf\left\{\|g\|_{H_p^c(\mathbb{R}^n,\, \mathcal{M})} + \|h\|_{H_p^r(\mathbb{R}^n,\, \mathcal{M})} : f = g + h,\, g \in H_p^c(\mathbb{R}^n,\, \mathcal{M}),\, h \in H_p^r(\mathbb{R}^n,\, \mathcal{M})\right\}.\end{aligned}$$

For $p \in [2,\, \infty)$, define

$$H_p(\mathbb{R}^n,\, \mathcal{M}) := H_p^c(\mathbb{R}^n,\, \mathcal{M}) \cap H_p^r(\mathbb{R}^n,\, \mathcal{M})$$

equipped with the intersection norm

$$\|f\|_{H_p(\mathbb{R}^n,\, \mathcal{M})} := \max\left\{\|f\|_{H_p^c(\mathbb{R}^n,\, \mathcal{M})},\, \|f\|_{H_p^r(\mathbb{R}^n,\, \mathcal{M})}\right\}.$$

A wavelet basis of $L_2(\mathbb{R}^n)$ is a complete orthonormal system $\{\omega_I\}_{I\in\mathcal{D}}$, where $\{\omega_I\}_{I\in\mathcal{D}}$ is a 1-regular basis and ω is a real-valued function on $\mathbb{R}^n$ satisfying the properties for Meyer's construction in [2], here and hereafter, let $\mathcal{D}$ denote the collection of all dyadic intervals in $\mathbb{R}^n$, that is,

$$\mathcal{D} := \left\{I_{j,\,k} : I_{j,\,k} = 2^{-j}([0,\,1)^n + k), j \in \mathbb{Z}, k \in \mathbb{Z}^n\right\}$$

and

$$\omega_{I_{j,\,k}}(x) := 2^{jn/2}\omega(2^j x - k), \quad \text{for any } j,\, k \in \mathbb{Z}^n.$$

The wavelet basis $\{\omega_I\}_{I\in\mathcal{D}}$ is called 1-regular if $|\partial^\alpha\omega(x)| \le C_m(1+|x|)^{-m}$ and $\int_{\mathbb{R}^n} x^\alpha \omega(x)\,dx = 0$ for all $|\alpha| \le 1$ and $m \in \mathbb{N}$.

[2] Y. Meyer, Wavelets and Operators, Cambridge Studies in Advanced Mathematics 37, Cambridge University Press, Cambridge, 1992.

As in [2], we may assume that there exists some cube $A^\gamma \subset [0,1)^n$ such that

$$|A^\gamma| = \gamma > 0 \quad \text{and} \quad |\omega(x)| \geq c_0, \qquad \forall x \in A^\gamma, \tag{2.1}$$

for some fixed positive constants c_0 and γ. In what follows, for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, define

$$\widetilde{I_{j,\,k}} := 2^{-j}(A^\gamma + k). \tag{2.2}$$

[3] G. Hong and Z. Yin, Wavelet approach to operator-valued Hardy spaces, Rev. Mat. Iberoam., 29 2013, 293-313.

Theorem 2.1 (Hong-Wang-Wu, IMRN, 2022)

Let $p \in [1, \infty)$ *and* $\{\omega_I\}_{I\in\mathcal{D}}$ *be a* 1*-regular wavelet basis of* $L_2(\mathbb{R}^n)$*. Then the following conditions are equivalent for any* $L_p(\mathcal{M})$*-valued distribution* f*,*

$$f = \sum_{I\in\mathcal{D}} \langle f, \omega_I\rangle \omega_I \ \ in\ the\ sense\ of\ distribution:$$

(i) $f \in H_p^c(\mathbb{R}^n, \mathcal{M})$;

(ii) $\| \left(\sum_{I\in\mathcal{D}} |\langle f, \omega_I\rangle|^2 \, |\omega_I|^2 \right)^{1/2} \|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})} < \infty$;

(iii) $\| \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, \omega_I\rangle|^2}{|I|} \chi_I \right)^{1/2} \|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})} < \infty$;

(iv) $\| \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, \omega_I\rangle|^2}{|I|} \chi_{\widetilde{I}} \right)^{1/2} \|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})} < \infty$;

where $\mathcal{D}$ *denotes the collection of all dyadic cubes in* $\mathbb{R}^n$ *and for every* $I \in \mathcal{D}$*, the definition of* $\widetilde{I}$ *is as in* (2.2)*.*

Let φ, $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\int \varphi = 0$, $\int \psi = 0$, and satisfy, for any $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$,

$$\int_0^\infty \widehat{\varphi}(t\xi)\overline{\widehat{\psi}(t\xi)}\frac{dt}{t} = 1, \tag{3.1}$$

where $\widehat{\varphi}$ denotes the Fourier transform of φ. In what follows, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $t > 0$ and $x \in \mathbb{R}^n$, let $\varphi_t(x) := t^{-n}\varphi\left(\frac{x}{t}\right)$.

For any $S_{\mathcal{M}}$-valued simple function f, the Lusin area functions of f is defined by

$$g_\varphi^c(f)(x) := \left(\int_0^\infty |f * \varphi_t(x)|^2 \frac{dt}{t}\right)^{1/2}$$

and

$$S_\varphi^c(f)(x) := \left(\int\int_{\Gamma_x} |f * \varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma_x := \{(y,\, t) \in \mathbb{R}^n \times \mathbb{R}_+ : |x - y| < t\}$ with $x \in \mathbb{R}^n$.

A New Characterization

Theorem 3.1 (Xia-Xiong-Xu, Adv. Math. 2016)

Let $p \in [1, \infty)$. *Then* $f \in H_p^c(\mathbb{R}^n, \mathcal{M})$ *iff* $S_\varphi^c(f) \in L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$ *iff* $s_\varphi^c(f) \in L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$, *and*

$$\|f\|_{H_p^c(\mathbb{R}^n,\mathcal{M})} \sim \|s_\varphi^c(f)\|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})} \sim \|S_\varphi^c(f)\|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})}$$

Similarly, these results also holds for row and mixture Hardy spaces.

