Sharp convergence for sequences of Schrödinger means and related generalizations

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Convergence for Schrödinger operators

The solution to the Schrödinger equation

$$i\partial_t u - \Delta u = 0, \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{1}$$

with initial datum $u(x,0) = f$, is formally written as

$$e^{it\Delta} f(x) := \int_{\mathbb{R}^N} e^{i\left(x\cdot\xi + t|\xi|^2\right)} \widehat{f}(\xi)\, d\xi.$$

Pointwise convergence problem: finding optimal s for which

$$\lim_{t\to 0^+} e^{it\Delta} f(x) = f(x) \ \ a.e. \ \ x \in \mathbb{R}^N. \tag{2}$$

whenever $f \in H^s\left(\mathbb{R}^N\right)$.

Convergence for Schrödinger operators

Carleson, 1980: True for $f \in H^{1/4}(\mathbb{R})$.
Dahlberg-Kenig, 1982: False for $s < 1/4$ in all dimensions.
Sjölin-Vega, 1985: Ture for $s > 1/2$ in any dimension.
Bourgain, 1995: Improved to $f \in H^{1/2-\epsilon}(\mathbb{R}^2)$ in dimension $N = 2$.
Later improved by Moyua-Vagas-Vega, Tao-Vagas-Vega, Tao-Vagas by bilinear restriction estimates.
Lee, 2006: True for $s > 3/8$ when $N = 2$ by Tao-Wolff's bilinear restriction method.
Lee, 2007: True for $s > \frac{1}{2} - \frac{1}{2N}$ when $N = 3$ on the basis of wave packet decomposition.

Convergence for Schrödinger operators

Bourgain, 2012: For $N \geq 3$ a sufficient condition is $s > \frac{1}{2} - \frac{1}{4N}$ by multilinear estimates for Fourier extension operators. For $N \geq 4$, a necessary condition is $s \geq \frac{1}{2} - \frac{1}{2N}$.

Bourgain, 2016: False for $s < \frac{N}{2(N+1)}$. A different counterexample was also raised by Lucà-Rogers.

Du-Guth-Li, 2017: When $N = 2$, convergence for $s > 1/3$ by the polynomial partitioning and ℓ^2-decoupling method. Sharp!

Du-Guth-Li-Zhang, 2018: For $N \geq 3$, $s > \frac{N+1}{2(N+2)}$ by linear refined Strichartz estimates.

Du-Zhang, 2019: For $N \geq 3$, $s > \frac{N}{2(N+1)}$ by a broad-narrow analysis, multilinear refined Strichartz estimate and decoupling. Sharp!

Convergence for Fractional Schrödinger operators

For the fractional Schrödinger operator

$$e^{it\Delta^{\frac{a}{2}}}f(x) := \int_{\mathbb{R}^N} e^{ix\cdot\xi+it|\xi|^a}\hat{f}(\xi)d\xi,$$

(1) When $a > 1$,

Sjolin,1987: shows that the convergence of $e^{it\triangle^{\frac{\alpha}{2}}}f(x)$ for $s \geq 1/4$ when $N = 1$, $s \geq 1/2$ when $N = 2$ and $s > 1/2$ when $N \geq 3$.

Miao-Yang-Zheng, 2015: By using the iterative argument developed in [Bourgain, 2012], improved this result to $s > 3/8$ for $N \geq 2$.

Cho-Ko, 2019: obtained an almost everywhere convergence for $s > \frac{N}{2(N+1)}$.

Convergence for Fractional Schrödinger operators

When $0 < a < 1$,
Walther, 2001: get the sufficient condition for $s > \alpha/4$ when $n = 1$, and $s > \alpha/4$ with addition requirement of being a radial function in higher dimension.
Zhang, 2014: removed the additional requirement and get the convergence for $f \in H^s(\mathbb{R}^n)$, $s > n\alpha/4$ in a uniform way.

Sharp results remains open when $N \geq 2$.

Convergence for non-elliptic Schrödinger operators

For the non-elliptic Schrodinger operator:

$$e^{itL}f(x) := \int_{\mathbb{R}^N} e^{ix\cdot\xi+it(\xi_1^2-\xi_2^2\pm\ldots\pm\xi_N^2)}\hat{f}(\xi)d\xi.$$

When $N = 2$,
Rogers-Vargas-Vega,2006: show that the pointwise convergence of the non-elliptic Schrödinger operators holds whenever $f \in H^s(\mathbb{R}^2)$ if and only if $s \geq 1/2$.

