L^p-improving bounds and weighted estimates for maximal functions associated with curvature

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Classical example

The Cauchy problem of the wave equation

$$\begin{cases} ((\partial/\partial t)^2 - \triangle)u = 0, \\ u|_{t=0} = f, \\ (\partial/\partial t)u|_{t=0} = g. \end{cases} \quad \text{in } \mathbb{R}^2. \tag{1}$$

Taking the Fourier transform in the spatial variables x only, then we have

$$\begin{cases} ((\partial/\partial t)^2 + |\xi|^2)\hat{u}(t,\xi) = 0, \\ \hat{u}(0,\xi) = \hat{f}(\xi), \\ (\partial/\partial t)\hat{u}(0,\xi) = \hat{g}(\xi). \end{cases} \quad \text{in } \mathbb{R}^2. \tag{2}$$

This is an ODE for each fixed $\xi \in \mathbb{R}^2$. It is easy to show that

$$\hat{u}(t,\xi) = \cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)\hat{g}(\xi)}{|\xi|}.$$

Classical example

Let J_σ be Bessel functions ($\sigma > -1/2$) and $m_\sigma(\xi) = C_\sigma |\xi|^{-\sigma} J_\sigma(|\xi|)$. Observe that

$$J_{-1/2}(r) = \lim_{k \to -1/2} J_k(r) = \sqrt{\frac{2}{\pi}} \frac{\cos r}{r^{1/2}},$$

$$J_{1/2}(r) = \sqrt{\frac{2}{\pi}} \frac{\sin r}{r^{1/2}},$$

then one can easily see that

$$\begin{aligned} \hat{u}(t,\xi) &= c_0 m_{-1/2}(t|\xi|)\hat{f}(\xi) + c_1 t m_{1/2}(t|\xi|)\hat{g}(\xi) \\ &=: c_0 \widehat{A_t^{-1/2} f}(\xi) + c_1 t \widehat{A_t^{1/2} g}(\xi), \end{aligned}$$

where $\widehat{A_t^\sigma f}(\xi) = m_\sigma(t|\xi|)\hat{f}(\xi)$.

Classical example

Taking the inverse Fourier transform, then we have

$$u(t,x) = c_0 A_t^{-1/2} f(x) + c_1 t A_t^{1/2} g(x),$$

where

$$\begin{aligned} A_t^{\sigma} f(x) &= \Big(m_\sigma(t|\cdot|) \Big)^{\vee} * f(x) \\ &= \frac{1}{\gamma(\sigma)} \int (1 - |\frac{y}{t}|^2)_+^{\sigma-1} f(x-y) dy. \end{aligned}$$

where $x_+^\delta = x$, if $x \geq 0$; $x_+^\delta = 0$, if $x < 0$.

Classical example

The existence almost everywhere of $\lim_{t\to 0} u(t,x)$ and $\lim_{t\to 0} \frac{\partial}{\partial t} u(t,x)$ to the wave equation (1) will follow from

$$\left\| \sup_{t>0} |A_t^{\sigma} f(x)| \right\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}.$$

Noticing that in the sense of distribution,

$$\lim_{\alpha\to 0^+} \frac{1}{\gamma(\alpha)} t_+^{\alpha-1} = \delta(t),$$

where $\delta(t)$ denotes the Dirac distribution at zero. Then it is not hard to see that

$$A_t f(x) := A_t^0 f(x) = \int_{S^1} f(x-ty)d\sigma(y),$$

which is well-known "Bourgain's circular maximal operators".

Maximal operator $\mathcal{M}$ related to hypersurfaces

Definition

Let S be a smooth hypersurface in $\mathbb{R}^n$ with a surface measure $d\mu$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ be a non-negative smooth function with compact support. Suppose that $\delta_t(x) = (t^{a_1}x_1, t^{a_2}x_2, \cdots, t^{a_n}x_n)$ is a family of dilations with $a_j > 0$. Then the associated full maximal operator $\mathcal{M}$ is given by

$$\mathcal{M}f(x) := \sup_{t>0} \left| \int_S f(x - \delta_t(y))\eta(y)d\mu(y) \right|, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

For which p, $\mathcal{M} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded operator?

