Bilinear decompositions and endpoint estimates of martingale commutators

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Joint with Yong Jiao et al.

Overview

Product of functions in H_1 and BMO

- Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi \neq 0$. The Hardy space $H_1(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H_1(\mathbb{R}^n)} := \|\mathcal{M}(f)\|_{L^1(\mathbb{R}^n)} := \Big\| \sup_{t>0} |f * \phi_t(x)| \Big\|_{L^1(\mathbb{R}^n)} < \infty.$$

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- For Q a cube of $\mathbb{R}^n$ and $f \in L^1_{\rm loc}(\mathbb{R}^n)$, let $f_Q := \frac{1}{|Q|} \int_Q f$. A function f is in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| \, dx < \infty.$$

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$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|}\int_Q |f - f_Q|\, dx < \infty.$$

- For any $f \in H_1(\mathbb{R}^n)$ and $g \in \mathrm{BMO}(\mathbb{R}^n)$, the product $f \times g$ is defined to be a Schwartz distribution in $\mathcal{S}'(\mathbb{R}^n)$ such that, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle f \times g, \phi \rangle := \langle \phi g, f \rangle.$$

- There exist two bounded bilinear operators

$$L: \ H_1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

and

$$G: \ H_1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \to H_\varphi(\mathbb{R}^n)$$

such that

$$f \times g = L(f,g) + G(f,g), \qquad \forall\, f \in H_1(\mathbb{R}^n),\ g \in \mathrm{BMO}(\mathbb{R}^n).$$

Here, $H_\varphi(\mathbb{R}^n)$ is a Musielak–Orlicz Hardy space defined by

$$\varphi(x,t) := t/[\log(e+|x|) + \log(e+t)], \qquad \forall\,(x,t) \in \mathbb{R}^n \times [0,\infty).$$

A. Bonami, T. Iwaniec, P. Jones and M. Zinsmeister, *On the product of functions in BMO and* H^1, Ann. Inst. Fourier (Grenoble) **57** (2007), 1405–1439.

A. Bonami, S. Grellier and L. D. Ky, *Paraproducts and products of functions in* $BMO(\mathbb{R}^n)$ *and* $\mathcal{H}^1(\mathbb{R}^n)$ *through wavelets*, J. Math. Pures Appl. (9) **97** (2012), 230–241.

- Motivation: geometric function theory and nonlinear elasticity.
- Sharpness: $H_\varphi(\mathbb{R}^n)$ is the smallest Banach space $\mathcal{Y}$ satisfying $H_1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + \mathcal{Y}$.
- Applications: commutators, Div-Curl Lemma.

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A. Bonami and L. D. Ky, *Factorization of some Hardy-type spaces of holomorphic functions*, C. R. Math. Acad. Sci. Paris **352** (2014), 817–821.

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Further study:

L. Liu, D. Yang and W. Yuan, *Bilinear decompositions for products of Hardy and Lipschitz spaces on spaces of homogeneous type*, Dissertationes Math. **533** (2018), 1–93.

A. Bonami et al., *Multiplication between Hardy spaces and their dual spaces,* J. Math. Pures Appl. (9) **131** (2019), 130–170.

O. Bakas, S. Pott, S. Rodríguez-López and A. Sola, *Notes on $H^{\log}$: structural properties, dyadic variants, and bilinear H^1-BMO mappings*, Ark. Mat. (to appear) or arXiv: 2012.02872 (2020).

D. Yang, W. Yuan and Y. Zhang, *Bilinear decomposition and divergence-curl estimates on products related to local Hardy spaces and their dual spaces*, J. Funct. Anal. **280** (2021), Paper No. 108796, 74 pp.

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Product of martingales in H_1 and BMO

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$ with the associated conditional expectations $(\mathbb{E}_n)_{n\in\mathbb{Z}_+}$.