When $N > 2$,
similar results hold except the endpoint.

Some interesting observations

Let $N = 1$ and $t_n = 1/n$, $n = 1, 2, \cdots$. Consider

$$\lim_{n\to\infty} e^{it_n\Delta} f(x) = f(x) \;\; a.e. \;\; x \in \mathbb{R}, \tag{3}$$

One may expect that less regularity on f is enough to ensure convergence in this discrete case.

Observation:
Carleson, 1980: True for $s > 1/4$ but false for $s < 1/8$.
Lee-Rogers, 2012: False for $s < 1/4$ (implied by the counterexample of Dahlberg-Kenig, 1982).

Further results

When $N = 1$

Dimou-Seeger, 2020: $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$, $0 < r < \infty$, i.e.,

$$\sup_{b>0} b^r \sharp \left\{ n \in \mathbb{N} : t_n > b \right\} < \infty, \tag{4}$$

$e^{it_n\Delta}f$ pointwisely converges to f if and only if $s \geq min\{\frac{r}{2r+1}, \frac{1}{4}\}$.

When $N \geq 2$,

partially convergence results obtained by Sjölin, Sjölin-Strömberg but they are far from sharp.

Question

One of the natural generalizations of the pointwise convergence problem is to ask a.e. convergence of the Schrödinger means where the limit is taken over decreasing sequences $\{t_n\}_{n=1}^{\infty}$ converging to zero.

Question: What is the relationship between optimal s and properties of $\{t_n\}_{n=1}^{\infty}$ such that for each function $f \in H^s(\mathbb{R}^N)$,

$$\lim_{n\to\infty} e^{it_n\Delta} f(x) = f(x) \ \ a.e. \ \ x \in \mathbb{R}^N. \tag{5}$$

Sequence of Schrödinger means

This problem was first considered by Sjölin. Define

$$S_t^a f(x) = \int_{\mathbb{R}^n} e^{i\xi\cdot x} e^{it|\xi|^a} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^N, \quad t \geq 0. \tag{6}$$

and

$$1 > t_1 > t_2 > t_3 > \cdots > 0 \text{ and } \lim_{k\to\infty} t_k = 0. \tag{7}$$

Theorem A (Sjölin, JFAA, 2018)

Assume $N \geq 1$ and $a > 1$ and $s > 0$. We assume that (7) holds and that $\sum_{k=1}^{\infty} t_k^{2s/a} < \infty$ and $f \in H^s(\mathbb{R}^N)$. Then

$$\lim_{k\to\infty} S_{t_k}^a f(x) = f(x) \tag{8}$$

for almost every $x \in \mathbb{R}^N$.

Sequence of Schrödinger means

Theorem B (Sjölin-Strömberg, JMAA, 2020)

Assume $N \geq 1$ and $a > 1$ and $0 < s \leq 1/2$. We assume that (7) holds and that $\sum_{k=1}^{\infty} t_k^{2s/(a-s)} < \infty$. If $f \in H^s(\mathbb{R}^N)$, then

$$\lim_{k \to \infty} S_{t_k}^a f(x) = f(x) \tag{9}$$

for almost every $x \in \mathbb{R}^N$.

Sequence of Schrödinger means

In order to characterize the convergence of $\{t_n\}_{n=1}^{\infty}$, the Lorentz space $\ell^{r,\infty}(\mathbb{N})$, $r > 0$ is involved. The sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ if and only if

$$\underset{b>0}{sup}\, b^r \sharp \Big\{ n : t_n > b \Big\} < \infty. \tag{10}$$

Note that $\ell^{r_1,\infty}(\mathbb{N}) \subset \ell^{r_2}(\mathbb{N}) \subset \ell^{r_2,\infty}(\mathbb{N})$ if $r_1 < r_2 < \infty$ and all inclusions are strict.

Sequence of Schrödinger means

Dimou-Seeger proved complete characteraization of when sequence convergence holds for all $f \in H^s(\mathbb{R})$ when $N = 1$.