Let φ, $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\int \varphi = 0$, $\int \psi = 0$ and satisfy, for any $\xi \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$,

$$\sum_{j=-\infty}^{\infty} \widehat{\varphi}(2^j \xi)\overline{\widehat{\psi}(2^j \xi)}\frac{dt}{t} = 1, \tag{3.2}$$

where $\widehat{\varphi}$ denotes the Fourier transform of φ. In what follows, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_{2^j}(x) := 2^{-jn}\varphi\left(\frac{x}{2^j}\right)$.

For any $S_{\mathcal{M}}$-valued simple function f, the Lusin area functions of f is defined by

$$g_{\varphi}^{c,D}(f)(x) := \left(\sum_{j=-\infty}^{\infty} |f * \varphi_{2^j}(x)|^2\right)^{1/2}$$

and

$$S_{\varphi}^{c,D}(f)(x) := \left(\sum_{j=-\infty}^{\infty} 2^{-nj} \int_{B(x,\,2^j)} |f * \varphi_{2^j}(y)|^2 \, dy\right)^{1/2}.$$

Theorem 3.2 (Xia-Xiong-Xu, Adv. Math. 2016)

Let $p \in [1, \infty)$. *Then* $f \in H_p^c(\mathbb{R}^n, \mathcal{M})$ *iff* $S_\varphi^c(f) \in L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$ *iff* $s_\varphi^c(f) \in L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$, *and*

$$\|f\|_{H_p^c(\mathbb{R}^n, \mathcal{M})} \sim \|s_\varphi^{c,D}(f)\|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})} \sim \|S_\varphi^{c,D}(f)\|_{L_p(L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})}$$

Similarly, these results also holds for row and mixture Hardy spaces.

Anisotropic dilations

(I) Isotropic ball cover $\{x + 2^k\mathbb{B}^n : x \in \mathbb{R}^n, k \in \mathbb{Z}\}$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$, $\rho(x, y) = |x - y|^n$.

(II) Anisotropic ellipsoid cover [Bownik, Mem. Amer. Math. Soc., 2003] $\{x + A^k B_*^n : x \in \mathbb{R}^n, k \in \mathbb{Z}\}$, where A is a fixed matrix with all eigenvalues $|\lambda| > 1$ and B_*^n is some fixed ellipsoid.

For example,

$$A := \begin{pmatrix} 2^{a_1} & 0 & \cdots & 0 \\ 0 & 2^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{a_n} \end{pmatrix}, \quad a_i > 0.$$

♠ Classical isotropic Hardy space (Fefferman, Stein)

♠ Parabolic Hardy space (Calderón and Torchinsky, Adv. Math. 1975)

♠ Anisotropic Hardy space (Bownik, Mem. Amer. Math. Soc, 2003)

♠ Anisotropic weak Hardy space, (Y. Ding, S. Lan, Sci. China Ser. A, 2008)

♠ Weighted anisotropic Hardy space (Bownik, B. Li, D. Yang, Y. Zhou, Indiana Univ. Math. J. 2008)

♠ Weighted anisotropic product Hardy spaces, (B. Li, Bownik, D. Yang, JFA, 2014)

♠ Anisotropic Hardy spaces of Musielak-Orlicz type (B. Li, D. Yang, W. Yuan, The Scientific World Journal, 2014)

♠ Anisotropic Hardy space with variable exponent (J. Liu, F. Weisz, D. Yang, W. Yuan, Taiwanese J. Math. 2018)

♠ Anisotropic mixed-norm Hardy space (L. Huang, J. Liu, D. Yang, W. Yuan, CPAA, 2020)

♠ ....

♠ Dahmen, Dekel and Petrushev, [Numer. Math., 2007]: Anisotropic ellipsoid cover applied to solve elliptic boundary value problems.

♠ Jakab and Mitrea, [Math. Res. Lett., 2006]: Parabolic initial boundary value problems in nonsmooth cylinders with data in anisotropic Besov spaces.

♠ Zhang and Li, [Turkish J. Math., 2018]: Unconditional wavelet bases in Lebesgue spaces.

♠ Bownik and Wang, [arXiv:2011.10651, 2020]: A PDE Characterization of Anisotropic Hardy Spaces.

♠ . . . .

A real $n \times n$ matrix A is called an *expansive matrix*, if $\min_{\lambda\in\sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of A. Let λ_- and λ_+ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \ \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \ \lambda \in \sigma(A)\} < \lambda_+.$$

Bownik [4, Lemma 2.2] proved that, for a fixed dilation A, there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : \ |Px| < 1\}$, where P is some non-degenerate $n \times n$ matrix, such that $\Delta \subset r\Delta \subset A\Delta$, and we can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-*dimensional Lebesgue measure* of the set Δ. For $k \in \mathbb{Z}$, let $B_k := A^k\Delta$. Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$, here and hereafter, $b := |\det A|$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Define

$$\mathfrak{B} := \{x + B_k : \ x \in \mathbb{R}^n, k \in \mathbb{Z}\}. \tag{3.3}$$

[4] M. Bownik, Anisotropic Hardy spaces and wavelets, [M]. Mem. Amer. Math. Soc., 2003, 164: 781.

Lemma 3.1

Let σ be the smallest integer such that $2B_0 \subset A^\sigma B_0$. Then, for all $k, j \in \mathbb{Z}$ with $k \leq j$, it holds that

$$B_k + B_j \subset B_{j+\sigma}, \tag{3.4}$$

$$B_k + (B_{k+\sigma})^\complement \subset (B_k)^\complement, \tag{3.5}$$

where $E + F$ denotes the algebraic sum $\{x + y : \ x \in E, \, y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

For any A and $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\} \subset \mathbb{R}^n$, we have $AB(x,r) \supset B(x,r)$. However, for any $x \in \mathbb{R}^n$,

$$|Ax|^n = |\det A||x|^n?$$

In other words, $|\cdot|$ is not valid.