- A martingale $f := (f_n)_{n\in\mathbb{Z}_+}$ is in martingale Hardy space H_1 if

$$\|f\|_{H_1} := \|S(f)\|_{L^1} := \Big\|\Big(\sum_{n\in\mathbb{N}} |f_n - f_{n-1}|^2\Big)^{\frac{1}{2}}\Big\|_{L^1} < \infty.$$

- The martingale space BMO is defined to be the set of all the martingales $f \in L^1$ with the norm

$$\|f\|_{\mathrm{BMO}} := \sup_{n\in\mathbb{Z}_+} \|\mathbb{E}_n(|f - f_{n-1}|)\|_{L^\infty} < \infty.$$

- It is well known that $(H_1)^* = \mathrm{BMO}$.

How to define the product of martingales?

- $f = (f_n)_{n\in\mathbb{Z}_+}$ belongs to H_1 and $g = (g_n)_{n\in\mathbb{Z}_+}$ belongs to BMO, the product $f_\infty g_\infty$ is not integrable in general.

- It is natural to define $f \cdot g$ as the discrete process $(f_n g_n)_{n\in\mathbb{Z}_+}$. Then

$$f_n g_n - f_{n-1}g_{n-1} = f_{n-1}d_n(g) + g_{n-1}d_n(f) + d_n(f)d_n(g).$$

  The first two terms are differences of martingales, while the third one is the difference of a process of bounded variation. Thus,

$$\begin{aligned} f_n g_n &= \sum_{k=0}^{n} f_{k-1}d_k(g) + \sum_{k=0}^{n} g_{k-1}d_k(f) + \sum_{k=0}^{n} d_k(f)d_k(g) \\ &=: \Pi_{1,n}(f,g) + \Pi_{2,n}(f,g) + \Pi_{3,n}(f,g). \end{aligned}$$

- We call $\mathcal{B}V$ the space of adapted sequences of random variables $f := (f_n)$ such that $E(\sum |f_n - f_{n-1}|) < \infty$.

- $(\Pi_{3,n})_{n\in\mathbb{Z}_+} \in \mathcal{B}V$. Indeed, according to the duality $(H_1)^* = \mathrm{BMO}$,

$$E\left(\sum |\Pi_{3,n} - \Pi_{3,n-1}|\right) = \sum E(|d_n(f)d_n(g)|) \leq \sqrt{2}\|f\|_{H_1}\|g\|_{\mathrm{BMO}}.$$

- Clearly, $(\Pi_{1,n})_{n\in\mathbb{Z}_+}$ and $(\Pi_{2,n})_{n\in\mathbb{Z}_+}$ are martingales. $(\Pi_{3,n})_{n\in\mathbb{Z}_+}$ is not a martingale. So, $(f_n g_n)_{n\in\mathbb{Z}_+}$ is not a martingale. Fortunately, it is a semi-martingale, which is the sum of a martingale and a process with bounded variation.

- Note that

$$\begin{aligned} f_\infty g_\infty = \lim_n f_n g_n &= \sum_{k=0}^{\infty} f_{k-1} d_k(g) + \sum_{k=0}^{\infty} g_{k-1} d_k(f) + \sum_{k=0}^{\infty} d_k(f) d_k(g) \\ &= \lim_n \Pi_{1,n}(f,g) + \lim_n \Pi_{2,n}(f,g) + \lim_n \Pi_{3,n}(f,g). \end{aligned}$$

- A. M. Garsia, *Martingale Inequalities: Seminar Notes on Recent Progress*, Mathematics Lecture Note Series, Mass.-London-Amsterdam, 1973.
  C. Herz, *H_p-spaces of martingales,* $0 < p \leq 1$, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **28** (1973/74), 189–205.

Another viewpoint

- If we define

$$\widetilde{\Pi_{3,n}}(f,g) := \mathbb{E}_n\Big(\sum_{k\in\mathbb{Z}_+} d_k(f)d_k(g)\Big),$$

then the product

$$(f\times g)_n := \Pi_{1,n}(f,g) + \Pi_{2,n}(f,g) + \widetilde{\Pi_{3,n}}(f,g)$$

is a martingale.