Theorem C (Dimou-Seeger, Mathematika, 2020)

Let $a > 0$, $a \neq 1$, and $0 < s < \min\{a/4, 1/4\}$. Assume that (7) holds. Then for every $f \in H^s(\mathbb{R})$, we have

$$\lim_{n\to\infty} S^a_{t_n} f(x) = f(x) \quad a.e. x \in \mathbb{R} \tag{11}$$

holds if and only if the sequence $\{t_n\} \in \ell^{\frac{2s}{a-4s},\infty}(\mathbb{N})$.

Sequence of Schrödinger means

Remark: Specially, let $a = 2$,
(1) Since $\{t_n\} \in \ell^{r,\infty}(\mathbb{N}) \Rightarrow \{t_n\} \in \ell^{r(s),\infty}(\mathbb{N})$ if $r \leq r(s)$, which means that Dimou-Seeger proved that when $N = 1$, $s \geq \min\{\frac{r}{2r+1}, \frac{1}{4}\}$ is necessary and sufficient for inequality (12) to hold.

(2) For $N > 1$, it follows from [Sjölin, 2018] that $s > min\{r, \frac{N}{2(N+1)}\}$ is sufficient. This was later improved by [Sjölin-Strömberg, 2020] where $s > min\{\frac{r}{r+1}, \frac{N}{2(N+1)}\}$ is shown to be enough for pointwise convergence.

Preliminary Attempt

Theorem (Li-Wang-Yan, 2020)

Given a decreasing sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ converging to zero and $\{t_n\}_{n=1}^{\infty} \subset (0,1)$, then for any $s > s_0 = min\{\frac{r}{\frac{4}{3}r+1}, \frac{1}{3}\}$,

$$\lim_{n\to\infty} e^{it_n\Delta} f(x) = f(x) \quad a.e. x \in \mathbb{R}^2 \tag{12}$$

holds whenever $f \in H^s(\mathbb{R}^2)$.

Open question

Question: Let $N \geq 2$. What is the relationship between optimal s and properties of $\{t_n\}_{n=1}^{\infty}$ such that for each function $f \in H^s(\mathbb{R}^N)$,

$$\lim_{n\to\infty} e^{it_n\Delta}f(x) = f(x) \ \ a.e. \ \ x \in \mathbb{R}^N. \tag{13}$$

Next we will give our recent results on this problem and state some related generalizations.

Main results and related generalizations

Main results

Theorem 1 (Li-Wang-Yan, 2022)

Let $N \geq 2$ and $r \in (0, \infty)$. For any decreasing sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ converging to zero and $\{t_n\}_{n=1}^{\infty} \subset (0, 1)$, we have

$$\lim_{n \to \infty} e^{it_n \Delta} f(x) = f(x) \;\; a.e. \;\; x \in \mathbb{R}^N \tag{14}$$

whenever $f \in H^s(\mathbb{R}^N)$ and $s > s_0 = min\{\frac{r}{\frac{N+1}{N} r+1}, \frac{N}{2(N+1)}\}$.

Main results

By standard arguments, it is sufficient to show the corresponding maximal estimate in $\mathbb{R}^N$.

Theorem 2 (Li-Wang-Yan, 2022)

Under the assumptions of Theorem 1, we have

$$\left\| \sup_{n\in\mathbb{N}} |e^{it_n\Delta} f| \right\|_{L^2(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R}^N)}, \tag{15}$$

whenever $f \in H^s(\mathbb{R}^N)$ and $s > s_0 = min\{\frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\}$, where the constant C does not depend on f.

Main results

The L^2 maximal estimate in Theorem 2 is sharp up to the endpoints.

Theorem 3 (Li-Wang-Yan, 2022)

For each $r \in (0, \infty)$, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ which belongs to $\ell^{r,\infty}(\mathbb{N})$, the corresponding maximal estimate (15) fails if $s < s_0 = min\{\frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\}$.

Examples

Example: We pose two examples for $\{t_n\}_{n=1}^{\infty}$. It is not hard to check that $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ if we take (E1): $t_n = \frac{1}{n^{1/r}}$, $n \geq 1$. It is obvious that when $r < \frac{N}{N+1}$, there is a gain over the general pointwise convergence result for $s > \frac{N}{2(N+1)}$ in $\mathbb{R}^N$.