Definition 3.1

A quasi-norm, associated with dilation A, is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$, for simplicity, denoted by ρ, satisfying

(i) $\rho(x) > 0$ *for all* $x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$, *here and hereafter,* $\mathbf{0}_n$ *denotes the origin of* $\mathbb{R}^n$;

(ii) $\rho(Ax) = b\rho(x)$ *for any* $x \in \mathbb{R}^n$, *where* $b := |\det A|$;

(iii) $\rho(x + y) \le C_A [\rho(x) + \rho(y)]$ *for all* $x, y \in \mathbb{R}^n$, *where* $C_A \in [1, \infty)$ *is a constant independent of* x *and* y.

When $A := 2\mathrm{I}_{n\times n}$, $\rho_A(x) := |x|^n$, for any $x \in \mathbb{R}^n$, ρ_A is a quasi-norm, associated with dilation A.

In [4, Lemma 2.4], M. Bownik also showed that all homogeneous quasi-norms associated with a fixed dilation A are equivalent. Therefore, for a fixed dilation A, in what follows, we always use the *step homogeneous quasi-norm* ρ defined by

$$\rho(x) := \sum_{k\in\mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \text{ if } x \neq \mathbf{0}_n, \quad \text{or else} \quad \rho(\mathbf{0}_n) := 0.$$

Definition 3.2

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be a radial real-valued function satisfying*

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 0 \tag{3.6}$$

and, for any $\xi \in \mathbb{R}^n \backslash \{\mathbf{0}_n\}$,

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\varphi}((A^T)^j \xi)\right|^2 = 1, \tag{3.7}$$

where $\widehat{\varphi}$ *denotes the Fourier transform of* φ *and* A^T *is the transpose of* A. *In what follows, every* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *always satisfies* (3.6) *and* (3.7). *For* $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $k \in \mathbb{Z}$ *and* $x \in \mathbb{R}^n$, *let* $\varphi_k(x) := b^{-k}\varphi\left(A^{-k}x\right)$.

For any $S_{\mathcal{M}}$-valued simple function f, the anisotropic Lusin area functions of f is defined by

$$S_{\varphi}^{c}(f)(x) := \left(\sum_{k\in\mathbb{Z}} b^{-k} \int_{x+B_k} |f * \varphi_k(y)|^2 \, dy\right)^{1/2}$$

and

$$S_{\varphi}^{r}(f)(x) := \left(\sum_{k\in\mathbb{Z}} b^{-k} \int_{x+B_k} |f^{*} * \varphi_k(y)|^2 \, dy\right)^{1/2}.$$

Definition 3.3

Let $p \in [1, \infty)$ *and* f *be an* $S_{\mathcal{M}}$*-valued simple function. We define the* $\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M})$ *and* $\mathcal{H}_A^{p,r}(\mathbb{R}^n, \mathcal{M})$ *norms of* f *by*

$$\|f\|_{\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M})} := \left\|S_\varphi^c(f)\right\|_{L^p(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})}, \quad \|f\|_{\mathcal{H}_A^{p,r}(\mathbb{R}^n, \mathcal{M})} := \|f^*\|_{\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M})}.$$

Define the anisotropic column Hardy space $\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M})$ *(resp. anisotropic row Hardy space* $\mathcal{H}_A^{p,r}(\mathbb{R}^n, \mathcal{M})$*) to be the completion of the space of all* $S_{\mathcal{M}}$*-valued simple functions with finite* $\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M})$ *(resp.* $\mathcal{H}_A^{p,r}(\mathbb{R}^n, \mathcal{M})$*) norm.*

Definition 3.4

Define the anisotropic mixture space $\mathcal{H}^p_A(\mathbb{R}^n,\,\mathcal{M})$ *as follows: if* $p \in [1,\,2)$,

$$\mathcal{H}^p_A(\mathbb{R}^n,\,\mathcal{M}) := \mathcal{H}^{p,c}_A(\mathbb{R}^n,\,\mathcal{M}) + \mathcal{H}^{p,r}_A(\mathbb{R}^n,\,\mathcal{M})$$

equipped with the sum norm

$$\begin{aligned}&\|f\|_{\mathcal{H}^p_A(\mathbb{R}^n,\,\mathcal{M})}\\ &:= \inf\left\{\|g\|_{\mathcal{H}^{p,c}_A(\mathbb{R}^n,\,\mathcal{M})} + \|h\|_{\mathcal{H}^{p,r}_A(\mathbb{R}^n,\,\mathcal{M})} :\ f = g + h,\ g \in \mathcal{H}^{p,c}_A(\mathbb{R}^n,\,\mathcal{M}),\ h \in \mathcal{H}^{p,r}_A(\mathbb{R}^n,\,\mathcal{M})\right\}\end{aligned}$$

where the infimum is taken over all the decompositions of f *as above. If* $p \in [2,\,\infty)$, *define*

$$\mathcal{H}^p_A(\mathbb{R}^n,\,\mathcal{M}) := \mathcal{H}^{p,c}_A(\mathbb{R}^n,\,\mathcal{M}) \cap \mathcal{H}^{p,r}_A(\mathbb{R}^n,\,\mathcal{M})$$

equipped with the intersection norm $\|f\|_{\mathcal{H}^p_A(\mathbb{R}^n,\,\mathcal{M})} := \max\left\{\|f\|_{\mathcal{H}^{p,c}_A(\mathbb{R}^n,\,\mathcal{M})},\ \|f\|_{\mathcal{H}^{p,r}_A(\mathbb{R}^n,\,\mathcal{M})}\right\}.$

Remark 3.1

(i) *When it comes back to the commutative setting, i.e.,* $\mathcal{M} := \mathbb{C}$, *these spaces are reduced to the anisotropic Hardy space* $H^p_A(\mathbb{R}^n)$ *studied by Bownik, where* $p \in [1, \infty)$.