- J. Chao and R. Long, *Martingale transforms with unbounded multipliers*, Proc. Amer. Math. Soc. **114** (1992), 831–838.

  J. Chao and R. Long, *Martingale transforms and Hardy spaces*, Probab. Theory Related Fields **91** (1992), 399–404.

  R. Long, *Martingale Spaces and Inequalities*, Peking university press, Beijing, 1993.

Bilinear decomposition in the context of martingales

- $H_{\log}$ is the martingale Orlicz Hardy space associated with the Orlicz function $\Phi(t) := \frac{t}{\log(e+t)}$ for any $t \in [0, \infty)$.

Theorem 1

One can write

$$f \cdot g = L(f, g) + G(f, g),$$

where L and G are two bounded bilinear operators, with

$$L : \ H_1 \times \mathrm{BMO} \to \mathcal{B}V$$

and

$$G : \ H_1 \times \mathrm{BMO} \to H_{\log}.$$

Remark

- Theorem 1 holds ture when the spaces H_1 and $H_{\log}$ therein are replaced, respectively, by the spaces H_1^M and $H_{\log}^M$.
- Theorem 1 is sharp in the sense of duality. Precisely, if Theorem 1 holds true for $\mathcal{Y}$ with $\mathcal{Y} \subset H_{\log}$, then

$$\left(L^1 + \mathcal{Y}\right)^* = \left(L^1 + H_{\log}\right)^*.$$

- Theorem 1 is comparable with the corresponding one in harmonic analysis when $\Omega = [0,1]$. Indeed, for any $x \in [0,1]$ and $t \in (0,\infty)$,

$$\frac{t}{3\log(e+t)} < \frac{t}{\log(e+|x|)+\log(e+t)} < \frac{t}{\log(e+t)}.$$

E. Nakai and G. Sadasue, *Pointwise multipliers on martingale Campanato spaces*, Studia Math. **220** (2014), 87–100.

Sketch of the proof

Note that

$$
\begin{aligned}
f\cdot g &= (\Pi_{1,n}(f,g))_{n\in\mathbb{Z}_+} + (\Pi_{2,n}(f,g))_{n\in\mathbb{Z}_+} + (\Pi_{3,n}(f,g))_{n\in\mathbb{Z}_+} \\
&=: \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g).
\end{aligned}
$$

- The bilinear operator $\Pi_1 : \; H_1 \times \mathrm{BMO} \to H_1$ is bounded.
- The bilinear operator $\Pi_3 : \; H_1 \times \mathrm{BMO} \to \mathcal{B}V$ is bounded.
- Let $L := \Pi_3$ and $G := \Pi_1 + \Pi_2$. Observe that $H_1 \subset H_{\log}$. Hence, it is sufficient to prove that $\Pi_2 : \; H_1 \times \mathrm{BMO} \to H_{\log}$ is bounded.

- J. Chao and R. Long, *Martingale transforms with unbounded multipliers*, Proc. Amer. Math. Soc. **114** (1992), 831–838.

  J. Chao and R. Long, *Martingale transforms and Hardy spaces*, Probab. Theory Related Fields **91** (1992), 399–404.

Sketch of the proof

Using the John–Nirenberg inequality and

$$st \leq e^{t-1} + s\log^+ s, \quad \forall\, s,\ t \in (0,\infty)$$

and the idea from BIJZ, one can deduce

Lemma

Let f be a measurable function from L^1, and $g := (g_n)_{n\in\mathbb{Z}_+} \in \mathrm{BMO}$ a martingale. Then

$$\|fg_\infty\|_{L^{\log}} \lesssim \|f\|_{L^1}\|g\|_{\mathrm{BMO}}.$$

- The bilinear operator $\Pi_2 : \ H_1 \times \mathrm{BMO} \to H_{\log}$ is bounded.