Another example is the lacunary sequence (E2): $t_n = 2^{-n}$, $n \geq 1$. For this example, it is worth to mention that $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ for each $r > 0$. Therefore, inequality (14) holds whenever $f \in H^s(\mathbb{R}^2)$ for any $s > 0$.

Remark

Theorem 2 and Theorem 3 reveal a perhaps surprising phenomenon, namely if $0 < r < \frac{N}{N+1}$, there is a gain over the pointwise convergence result from [Du-Guth-Li, 2017] [Du-Zhang, 2019] [Bourgain, 2016] [Luca-Rogers, 2018] when time tends continuously to zero, but not when $r \geq \frac{N}{N+1}$.

Related generalizations

The method applies to or the fractional case. We have the following maximal estimate. When $a = 2$, it coincides with Theorem 2.

Theorem 4 (Li-Wang-Yan, 2022)

Under the conditions of Theorem 2, for $1 < a < \infty$, we have

$$\left\| \sup_{n \in \mathbb{N}} |e^{it_n \Delta^{\frac{a}{2}}} f| \right\|_{L^2(B(0,1))} \leq C \|f\|_{H^s(\mathbb{R}^N)}, \tag{16}$$

whenever $f \in H^s(\mathbb{R}^N)$ and $s > s_0 = min\{\frac{a}{2} \cdot \frac{r}{\frac{N+1}{N} r+1}, \frac{N}{2(N+1)}\}$, where the constant C does not depend on f.

Related generalizations

For the non-elliptic case, we have the following estimate.

Theorem 5 (Li-Wang-Yan, 2022)

Under the conditions of Theorem 2, we have

$$\left\| \sup_{n\in\mathbb{N}} |e^{it_nL} f| \right\|_{L^2(B(0,1))} \le C\|f\|_{H^s(\mathbb{R}^N)}, \tag{17}$$

whenever $f \in H^s(\mathbb{R}^N)$ and $s > s_0 = min\{\frac{r}{r+1}, \frac{1}{2}\}$, where the constant C does not depend on f.

Especially, the endpoint can be achieved when $N = 2$.

Known results and open problems

Operators type	Spatial dimensions	Continuous case $t \to 0$	Discrete case $t_n \to 0$
Schrödinger operator	$N = 1$	$s \geq \frac{1}{4}$	$s \geq \min\{\frac{1}{4}, \frac{r}{2r+1}\}$
	$N \geq 2$	$s > \frac{N}{2(N+1)}$	$s > \min\{\frac{N}{2(N+1)}, \frac{r}{\frac{N+1}{N}r+1}\}$
Nonelliptic Schrödinger	$N = 2$	$s \geq \frac{1}{2}$	$s \geq \min\{\frac{1}{2}, \frac{r}{r+1}\}$
	$N \geq 3$	$s > \frac{1}{2}$	$s > \min\{\frac{1}{2}, \frac{r}{r+1}\}$
Fractional $a > 1$	$N = 1$	$s \geq \frac{1}{4}$	$s \geq \min\{\frac{1}{4}, \frac{a}{2}\frac{r}{2r+1}\}$
	$N \geq 2$	$s > \frac{N}{2(N+1)}$	$s > \min\{\frac{N}{2(N+1)}, \frac{a}{2}\frac{r}{\frac{N+1}{N}r+1}\}$
Fractional $0 < a < 1$	$N = 1$	$s > \frac{a}{4}$	$s > \min\{\frac{a}{4}, \frac{a}{2}\frac{r}{2r+1}\}$
	$N \geq 2$	sharp result is open	sharp result is open

Known results and open problems

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- Thu, 14 Jul 2022

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[1] **arXiv:2207.09219** [**pdf**, **ps**, **other**]
Pointwise convergence of sequential Schrödinger means
Chu-Hee Cho, Hyerim Ko, Youngwoo Koh, Sanghyuk Lee
Subjects: **Classical Analysis and ODEs (math.CA)**