(ii) *When it comes back to the isotropic setting, i.e.,* $A := 2\mathrm{I}_{\mathrm{n}\times\mathrm{n}}$, *the operator-valued Hardy spaces* $\mathcal{H}^{p,c}_A(\mathbb{R}^n, \mathcal{M})$, $\mathcal{H}^{p,r}_A(\mathbb{R}^n, \mathcal{M})$ *and* $\mathcal{H}^p_A(\mathbb{R}^n, \mathcal{M})$, *introduced in this article, coincide the operator-valued Hardy spaces* $\mathcal{H}^{p,c}(\mathbb{R}^n, \mathcal{M})$, $\mathcal{H}^{p,r}(\mathbb{R}^n, \mathcal{M})$ *and* $\mathcal{H}^p(\mathbb{R}^n, \mathcal{M})$ *with equivalent norms, respectively, where* $p \in [1, \infty)$.

(iii) *When* $p = 2$, *we know that*

$$\mathcal{H}^{2,c}_A(\mathbb{R}^n, \mathcal{M}) = \mathcal{H}^{2,r}_A(\mathbb{R}^n, \mathcal{M}) = \mathcal{H}^2_A(\mathbb{R}^n, \mathcal{M}) = L^2(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M}).$$

Operator-Valued BMO Spaces Associated with Anisotropic Dilations

In what follows, for any $B \in \mathfrak{B}$ and function g with values in $\mathcal{M}$, g_B denotes its mean over B, that is,

$$g_B := \frac{1}{|B|} \int_B g(x)\, dx.$$

Definition 3.5

Let A be a dilation. The anisotropic column BMO *space* $\mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M})$ *is defined as*

$$\mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M}) := \{g \in L^\infty(\mathcal{M};\, L_c^2(\mathbb{R}^n, \frac{dx}{1+[\rho(x)]^2})) : \|g\|_{\mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M})} < \infty\},$$

where

$$\|g\|_{\mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M})} := \sup_{B\in\mathfrak{B}} \Big\| \left(\frac{1}{|B|} \int_B |g(y) - g_B|^2\, dy \right)^{1/2} \Big\|_{\mathcal{M}}.$$

Similarly, we define the anisotropic row BMO *space* $\mathcal{BMO}_A^r(\mathbb{R}^n,\mathcal{M})$ *as the space of* g *such that* $g^* \in \mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M})$ *with the norm* $\|g\|_{\mathcal{BMO}_A^r(\mathbb{R}^n,\mathcal{M})} := \|g^*\|_{\mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M})}$*, and*

$$\mathcal{BMO}_A(\mathbb{R}^n,\mathcal{M}) := \mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M}) \cap \mathcal{BMO}_A^r(\mathbb{R}^n,\mathcal{M})$$

with the norm $\|g\|_{\mathcal{BMO}_A(\mathbb{R}^n,\mathcal{M})} := \max\left\{\|g\|_{\mathcal{BMO}_A^c(\mathbb{R}^n,\mathcal{M})},\, \|g\|_{\mathcal{BMO}_A^r(\mathbb{R}^n,\mathcal{M})}\right\}.$

Theorem 3.3

Let A be a dilation. Then we have

$$(\mathcal{H}_A^{1,c}(\mathbb{R}^n,\, \mathcal{M}))^* = \mathcal{BMO}_A^c(\mathbb{R}^n,\, \mathcal{M})$$

in the following sense:

(i) *Every* $g \in \mathcal{BMO}_A^c(\mathbb{R}^n,\, \mathcal{M})$ *defines a continuous linear functional* $\mathcal{L}_g$ *on* $\mathcal{H}_A^{1,c}(\mathbb{R}^n,\, \mathcal{M})$ *by*

$$\mathcal{L}_g(f) := \tau \int_{\mathbb{R}^n} f(x) g^*(x)\, dx, \quad for\ any\ S_{\mathcal{M}} - valued\ simple\ function\ f.$$

(ii) *For any* $\mathcal{L} \in (\mathcal{H}_A^{1,c}(\mathbb{R}^n,\, \mathcal{M}))^*$*, then there exists some* $g \in \mathcal{BMO}_A^c(\mathbb{R}^n,\, \mathcal{M})$ *such that* $\mathcal{L} = \mathcal{L}_g$*.*

Moreover, there exists an universal positive constant C *such that*

$$C^{-1} \|g\|_{\mathcal{BMO}_A^c(\mathbb{R}^n,\, \mathcal{M})} \leq \|\mathcal{L}_g\|_{(\mathcal{H}_A^{1,c}(\mathbb{R}^n,\, \mathcal{M}))^*} \leq C \|g\|_{\mathcal{BMO}_A^c(\mathbb{R}^n,\, \mathcal{M})}.$$

Let us recall the definition of maximal norms. Let $p \in (0, \infty]$ and $x = \{x_i\}_{i\in\mathbb{N}}$ be a sequence of elements in $L^p(\mathcal{M})$. Define

$$\|x\|_{L^p(\mathcal{M};\,\ell^\infty)} := \inf_{x_i = a y_i b} \left\{ \|a\|_{L^{2p}(\mathcal{M})} \|b\|_{L^{2p}(\mathcal{M})} \sup_{i\in\mathbb{N}} \|x_i\|_{\mathcal{M}} \right\},$$

where the infimum is taken over all $a, b \in L^{2p}(\mathcal{M})$ and $\{y_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$ such that $x_i = a y_i b$. As usual, $\|x\|_{L^p(\mathcal{M};\,\ell^\infty)}$ is conventionally denoted by $\left\|\sup^+_{i\in\mathbb{N}} x_i\right\|_{L^p(\mathcal{M})}$. However, we should point out that $\sup^+_{i\in\mathbb{N}} x_i$ is just a notation, since it does not make any sense in the non-commutative setting. We just use this notation for convenience. If $p \in (1, \infty)$ and $\{x_i\}_{i\in\mathbb{N}}$ is a sequence of positive operators, it was proved by Junge that

$$\left\| \sup_{i\in\mathbb{N}}{}^{+} x_i \right\|_{L^p(\mathcal{M})} = \sup \left\{ \sum_{i\in\mathbb{N}} \tau(x_i y_i) : y_i \in L^q(\mathcal{M}),\, y_i \geq 0,\, \left\| \sum_{i\in\mathbb{N}} y_i \right\|_{L^q(\mathcal{M})} \leq 1 \right\}. \qquad (3.8)$$