Proof basing on atomic decompositions and Davis decomposition

simple atom

A measurable function a is called a *simple* (s, ∞)-*atom* if there exist an integer $n \in \mathbb{Z}_+$ and a set $A \in \mathcal{F}_n$ such that

(i) $a_n := \mathbb{E}_n(a) = 0$;

(ii) $\operatorname{supp}(a) \subset A$;

(iii) $\|s(a)\|_{L^\infty} \leq [\mathbb{P}(A)]^{-1}$.

simple atomic decomposition

Let $f \in h_1$. Then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of simple (s, ∞)-atoms, a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of real numbers, and a positive constant C such that, for any $n \in \mathbb{Z}_+$,

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n\left(a^k\right) \quad \text{a. e. and} \sum_{k\in\mathbb{Z}} |\mu_k| \leq C\|f\|_{h_1}.$$

weak ∞-atom

A measurable function w is called a *weak ∞-atom* if there exist an integer $k \in \mathbb{Z}_+$, a set $A \in \mathcal{F}_k$, and an $\mathcal{F}_k$-measurable function φ, with $|\varphi| \le 1$ and $\operatorname{supp}(\varphi) \subset A$, such that

$$w = \frac{\varphi - \mathbb{E}_{k-1}(\varphi)}{\mathbb{P}(A)}.$$

- A martingale f is in the Hardy space h_1^d if

$$\|f\|_{h_1^d} := \sum_{k \in \mathbb{Z}_+} \|d_k f\|_{L^1} < \infty.$$

- J. M. Conde-Alonso and J. Parcet, *Atomic blocks for noncommutative martingales*, Indiana Univ. Math. J. **65** (2016), 1425–1443.

Proof basing on atomic decompositions and Davis decomposition

weak ∞-atomic decomposition

Let $f \in h_1^d$. Then there exist a sequence $(w^k)_{k\in\mathbb{Z}}$ of weak ∞-atoms, a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of real numbers, and a positive constant C such that, for any $n \in \mathbb{Z}_+$,

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n\left(w^k\right) \quad \text{a. e. and} \quad \sum_{k\in\mathbb{Z}} |\mu_k| \le C\|f\|_{h_1^d}.$$

Davis decomposition

For any $f \in H_1$, there exist a positive constant C and two martingales $f^1 \in h_1$ and $f^d \in h_1^d$ such that $f = f^1 + f^d$ and

$$\left\|f^1\right\|_{h_1} \le C\|f\|_{H_1} \quad \text{and} \quad \left\|f^d\right\|_{h_1^d} \le C\|f\|_{H_1}.$$

The endpoint estimates of commutators

Commutators in harmonic analysis

- Let b be a BMO-function. The linear commutator $[b, T]$ of a Calderón–Zygmund operator T does not, in general, map continuously $H_1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.
- One can find a subspace $H^1_b(\mathbb{R}^n)$ of $H_1(\mathbb{R}^n)$ such that $[b, T]$ maps continuously $H_1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.
- The largest subspace $H^1_b(\mathbb{R}^n)$ such that $[b, T]$ is continuous from $H_1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

R. R. Coifman, R. Rochberg and G.Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103**, 611–635 (1976).

C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128** (1995), 163–185.

L. D. Ky, *Bilinear decompositions and commutators of singular integral operators*, Trans. Amer. Math. Soc. **365** (2013), 2931–2958.

Commutators in martingale theory

- The L^p-boundeness of commutators of martingale transforms were first investigated by Janson for regular r-adic martingales, and then studied by Chao and Peng for regular martingales.
- For non-regular martingales (non-homogeneous martingales), it was recently investigated by Treil.
- The boundedness of commutators of martingale fractional integrals were also developed by Chao and Ombe and Nakai et al..