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[2] **arXiv:2207.08709** [**pdf**, **ps**, **other**]
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[4] **arXiv:2207.08440** [**pdf**, **ps**, **other**]
Sharp convergence for sequences of Schrödinger means and related generalizations
Wenjuan Li, Huiju Wang, Dunyan Yan
Subjects: **Classical Analysis and ODEs (math.CA)**; Analysis of PDEs (math.AP)

[5] **arXiv:2207.08308** [**pdf**, **ps**, **other**]
Strong asymptotics of multi-level Hermite-Padé polynomials
L.G. González Ricardo, G. López Lagomasino
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Subjects: **Classical Analysis and ODEs (math.CA)**

[6] **arXiv:2207.08252** [**pdf**, **ps**, **other**]
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Comments: 7 pages
Subjects: **Classical Analysis and ODEs (math.CA)**

Proof of the L^2 maximal estimate: Theorem 2

Sketch for the proof of Theorem 2

By Littlewood-Paley decomposition and standard argument, we just concentrate on the case when supp$\hat{f} \subset \{\xi : |\xi| \sim 2^k\}$, $k \gg 1$. We consider the maximal function

$$\sup_{n\in\mathbb{N}:t_n\geq 2^{-\frac{2k}{(N+1)r/N+1}}} |e^{it_n\Delta}f|$$

and

$$\sup_{n\in\mathbb{N}:t_n< 2^{-\frac{2k}{(N+1)r/N+1}}} |e^{it_n\Delta}f|,$$

respectively. We deal with the first term by the assumption that the decreasing sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ and Plancherel's theorem.

Sketch for the proof of Theorem 2

For the second term, since $k < \frac{2k}{\frac{N+1}{N}r+1} < 2k$, the proof can be completed by the following theorem.

Theorem 6 (Li-Wang-Yan, 2022)

Let $j \in \mathbb{R}$ with $k < j < 2k$. For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f| \right\|_{L^2(B(0,1))} \leq C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)}+\epsilon k} \|f\|_{L^2(\mathbb{R}^N)}, \qquad (18)$$

for all f with supp $\hat{f} \subset \{\xi : |\xi| \sim 2^k\}$. The constant C_ϵ does not depend on f, j and k.

Remarks for Theorem 6

We have the following remarks for Theorem 6:

• Theorem 6 is sharp when $j = k$ and $j = 2k$.

• The presence of $2^{\epsilon k}$ leads us to lose the endpoint results in Theorem 2.

• In the case $N = 1$, similar result was built in [Dimou-Seeger, 2020] by TT^* argument and stationary phase method. But their method seems not to work well in the higher dimensional case.

Sketch for the proof of Theorem 6

Notice that for any function g with supp $\hat{g} \subset \{\xi : |\xi| \sim 2^{2k-j}\}$, it holds

$$\left\| \sup_{t \in (0, 2^{-(2k-j)})} |e^{it\Delta} g| \right\|_{L^2(B(0,1))} \leq C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)} + \epsilon k} \|g\|_{L^2(\mathbb{R}^N)}.$$

By scaling, we have

$$\left\| \sup_{t \in (0, 2^{-j})} |e^{it\Delta} g| \right\|_{L^2(B(0,2^{k-j}))} \leq C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)} + \epsilon k} \|g\|_{L^2(\mathbb{R}^N)} \tag{19}$$

whenever supp $\hat{g} \subset \{\xi : |\xi| \sim 2^k\}$.

Sketch for the proof of Theorem 6

Then we obtain the following lemma by translation invariance in the x-direction.

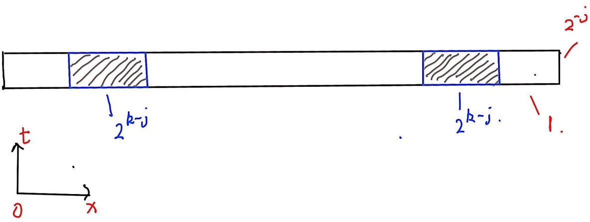
Lemma 7 (Li-Wang-Yan, 2022)

When $k < j < 2k$, for any $\epsilon > 0$ and $x_0 \in \mathbb{R}^N$, there exists a constant $C_\epsilon > 0$ such that

$$\left\| \sup_{t \in (0, 2^{-j})} \left| e^{it\Delta} f \right| \right\|_{L^2(B(x_0, 2^{k-j}))} \leq C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)} + \epsilon k} \|f\|_{L^2(\mathbb{R}^N)}, \tag{20}$$

whenever supp $\hat{f} \subset \{\xi : |\xi| \sim 2^k\}$. The constant C_ϵ does not depend on x_0 and f.