Let $g \in L^q(\mathcal{M}; L_c^2(\mathbb{R}^n, \frac{dx}{1+[\rho(x)]^2}))$. For any $x + B_k \in \mathfrak{B}$ with $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $\mathfrak{B}$ as in (3.3), denote

$$g_k^\sharp(x) := \frac{1}{|x + B_k|} \int_{x+B_k} |g(y) - g_{x+B_k}|^2 \, dy,$$

where $g_{x+B_k} := \frac{1}{|x+B_k|} \int_{x+B_k} g(y) \, dy$. For $q \in (2, \infty)$, define

$$\|g\|_{L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})} := \left\| {\sup_{k\in\mathbb{Z}}}^+ g_k^\sharp \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})}^{1/2}$$

and

$$\|g\|_{L^q\mathcal{MO}_A^r(\mathbb{R}^n, \mathcal{M})} := \|g^*\|_{L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})}.$$

Obviously, these are two norms. Therefore, we define two spaces

$$L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M}) := \left\{ g \in L^q(\mathcal{M}; L_c^2(\mathbb{R}^n, \frac{dx}{1 + [\rho(x)]^2})) : \|g\|_{L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})} < \infty \right\}$$

and

$$L^q\mathcal{MO}_A^r(\mathbb{R}^n, \mathcal{M}) := \left\{ g \in L^q(\mathcal{M}; L_r^2(\mathbb{R}^n, \frac{dx}{1 + [\rho(x)]^2})) : \|g\|_{L^q\mathcal{MO}_A^r(\mathbb{R}^n, \mathcal{M})} < \infty \right\}.$$

Moreover, we also define the mixture space

$$L^q\mathcal{MO}_A(\mathbb{R}^n, \mathcal{M}) := L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M}) \cap L^q\mathcal{MO}_A^r(\mathbb{R}^n, \mathcal{M}),$$

equipped with the norm

$$\|g\|_{L^q\mathcal{MO}_A^c(\mathbb{R}^n,\mathcal{M})} := \max\left\{\|g\|_{L^q\mathcal{MO}_A^c(\mathbb{R}^n,\mathcal{M})}, \|g\|_{L^q\mathcal{MO}_A^r(\mathbb{R}^n,\mathcal{M})}\right\}.$$

Theorem 3.4

Let $p \in (1, 2)$ and q be the conjugate index of p. Then we have

$$(\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M}))^* = L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})$$

in the following sense:

(i) *Every $g \in L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})$ defines a continuous linear functional $\mathcal{L}_g$ on $\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M})$ by*

$$\mathcal{L}_g(f) := \tau \int_{\mathbb{R}^n} f(x) g^*(x)\, dx, \quad \text{for any } S_{\mathcal{M}}\text{−valued simple function } f.$$

(ii) *For any $\mathcal{L} \in (\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M}))^*$, then there exists some $g \in L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})$ such that $\mathcal{L} = \mathcal{L}_g$. Moreover, there exists an universal positive constant C such that*

$$C^{-1}\|g\|_{L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})} \le \|\mathcal{L}_g\|_{(\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M}))^*} \le C\|g\|_{L^q\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})}.$$

Theorem 3.5

Let $p \in (2, \infty)$. *Then*

$$\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M}) = L^p\mathcal{MO}_A^c(\mathbb{R}^n, \mathcal{M})$$

with equivalent norms. Similarly, $\mathcal{H}_A^{p,r}(\mathbb{R}^n, \mathcal{M}) = L^p\mathcal{MO}_A^r(\mathbb{R}^n, \mathcal{M})$ *and* $\mathcal{H}_A^{p}(\mathbb{R}^n, \mathcal{M}) = L^p\mathcal{MO}_A(\mathbb{R}^n, \mathcal{M})$ *with equivalent norms.*

Corollary 3.1

Let $p \in (1, \infty)$ *and* q *be the conjugate index of* p. *Then*

$$(\mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M}))^* = \mathcal{H}_A^{q,c}(\mathbb{R}^n, \mathcal{M})$$

with equivalent norms.

Similarly, $(\mathcal{H}_A^{p,r}(\mathbb{R}^n, \mathcal{M}))^* = \mathcal{H}_A^{q,r}(\mathbb{R}^n, \mathcal{M})$ *and* $(\mathcal{H}_A^{p}(\mathbb{R}^n, \mathcal{M}))^* = \mathcal{H}_A^{q}(\mathbb{R}^n, \mathcal{M})$ *with equivalent norms.*

Theorem 3.6

Let $p \in (1, \infty)$. *Then*

$$\mathcal{H}_A^p(\mathbb{R}^n, \mathcal{M}) = L^p(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$$

with equivalent norms.

Theorem 3.7

Let $p \in (1, \infty)$. *Then*

$$\left[\mathcal{BMO}_A^c(\mathbb{R}^n, \mathcal{M}), \mathcal{H}_A^{1,c}(\mathbb{R}^n, \mathcal{M})\right]_{\frac{1}{p}} = \mathcal{H}_A^{p,c}(\mathbb{R}^n, \mathcal{M})$$

and

$$[X_1, X_2]_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M}),$$

where $X_1 = \mathcal{BMO}_A(\mathbb{R}^n, \mathcal{M})$ *or* $L^\infty(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$, $X_2 = \mathcal{H}_A^1(\mathbb{R}^n, \mathcal{M})$ *or* $L^1(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$.

Theorem 3.8

Let $p \in [1, \infty)$. *Then*

$$[X_0, X_1]_{\frac{1}{p}, p} = L^p(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M}),$$

where $X_0 = \mathcal{BMO}_A(\mathbb{R}^n, \mathcal{M})$ *or* $L^\infty(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$, $X_1 = \mathcal{H}_A^1(\mathbb{R}^n, \mathcal{M})$ *or* $L^1(L^\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M})$.