S. Janson, *BMO and commutators of martingale transforms*, Ann. Inst. Fourier (Grenoble) **31** (1981), 265–270.

J.-A. Chao and L. Peng, *Schatten classes and commutators on simple martingales*, Colloq. Math. **71** (1996), 7–21.

J.-A. Chao and H. Ombe, *Commutators on dyadic martingales*, Proc. Japan Acad. Ser. A Math. Sci. **61** (1985), 35–38.

S. Treil, *Commutators, paraproducts and BMO in non-homogeneous martingale settings*, Rev. Mat. Iberoam. **29** (2013), 1325–1372.

R. Arai, E. Nakai and G. Sadasue, *Fractional integrals and their commutators on martingale Orlicz spaces*, J. Math. Anal. Appl. **487** (2020), No. 123991, 35 pp.

Class $\mathcal{K}_q$

Let $q \in [1, \infty)$. Denote by $\mathcal{K}_q$ the set of all the sublinear operators T satisfying

(i) T is bounded from H_1 to L^q;

(ii) T is bounded from L^1 to $L^{q,\infty}$;

(iii) if a is a simple (s, ∞)-atom or a weak ∞-atom with respect to some $n \in \mathbb{Z}_+$, then, for any $b \in \mathrm{BMO}$,

$$\|(b - b_{n-1})T(a)\|_{L^q} \leq C\|b\|_{\mathrm{BMO}}.$$

Denote by $\mathcal{K}_H$ the set of all $T \in \mathcal{K}_1$ such that $T(f) \in L^1$ if and only if $f \in H_1$.

- Examples: The Doob maximal operator M or the square function S is in $\mathcal{K}_H$; martingale transforms are in $\mathcal{K}_1$; the fractional integral operator I_α belongs to $\mathcal{K}_{\frac{1}{1-\alpha}}$ for any $\alpha \in (0, 1)$.

Bilinear decomposition for commutators

Let $b \in \mathrm{BMO}$, $q \in [1, \infty)$, and $T \in \mathcal{K}_q$. For any $f \in H_1$ and $x \in \Omega$,

$$[T, b](f)(x) := T\left(b(x)f - bf\right)(x).$$

Moreover, if T is linear, then $[T, b](f) = bT(f) - T(bf)$.

Bilinear decomposition–main result

Let $q \in [1, \infty)$ and $T \in \mathcal{K}_q$. Then there exists a bounded bilinear operator $R: \ H_1 \times \mathrm{BMO} \to L^q$ such that, for any $(f, b) \in H_1 \times \mathrm{BMO}$,

$$[T, b](f) = R(f, b) - T(\Pi_3(f, b)).$$

Bilinear decomposition for commutators

Let $b \in \mathrm{BMO}$, $q \in [1, \infty)$, and $T \in \mathcal{K}_q$. For any $f \in H_1$ and $x \in \Omega$,

$$[T, b](f)(x) := T\left(b(x)f - bf\right)(x).$$

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Bilinear decomposition–main result

Let $q \in [1, \infty)$ and $T \in \mathcal{K}_q$. Then there exists a bounded bilinear operator $R: \ H_1 \times \mathrm{BMO} \to L^q$ such that, for any $(f, b) \in H_1 \times \mathrm{BMO}$,

$$[T, b](f) = R(f, b) - T(\Pi_3(f, b)).$$

Endpoint estimate

Let $q \in [1, \infty)$, $T \in \mathcal{K}_q$, and $b \in \mathrm{BMO}$. Then there exists a positive constant C such that, for any $f \in H_1$, $\|[T, b](f)\|_{L^{q,\infty}} \leq C\|f\|_{H_1}$.