Sketch for the proof of Theorem 6



$$\left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f(x)| \right\|^2_{L^2(B(0,1))} \le \sum_{\nu'} \left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f(x)| \right\|^2_{L^2(B(c(\nu'),2^{k-j}))}$$

$$\le C_\epsilon 2^{(2k-j)\frac{N}{N+1}+\epsilon k} \sum_{\nu'} \|f\|^2_{L^2}$$

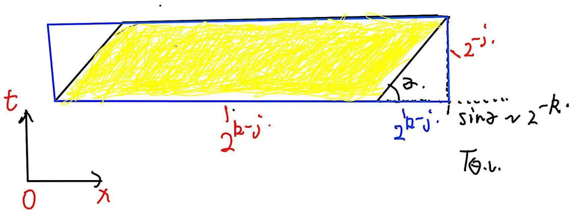
Sketch for the proof of Theorem 6

- Wave packets decomposition

$$f=\sum_\nu\sum_\theta f_{\theta,\nu}=\sum_\nu\sum_\theta\langle f,\varphi_{\theta,\nu}\rangle\varphi_{\theta,\nu},$$

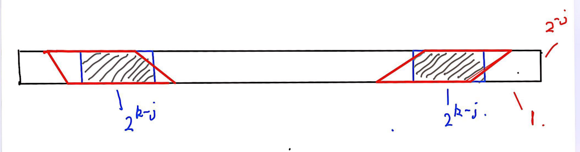
$e^{it\Delta}f_{\theta,\nu}$ is essentially supported in $T_{\theta,\nu}$ defined by

$$T_{\theta,\nu}:=\{(x,t),|x-c(\nu)+2tc(\theta)|\le 2^{(j-k)(-1+\delta)},0\le t\le 2^{-j}\},$$



Sketch for the proof of Theorem 6

- Orthogonality argument



$$\left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta}f(x)| \right\|^2_{L^2(B(0,1))} \leq \sum_{\nu'} \left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta}f_1(x)| \right\|^2_{L^2(B(c(\nu'),2^{k-j}))} + 2^{-2000k}\|f\|^2_{L^2} \leq \sum_{\nu'} C_\epsilon^2 2^{(2k-j)\frac{N}{N+1}+2\epsilon k}\|f_1\|^2_{L^2} + 2^{-2000k}\|f\|^2_{L^2},$$

where $f_1 = \sum_\theta \sum_{\nu:|c(\nu)-c(\nu')|\leq 2^{(j-k)(-1+10\delta)}} f_{\theta,\nu}$ and $f_2 = f - f_1$.

Construction of counterexample: Theorem 3

Sketch for the proof of Theorem 3

We notice that the counterexample for $r = \frac{N}{N+1}$ can be also applied to the case when $r > \frac{N}{N+1}$, since $\ell^{N/(N+1),\infty}(\mathbb{N}) \subset \ell^{r,\infty}(\mathbb{N})$ and $min\{\frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\} = \frac{N}{2(N+1)}$ when $r > \frac{N}{N+1}$.

Therefore, next we always assume $r \in (0, \frac{N}{N+1}]$.

Sketch for the proof of Theorem 3

Put $\beta = \frac{2}{\frac{N+1}{N}r+1}$. Let $R_1 = 2$ and for each positive integer n,

$$R_{n+1}^{-\beta} \leq \frac{1}{2} R_n^{-\beta(r+1)},$$

Denote the intervals $I_n = [R_n^{-\beta(r+1)}, R_n^{-\beta})$, $n \in \mathbb{N}^+$.
On each I_n, we get an equidistributed subsequence $t_{n_j}, j = 1, 2, ..., j_n$ such that

$$\{t_{n_j}, 1 \leq j \leq j_n\} =: R_n^{-\beta(r+1)} \mathbb{Z} \cap I_n,$$

and $t_{n_j} - t_{n_{j+1}} = R_n^{-\beta(r+1)}$.