Spaces of Homogeneous Type

Let us recall the definition of $(\mathbb{X},\, d,\, \mu)$ is a *space of homogeneous type* in the sense of Coifman and Weiss [5], which means that $\mathbb{X}$ is a metric space with distance function d, and endows a nonnegative, Borel, doubling measure μ. In what follows, for any ball $B_d(x,\, r) := \{y \in \mathbb{X} : d(x,\, y) < r\} \subset \mathbb{X}$, we define the volume functions

$$V_r(x) := \mu(B_d(x,\, r)) \ \text{ and } \ V(x,\, y) := \mu(B_d(x,\, d(x,\, y))).$$

We say the measure μ is doubling if there exists a positive constant C_0 such that, for any $x \in \mathbb{X}$ and $r > 0$,

$$V_{2r}(x) \le C_0 V_r(x).$$

[5] R.R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, [M]. Bull. Amer. Math. Soc., 1977, 83: 569-645.

Lemma 4.1

Let $(\mathbb{X}, d, \mu)$ *is a space of homogeneous type. Then*

(i) *For any* $x \in \mathbb{X}$ *and* $r > 0$, $V(x, y) \sim V(y, x)$ *and*

$$V_r(x) + V_r(y) + V(x, y) \sim V_r(y) + V(x, y) \sim V_r(x) + V(x, y) \sim \mu(B(x, r + d(x, y))),$$

where the equivalent positive constant are independently of x, y *and* r.

(ii) *There exist two constants* $C > 0$ *and* $0 \le \gamma \le n$ *such that*

$$V_{r_1}(x) \le C \left[\frac{r_1 + d(x, y)}{r_2} \right]^{\gamma} V_{r_2}(y)$$

uniformly for any $x, y \in \mathbb{X}$ *and* $r_1, r_2 > 0$.

Main Assumptions. Let $(\mathbb{X}, d, \mu)$ be a space of homogeneous space type and $\epsilon_1 \in (0, 1]$ and $\epsilon_2 \in (0, \infty)$. Then there exists a *calderón reproducing formula of order* (ϵ_1, ϵ_2) on $L_2(\mathbb{X})$, that is, there exists a family of bounded linear operators, $\{\mathbf{D}_t\}_{t>0}$, on $L_2(\mathbb{X})$ is called a Calderón reproducing formula of order (ϵ_1, ϵ_2) (for short, (ϵ_1, ϵ_2)-**CRF**) in $L_2(\mathbb{X})$ if, for all $f \in L_2(\mathbb{X})$,

$$f = \int_0^\infty \mathbf{D}_t^2(f)\frac{dt}{t},$$

and moreover, for all $f \in L_2(\mathbb{X})$ and $x \in \mathbb{X}$,

$$\mathbf{D}_t(f)(x) = \int_{\mathbb{X}} \mathbf{D}_t(x, y) f(y)\, d\mu(y),$$

[6] Y. Han, D. Müller and D. Yang, Littlewood-Paley-Stein characterizations for Hardy spaces on spaces of homogeneous type, [M]. Math. Nachr., 2006, 279: 1505-1537.

where $\mathbf{D}_t(\cdot,\,\cdot)$ is a measurable function from $\mathbb{X}\times\mathbb{X}$ to $\mathbb{R}$ satisfying the following conditions: there exists a positive constant C_1 such that, for all $t\in\mathbb{R}_+$ and all $x,\,x',\,y,\,y'\in\mathbb{X}$ with $d(x,\,x')\le\frac{[t+d(x,y)]}{2}$,

$(\mathbf{H}_1)$ $|\mathbf{D}_t(x,\,y)|\le C_1\frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^{\epsilon_2}$;

$(\mathbf{H}_2)$ $|\mathbf{D}_t(x,\,y)-\mathbf{D}_t(x',\,y)|\le C_1\left[\frac{d(x,x')}{t+d(x,y)}\right]^{\epsilon_1}\frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^{\epsilon_2}$;

$(\mathbf{H}_3)$ Property $(\mathbf{H}_2)$ still holds true with the roles of x and y interchanged, and $\mathbf{D}_t(x,\,y)=\mathbf{D}_t(y,\,x)$;

$(\mathbf{H}_4)$ $\int_{\mathbb{X}}\mathbf{D}_t(x,\,y)\,d\mu(x)=0=\int_{\mathbb{X}}\mathbf{D}_t(x,\,y)\,d\mu(y)$.

Let $\{\mathbf{D}_t\}_{t>0}$ be an (ϵ_1, ϵ_1)-**CRF** with $\epsilon_1 \in (0, 1]$ and $\epsilon_2 \in (\frac{1}{2}, \infty)$. For any $S_{\mathcal{M}}$-valued simple function f, and $x \in \mathbb{X}$, the *Lusin area functions* of f is defined by

$$\mathcal{S}^c(f)(x) := \left(\int\int_{\Gamma_x} |\mathbf{D}_t(f)(y)|^2 \frac{d\mu(y)dt}{V_t(x)t} \right)^{1/2} \quad \text{and} \quad \mathcal{S}^r(f)(x) := \mathcal{S}^c(f^*)(x),$$

where $\Gamma_x := \{(y, t) \in \mathbb{X} \times \mathbb{R}_+ : d(x, y) < t\}$ with $x \in \mathbb{X}$.

Let $p \in [1, \infty)$. We define the $\mathcal{H}_p^c(\mathbb{X}, \mathcal{M})$ norms of f by

$$\|f\|_{\mathcal{H}_p^c(\mathbb{X}, \mathcal{M})} := \|\mathcal{S}^c(f)\|_{L_p(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M})}, \quad \|f\|_{\mathcal{H}_p^r(\mathbb{X}, \mathcal{M})} := \|\mathcal{S}^r(f)\|_{L_p(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M})}.$$

Define the *column Hardy space* $\mathcal{H}_p^c(\mathbb{X}, \mathcal{M})$ (resp. *row Hardy space* $\mathcal{H}_p^r(\mathbb{X}, \mathcal{M})$) to be the completion of the space of all $S_{\mathcal{M}}$-valued simple functions with finite $\mathcal{H}_p^c(\mathbb{X}, \mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathbb{X}, \mathcal{M})$) norm.