Sketch of the proof

Note that

$$\begin{aligned}[T,b](f) &= bT(f) - T(\Pi_2(f,b)) - T(\Pi_1(f,b)) - T(\Pi_3(f,b)) \\ &=: U(f,b) - T(\Pi_1(f,b)) - T(\Pi_3(f,b)) \\ &=: R(f,b) - T(\Pi_3(f,b)).\end{aligned}$$

- $\|T(\Pi_1(f,b))\|_{L^q} \lesssim \|\Pi_1(f,b)\|_{H_1} \lesssim \|f\|_{H_1}\|b\|_{\mathrm{BMO}}$;
- For a simple (s,∞)-atom a associated with positive integer n, $\Pi_2(a,b_{n-1}) = ab_{n-1}$. Thus,

  $$U(a,b) = (b-b_{n-1})T(a) - T(\Pi_2(a,b-b_{n-1})).$$

  Since $T \in \mathcal{K}_q$, it follows that

  $$\begin{aligned}\|U(a,b)\|_{L^q} &\le \|(b-b_{n-1})T(a)\|_{L^q} + \|T(\Pi_2(a,b-b_{n-1}))\|_{L^q} \\ &\lesssim \|b\|_{\mathrm{BMO}} + \|\Pi_2(a,b-b_{n-1})\|_{H_1} \lesssim \|b\|_{\mathrm{BMO}}.\end{aligned}$$

- Davis decompositions and atomic decompositions imply that U is bounded from $H_1 \times \mathrm{BMO}$ into L^q.

Martingale Hardy space H_1^b

Definition

Let $b \in \mathrm{BMO}$. The *martingale Hardy space* H_1^b is defined to be the set of all the martingales f such that

$$\|f\|_{H_1^b} := \|f\|_{H_1} \|b\|_{\mathrm{BMO}} + \Big\| \sup_{n \in \mathbb{Z}_+} |[\mathbb{E}_n, b](f)| \Big\|_{L^1} < \infty.$$

Characterizations of H_1^b

Let $b \in \mathrm{BMO}$ be non-constant. Then the following are equivalent:

(i) $f \in H_1^b$;

(ii) $\Pi_3(f, b) \in H_1$;

(iii) $[T, b](f) \in L^1$ with $T \in \mathcal{K}_H$.

Endpoint estimate for commutators

Theorem 2

Let $q \in [1, \infty)$, $T \in \mathcal{K}_q$, and $b \in \mathrm{BMO}$ be non-constant. Then there exists a positive constant C such that, for any $f \in H_1^b$,

$$\|[T, b](f)\|_{L^q} \leq C\|f\|_{H_1^b}.$$

Proof

$$\begin{aligned}\|[T, b](f)\|_{L^q} &\leq \|T(\Pi_3(f, b))\|_{L^q} + \|R(f, b)\|_{L^q} \\ &\lesssim \|\Pi_3(f, b)\|_{H_1} + \|f\|_{H_1}\|b\|_{\mathrm{BMO}} \lesssim \|f\|_{H_1^b}.\end{aligned}$$

Remark. The space H_1^b in Theorem 2 is sharp in the sense that $\mathcal{Y} := H_1^b$ is the largest subspace of H_1 such that, for any $T \in \mathcal{K}_H$, the commutator $[T, b]$ is bounded from $\mathcal{Y}$ to L^1.

Dyadic harmonic analysis

- Tuomas Hytönen's 2012 proof of the A_2 conjecture based on a representation formula for any Calderón–Zygmund operator as an average of appropriate dyadic operators.
- The boundedness of the dyadic Hilbert transform (also knwown as the dyadic shift) beyond doubling measures was first characterized by López-Sánchez et al..

- T. Hytönen, *The sharp weighted bound for general Calderón–Zygmund operators*, Ann. Math. **175**, (2012), 1473–1506.

  M. C. Pereyra, *Dyadic harmonic analysis and weighted inequalities: the sparse revolution*, New Trends in Applied Harmonic Analysis, 159–239, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2019.

  L. D. López-Sánchez, J. M. Martell and J. Parcet, *Dyadic harmonic analysis beyond doubling measures*, Adv. Math. **267** (2014), 44–93.