Sketch for the proof of Theorem 3

Our counterexample comes from the following lemma.

Lemma 8 (Li-Wang-Yan, 2022)

Let $R \gg 1$ and $I = [R^{-\beta(r+1)}, R^{-\beta})$. Assume that the sequence $\{t_j : 1 \le j \le j_0\} = R^{-\beta(r+1)}\mathbb{Z} \cap I$ and $t_j - t_{j+1} = R^{-\beta(r+1)}$ for each $1 \le j \le j_0 - 1$. Then there exists a function f with supp $\hat{f} \subset B(0, 2R)$ such that

$$\Big\| \sup_{1\le j\le j_0} \big|e^{i\frac{t_j}{2\pi}\Delta} f\big| \Big\|_{L^2(B(0,1))} \gtrsim R^{\frac{1-\beta}{2}} R^{\frac{\beta}{2}} R^{(N-1)(1-\frac{(r+1)\beta}{2})-\epsilon}, \tag{21}$$

and

$$\|f\|_{H^s(\mathbb{R}^N)} \lesssim R^s R^{\frac{\beta}{4}} R^{\frac{N-1}{2}(1-\frac{(r+1)\beta}{2})}. \tag{22}$$

Here $\epsilon > 0$ can be sufficiently small.

Sketch for the proof of Theorem 3

Assume that the maximal estimate

$$\left\| \sup_n \sup_j |e^{i\frac{t_{n_j}}{2\pi}\Delta} f| \right\|_{L^2(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R}^N)} \tag{23}$$

holds for some $s > 0$ and each $f \in H^s(\mathbb{R}^N)$, then for each $n \in \mathbb{N}^+$, we have

$$\left\| \sup_j |e^{i\frac{t_{n_j}}{2\pi}\Delta} f| \right\|_{L^2(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R}^N)} \tag{24}$$

whenever $f \in H^s(\mathbb{R}^N)$. Lemma 6 and (24) yield

$$R_n^{\frac{2-\beta}{4}} R_n^{\frac{N-1}{2}(1-\frac{(r+1)\beta}{2})-\epsilon} \leq CR_n^s. \tag{25}$$

Then we have $s \geq \frac{r}{\frac{N+1}{N}r+1}$.

Sketch for the proof of Lemma 8

Setting

$$\Omega_1 = \left(-\frac{1}{100}R^{\frac{\beta}{2}}, \frac{1}{100}R^{\frac{\beta}{2}}\right),$$

$$\Omega_2 = \left\{\bar{\xi} \in \mathbb{R}^{N-1} : \bar{\xi} \in 2\pi R^{\frac{(r+1)\beta}{2}} \mathbb{Z}^{N-1} \cap B(0, R^{1-\epsilon})\right\} + B(0, \frac{1}{1000}),$$

then we define $\hat{f}_1(\xi_1) = \hat{h}(\xi_1 + \pi R)$, $\hat{f}_2(\bar{\xi}) = \hat{g}(\bar{\xi} + \pi R\theta)$, where $\hat{h} = \chi_{\Omega_1}$, $\hat{g} = \chi_{\Omega_2}$, and some $\theta \in \mathbb{S}^{N-2}$ (when $N = 2$, we denote $\mathbb{S}^0 := (0,1)$) which will be determined later. Define f by $\hat{f} = \hat{f}_1 \hat{f}_2$.

It is easy to check that f satisfies (22). We are left to prove that inequality (21) holds for such f. Notice that

$$|e^{i\frac{t_j}{2\pi}\Delta} f(x_1, \bar{x})| = |e^{i\frac{t_j}{2\pi}\Delta} f_1(x_1)||e^{i\frac{t_j}{2\pi}\Delta} f_2(\bar{x})|. \tag{26}$$

Sketch for the proof of Lemma 8

For the first part:

- A change of variables implies

$$|e^{i\frac{t_j}{2\pi}\Delta}f_1(x_1)| = |e^{i\frac{t_j}{2\pi}\Delta}h(x_1 - Rt_j)|.$$

- $|e^{i\frac{t_j}{2\pi}\Delta}h(x_1)| \gtrsim |\Omega_1|$ for each j whenever $|x_1| \leq R^{-\frac{\beta}{2}}$.
- We have

$$|e^{i\frac{t_j}{2\pi}\Delta}f_1(x_1)| \gtrsim |\Omega_1|,$$

whenever $x_1 \in (0, \frac{1}{2}R^{1-\beta})$ and $Rt_j \in (x_1, x_1 + R^{-\frac{\beta}{2}})$.