Define the *mixture space* $\mathcal{H}_p^{cr}(\mathbb{R}^n,\, \mathcal{M})$ as follows: when $p \in [1,\, 2)$,

$$\mathcal{H}_p^{cr}(\mathbb{X},\, \mathcal{M}) := \mathcal{H}_p^{c}(\mathbb{X},\, \mathcal{M}) + \mathcal{H}_p^{r}(\mathbb{X},\, \mathcal{M})$$

equipped with the sum norm

$$\begin{aligned} &\|f\|_{\mathcal{H}_p^{cr}(\mathbb{X},\, \mathcal{M})} \\ &:= \inf\left\{\|f_1\|_{\mathcal{H}_p^{c}(\mathbb{X},\, \mathcal{M})} + \|f_2\|_{\mathcal{H}_p^{r}(\mathbb{X},\, \mathcal{M})} :\, f = f_1 + f_2,\, f_1 \in \mathcal{H}_p^{c}(\mathbb{X},\, \mathcal{M}),\, f_2 \in \mathcal{H}_p^{r}(\mathbb{X},\, \mathcal{M})\right\} \end{aligned}$$

where the infimum is taken over all the decompositions of f as above. When $p \in [2,\, \infty)$, define

$$\mathcal{H}_p^{cr}(\mathbb{X},\, \mathcal{M}) := \mathcal{H}_p^{c}(\mathbb{X},\, \mathcal{M}) \cap \mathcal{H}_p^{r}(\mathbb{X},\, \mathcal{M})$$

equipped with the intersection norm

$$\|f\|_{\mathcal{H}_p^{cr}(\mathbb{X},\, \mathcal{M})} := \max\left\{\|f\|_{\mathcal{H}_p^{c}(\mathbb{X},\, \mathcal{M})},\, \|f\|_{\mathcal{H}_p^{r}(\mathbb{X},\, \mathcal{M})}\right\}.$$

Let $\mathcal{V}$ denote the all locally integrable functions on $\mathbb{X}$ with values in $\mathcal{M}$. For any ball $B \subset \mathbb{X}$ and operator-valued function $g \in \mathcal{V}$, we define g_B to be the mean of g on B, that is,

$$g_B := \frac{1}{\mu(B)} \int_B g(y)\, d\mu(y).$$

Definition 4.1

The column BMO *space* $\mathcal{BMO}^c(\mathbb{X},\mathcal{M})$ *is defined as*

$$\mathcal{BMO}^c(\mathbb{X},\mathcal{M}) := \left\{g \in \mathcal{V} : \|g\|_{\mathcal{BMO}^c(\mathbb{X},\mathcal{M})} < \infty\right\},$$

where

$$\|g\|_{\mathcal{BMO}^c(\mathbb{X},\mathcal{M})} := \sup_{x\in\mathbb{X},\, r>0} \left\| \left(\frac{1}{V_r(x)} \int_{B_d(x,r)} |g(y) - g_{B_d(x,r)}|^2 \, d\mu(y) \right)^{1/2} \right\|_{\mathcal{M}}.$$

Similarly, we define the row BMO *space* $\mathcal{BMO}^r(\mathbb{X},\mathcal{M})$ *as the space of* $g \in \mathcal{V}$ *such that* $g^* \in \mathcal{BMO}^c(\mathbb{X},\mathcal{M})$ *with the norm* $\|g\|_{\mathcal{BMO}^r(\mathbb{X},\mathcal{M})} := \|g^*\|_{\mathcal{BMO}^c(\mathbb{X},\mathcal{M})}$*, and the mixture* BMO *space*

$$\mathcal{BMO}^{cr}(\mathbb{X},\mathcal{M}) := \mathcal{BMO}^c(\mathbb{X},\mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{X},\mathcal{M})$$

with the norm $\|g\|_{\mathcal{BMO}^{cr}(\mathbb{X},\mathcal{M})} := \max\left\{\|g\|_{\mathcal{BMO}^c(\mathbb{X},\mathcal{M})}, \|g\|_{\mathcal{BMO}^r(\mathbb{X},\mathcal{M})}\right\}.$

Theorem 4.1

We have

$$(\mathcal{H}_1^c(\mathbb{X},\, \mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{X},\, \mathcal{M})$$

in the following sense:

(i) *Each* $g \in \mathcal{BMO}^c(\mathbb{X},\, \mathcal{M})$ *defines a continuous linear functional* $\mathcal{L}_g$ *on* $\mathcal{H}_1^c(\mathbb{X},\, \mathcal{M})$ *by*

$$\mathcal{L}_g(f) := \tau \int_{\mathbb{X}} f(x) g^*(x)\, d\mu(x), \quad for\ any\ S_{\mathcal{M}} - valued\ simple\ function\ f.$$

(ii) *If* $\mathcal{L} \in (\mathcal{H}_1^c(\mathbb{X},\, \mathcal{M}))^*$, *then there exists some* $g \in \mathcal{BMO}^c(\mathbb{X},\, \mathcal{M})$ *such that* $\mathcal{L} = \mathcal{L}_g$ *as the above.*

Moreover, there exists a positive constant C *such that*

$$C^{-1}\|g\|_{\mathcal{BMO}^c(\mathbb{X},\, \mathcal{M})} \le \|\mathcal{L}_g\|_{(\mathcal{H}_1^c(\mathbb{X},\, \mathcal{M}))^*} \le C\|g\|_{\mathcal{BMO}^c(\mathbb{X},\, \mathcal{M})}.$$

Let $\mathcal{U}$ denote the all locally integrable functions on $\mathbb{X}$ with values in $L_q(\mathcal{M})$. For any $g \in \mathcal{U}$ and ball $B \subset \mathbb{X}$, set

$$g_B^{\sharp}(x) := \frac{1}{\mu(B)} \int_B |g(y) - g_B|^2 \, d\mu(y), \quad x \in B,$$

where $g_B := \frac{1}{\mu(B)} \int_B g(y) \, d\mu(y)$.