Dyadic Hilbert transform beyond doubling measures

Here, we work with $([0,1), \mathcal{F}; (\mathcal{F}_n)_{n\in\mathbb{Z}_+})$ equipped with a Borel measure μ. Given a dyadic interval I, we write I_- and I_+, respectively, for the left and the right dyadic children of I. Let

$$m(I) := \frac{\mu(I_-)\mu(I_+)}{\mu(I)} \quad \text{and} \quad h_I := \sqrt{m(I)}\Big[\frac{\mathbf{1}_{I_-}}{\mu(I_-)} - \frac{\mathbf{1}_{I_+}}{\mu(I_+)}\Big].$$

The Borel measure μ is said to be *m-increasing* if there exists a positive constant C such that, for any $I \in A(\mathcal{F})$,

$$m(I) \leq Cm(\widehat{I}),$$

where, the symbol $\widehat{I}$ stands for the *dyadic parent* of I.

The *dyadic Hilbert transform* is defined by setting, for any measurable function f on $[0,1)$ and any $x \in [0,1)$,

$$H_{\mathbb{D}}(f)(x) := \sum_{k\in\mathbb{N}} \sum_{I\in A(\mathcal{F}_k)} \delta(I)\langle f, h_{\widehat{I}}\rangle h_I(x),$$

where $\delta(I) := 1$ if $I := (\widehat{I})_-$, and $\delta(I) := -1$ if $I := (\widehat{I})_+$.

Result of LLP

Let μ be an m-increasing Borel measure on $[0,1)$. Then

(i) $H_{\mathbb{D}}$ is bounded on $L^2(\mu)$ and, moreover, for any $f \in L^2(\mu)$, $\|H_{\mathbb{D}}(f)\|_{L^2(\mu)} \le 2\|f\|_{L^2(\mu)}$;

(ii) $H_{\mathbb{D}}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.

L. D. López-Sánchez, J. M. Martell and J. Parcet, *Dyadic harmonic analysis beyond doubling measures*, Adv. Math. **267** (2014), 44–93.

Main Result: dyadic Hilbert transform

Theorem 3

Let μ be an m-increasing Borel measure on $[0,1)$. Then the dyadic Hilbert transform $H_{\mathbb{D}}$ belongs to $\mathcal{K}_1$.

Corollary

Let μ be an m-increasing Borel measure on $[0,1)$ and $b \in \mathrm{BMO}(\mu)$ non-constant. Then

(i) the commutator $[H_{\mathbb{D}}, b]$ is bounded from $H_1(\mu)$ to $L^{1,\infty}(\mu)$;

(ii) the commutator $[H_{\mathbb{D}}, b]$ is bounded from $H_1^b(\mu)$ to $L^1(\mu)$.

Remark Denote by $H_{\mathbb{D}}^*$ the adjoint operator of the dyadic Hilbert transform $H_{\mathbb{D}}$. Let μ be an m-decreasing Borel measure on $[0,1)$ and $b \in \mathrm{BMO}(\mu)$ non-constant. Similarly to the above corollary, we can show that the commutator $[H_{\mathbb{D}}^*, b]$ is bounded from $H_1(\mu)$ to $L^{1,\infty}(\mu)$ and from $H_1^b(\mu)$ to $L^1(\mu)$.

More applications

Cesàro means of Walsh–Fourier series

The maximal operator $\sigma := \sup_{n\in\mathbb{N}} |\sigma_n|$ is in $\mathcal{K}_1$.

corollary

Let $\sigma := \sup_{n\in\mathbb{N}} |\sigma_n|$, and let $b \in \mathrm{BMO}(0,1)$ be non-constant. Then

(i) the commutator $[\sigma, b]$ is bounded from $H_1(0,1)$ to $L^{1,\infty}(0,1)$;

(ii) the commutator $[\sigma, b]$ is bounded from $H_1^b(0,1)$ to $L^1(0,1)$.

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Thank You !