Sketch for the proof of Lemma 8

For the second part:

- we have

$$|e^{i\frac{t_j}{2\pi}\Delta}f_2(\bar{x})| = |e^{i\frac{t_j}{2\pi}\Delta}g(\bar{x} - Rt_j\theta)|.$$

- for each j and $\bar{x} \in U_0$,

$$|e^{i\frac{t_j}{2\pi}\Delta}g(\bar{x})| \gtrsim |\Omega_2|, \tag{27}$$

here

$$U_0 = \left\{\bar{x} \in \mathbb{R}^{N-1} : \bar{x} \in R^{-\frac{(r+1)\beta}{2}}\mathbb{Z}^{N-1} \cap B(0,2)\right\} + B(0, \frac{1}{1000}R^{-1+\epsilon}).$$

- we have

$$|e^{i\frac{t_j}{2\pi}\Delta}f_2(\bar{x})| \gtrsim |\Omega_2|, \tag{28}$$

if $\bar{x} \in U_{x_1} = \bigcup_{j: Rt_j \in R^{1-(r+1)\beta}\mathbb{Z} \cap (x_1, x_1 + R^{-\beta/2})} U_0 + Rt_j\theta.$

Sketch for the proof of Lemma 8

Next we need to select a $\theta \in \mathbb{S}^{N-2}$, such that $|U_{x_1}| \gtrsim 1$ for each $x_1 \in (0, \frac{1}{2}R^{1-\beta})$, which follows if we can prove that there exists a $\theta \in \mathbb{S}^{N-2}$ so that $B(0, 1/2) \subset U_{x_1}$ for all $x_1 \in (0, \frac{1}{2}R^{1-\beta})$.

It suffices to prove the claim that there exists a $\theta \in \mathbb{S}^{N-2}$ such that

$$\bigcup_{j: Rt_j \in R^{1-\beta(r+1)}\mathbb{Z} \cap (x_1, x_1 + R^{-\beta/2})} \left\{ \bar{x} \in \mathbb{R}^{N-1} : \bar{x} \in R^{-\frac{(r+1)\beta}{2}} \mathbb{Z}^{N-1} \cap B(0,2) \right\} + Rt_j\theta$$

is $\frac{1}{1000}R^{-1+\epsilon}$ dense in the ball $B(0, 1/2)$.

Sketch for the proof of Lemma 8

This can be proved by the following lemma.

Lemma 11 (Lucà-Rogers, 2017)

Let $d \geq 2$, $0 < \epsilon, \delta < 1$ and $\kappa > \frac{1}{d+1}$. Then, if $\delta < \kappa$ and $R > 1$ is sufficiently large, there is $\theta \in \mathbb{S}^{d-1}$ for which, given any $[y] \in \mathbb{T}^d$ and $a \in \mathbb{R}$, there is a $t_y \in R^\delta \mathbb{Z} \cap \{a, a+R\}$ such that

$$|[y] - [t_y \theta]| \leq \epsilon R^{(\kappa-1)/d},$$

where "$[\cdot]$" means taking the quotient $\mathbb{R}^d / \mathbb{Z}^d = \mathbb{T}^d$. Moreover, this remains true with $d = 1$, for some $\theta \in (0, 1)$.

Sketch for the proof of Lemma 8

Finally, we obtain that

$$\begin{aligned}\int_{B(0,1)} \sup_j |e^{i\frac{t_j}{2\pi}\Delta} f(x_1,\bar{x})|^2 d\bar{x}dx_1 &\geq \int_0^{\frac{R^{1-\beta}}{2}} \int_{U_{x_1}} \sup_j |e^{i\frac{t_j}{2\pi}\Delta} f(x_1,\bar{x})|^2 d\bar{x}dx_1 \\ &\gtrsim R^{1-\beta}|\Omega_1|^2|\Omega_2|^2,\end{aligned}$$

which completes the proof of Lemma 8.

Thanks!