Definition 4.2

Let $q \in (2, \infty)$. We define the column BMO-type space

$$L_q\mathcal{MO}^c(\mathbb{X}, \mathcal{M}) := \left\{ g \in \mathcal{U} : \|g\|_{L_q\mathcal{MO}^c(\mathbb{X}, \mathcal{M})} < \infty \right\},$$

where

$$\|g\|_{L_q\mathcal{MO}^c(\mathbb{X}, \mathcal{M})} := \left\| {\sup_{x \in B \subset \mathbb{X}}}^{+} g_B^{\sharp} \right\|_{L_{\frac{q}{2}}(L_\infty(\mathbb{X}) \overline{\otimes} \mathcal{M})}^{1/2}.$$

Theorem 4.2

Let $p \in (1, 2)$ *and* q *be the conjugate index of* p. *Then we have*

$$(\mathcal{H}_p^c(\mathbb{X}, \mathcal{M}))^* = L_q\mathcal{MO}^c(\mathbb{X}, \mathcal{M}).$$

Similarly, the duality holds between $\mathcal{H}_p^r(\mathbb{X}, \mathcal{M})$ *and* $L_q\mathcal{MO}^r(\mathbb{X}, \mathcal{M})$, *and between* $\mathcal{H}_p^{cr}(\mathbb{X}, \mathcal{M})$ *and* $L_q\mathcal{MO}^{cr}(\mathbb{X}, \mathcal{M})$ *with equivalent norms.*

Corollary 4.1

Let $p \in (1, \infty)$ *and* p' *be the conjugate index of* p. *Then*

$$(\mathcal{H}_p^c(\mathbb{X}, \mathcal{M}))^* = \mathcal{H}_{p'}^c(\mathbb{X}, \mathcal{M})$$

with equivalent norms. Similarly, $(\mathcal{H}_p^r(\mathbb{X}, \mathcal{M}))^* = \mathcal{H}_{p'}^r(\mathbb{X}, \mathcal{M})$ *and* $(\mathcal{H}_p^{cr}(\mathbb{X}, \mathcal{M}))^* = \mathcal{H}_{p'}^{cr}(\mathbb{X}, \mathcal{M})$ *with equivalent norms.*

Interpolations

Theorem 4.3

Let $1 \leq q < p < \infty$. *Then*

$$\left[\mathcal{BMO}^c(\mathbb{X}, \mathcal{M}), \mathcal{H}_q^c(\mathbb{X}, \mathcal{M})\right]_{\frac{q}{p}} = \mathcal{H}_p^c(\mathbb{X}, \mathcal{M}) \quad (4.1)$$

and

$$[\mathcal{X}, \mathcal{Y}]_{\frac{1}{p}} = L_p(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M}), \quad (4.2)$$

where $\mathcal{X} = \mathcal{BMO}^{cr}(\mathbb{X}, \mathcal{M})$ *or* $L_\infty(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M})$, $\mathcal{Y} = \mathcal{H}_1^{cr}(\mathbb{X}, \mathcal{M})$ *or* $L_1(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M})$.

Theorem 4.4

Let $1 \le q < p < \infty$. *Then*

$$\left[\mathcal{BMO}^c,\, \mathcal{H}_q^c(\mathbb{X},\, \mathcal{M})\right]_{\frac{q}{p},\, p} = \mathcal{H}_p^c(\mathbb{X},\, \mathcal{M})$$

with equivalent norms. Similar result also holds for row BMO and Hardy spaces.

Theorem 4.5

Let $p \in [1,\, \infty)$. *Then*

$$[\mathcal{X}_0,\, \mathcal{X}_1]_{\frac{1}{p},\, p} = L_p(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M}),$$

where $\mathcal{X}_0 = \mathcal{BMO}^{cr}(\mathbb{X},\, \mathcal{M})$ *or* $L_\infty(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M})$, $\mathcal{X}_1 = \mathcal{H}_1^{cr}(\mathbb{X},\, \mathcal{M})$ *or* $L_1(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M})$.

Let us recall that a linear operator T is a Calderón-Zygmund operator, if T is bounded on $L_2(L_\infty(\mathbb{X})\overline{\otimes}\mathcal{M})$ with kernel $\mathcal{K}$ coinciding with a locally integrable $\mathcal{M}$-valued function on $\mathbb{X} \setminus \{(x, x) : x \in \mathbb{X}\}$, and satisfying that there exists a positive constant C such that, for any $x, y \in \mathbb{X}$

$$\|\mathcal{K}(x, y)\|_{\mathcal{M}} \leq C \frac{1}{V(x, y)};$$

for any $x, x', y \in \mathbb{X}$ with $d(x', x) < \frac{d(x, y)}{2}$,

$$\|\mathcal{K}(x', y) - \mathcal{K}(x, y)\|_{\mathcal{M}} + \|\mathcal{K}(y, x') - \mathcal{K}(y, x)\|_{\mathcal{M}} \leq C \frac{[d(x', x)]^\delta}{V(x, y)[d(x, y)]^\delta}.$$

For any $S_{\mathcal{M}}$-valued simple function f on $\mathbb{X}$, define the δ-*type left Calderón-Zygmund operator* by

$$T^c(f)(x) := \int_{\mathbb{X}} \mathcal{K}(x, y) f(y)\, d\mu(y), \quad x \in \mathbb{X}.$$

In what follows, we let $\mathcal{BMO}^{c,0}(\mathbb{X}, \mathcal{M})$ to denote the subspace of $\mathcal{BMO}^{c}(\mathbb{X}, \mathcal{M})$ consisting of compactly supported functions.

Theorem 4.6

Let T^c be the left Calderón-Zygmund operator. Then T^c is bounded from $\mathcal{BMO}^{c,0}(\mathbb{X}, \mathcal{M})$ to $\mathcal{BMO}^{c}(\mathbb{X}, \mathcal{M})$. Moreover, there exists a positive constant C such that, for any $f \in \mathcal{BMO}^{c,0}(\mathbb{X}, \mathcal{M})$,

$$\|T^c(f)\|_{\mathcal{BMO}^c(\mathbb{X},\mathcal{M})} \leq C \|f\|_{\mathcal{BMO}^{c,0}(\mathbb{X},\mathcal{M})}.$$

Thanks