Results for $\mathcal{M}$ associated with isotopic dilations

The best understood class $\rightarrow \delta_t(x) = tx$:

I. Hypersurface: non-vanishing Gaussian curvature everywhere

- Stein, 1976: spherical maximal operator, $p > n/(n-1)$, $n \geq 3$;
- Greenleaf, 1981: non-vanishing Gaussian curvature everywhere, star-shaped with respect to the origin, $p > n/(n-1)$, $n \geq 3$;
- Bourgain, 1986: analogous results in dimension two, $p > 2$.
  ! An alternative approach by [Mockenhaupt-Seeger-Sogge, 1992]
  More general results about local smoothing estimates by [Mockenhaupt-Seeger-Sogge, 1993][Beltran-Hickman-Sogge, 2020] [Guth-Wang-Zhang, 2020][Gao-Liu-Miao-Xi,2020]

Results for $\mathcal{M}$ associated with isotopic dilations

II. Hypersurface: Gaussian curvature vanishes at some points

- Iosevich, 1994: curves of finite type, sharp results, $n = 2$;
- Iosevich-Sawyer, 1997: convex hypersurfaces of finite line type, sharp results, $n \geq 3$;
- Ikromov-Kempe-Müller, 2010: hypersurfaces of finite type satisfying the transversality assumption (in particular, $0 \notin S$), $p > max\{h(x_0, S), 2\}$ for a fixed point $x_0 \in S$, $n = 3$;
- Zimmermann, 2014: analytic hypersurfaces located at the origin, $p > 2$, $n = 3$.

Results associated with nonisotopic dilations

Generic dilations $\delta_t(x) = (t^{a_1}x_1, t^{a_2}x_2, \cdots, t^{a_n}x_n)$:

Some previously mentioned results have been extended with little change to maximal operators associated with nonisotropic dilations, such as by [Greenleaf, 1981], [Iosevich-Sawyer, 1997] and [Ikromov-Kempe-Müller, 2010].

Results associated with nonisotopic dilations

Open problem:

1. The $L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ estimates for maximal operator defined by the surface $S := \{(x_1, x_2, c_0 + x_2^d(1 + \mathcal{O}(x_2^m))) : (x_1, x_2) \in \Omega\}$, where $d \geq 2$, $m \geq 1$, $c_0 \in \mathbb{R}$ and Ω is an open neighborhood of the origin.

2. Weighted estimates.

3. Higher dimensions $n > 3$.

[Li, 2018, J. Math. Pure. Appl.]: confirmly answer the first question in $\mathbb{R}^3$.

Definition for weights

Definition

1. A weight $w \in \mathcal{A}_p$ (Muckenhoupt class) for $1 < p < \infty$ if

$$[\omega]_{\mathcal{A}_p} := \sup_{Q \in \mathcal{Q}} \langle \omega \rangle_Q \langle \omega^{1-p'} \rangle_Q^{p-1} < \infty.$$

2. A weight $w \in RH_p$ (reverse Hölder class) for $1 < p < \infty$ if

$$[\omega]_{RH_p} := \sup_{Q \in \mathcal{Q}} \langle \omega \rangle_Q^{-1} \langle \omega \rangle_{Q,p} < \infty.$$

Weighted estimates for spherical maximal functions

Theorem [Lacey, J. D'Analyse Math., 2019]

Set $\mathcal{F}_p$ to be those weights ω for which $\mathcal{M}_{\mathbb{S}^{n-1}}$ maps $L^p(\omega)$ to $L^p(\omega)$, for $1 < p < \infty$. Define $\frac{1}{\phi(1/r)}$ to be the piecewise linear function on $[0, \frac{n-1}{n}]$ whose graph connects the points $P_1 = (0, 1)$, $P_4 = (\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1})$ and $P_3 = (\frac{n-1}{n}, \frac{n-1}{n})$. Assuming $\frac{n}{n-1} < r < p < \phi(r)'$, we have

$$A_{p/r} \cap RH_{(\phi(r)'/p)'} \subset \mathcal{F}_p. \tag{3}$$

Weighted estimates for spherical maximal functions

Definition

We say that a collection of cubes $\mathcal{S}$ is *sparse* if there are sets $\{E_S \subseteq S : S \in \mathcal{S}\}$ such that they are pairwise disjoint and $|E_S| > \frac{1}{4}|S|$ for all $S \in \mathcal{S}$. For any cube Q and $1 \leq r < \infty$, define the r-average of a function f on Q by $\langle f \rangle_{Q,r} := \left(\frac{1}{|Q|}\int_Q |f(x)|^r dx\right)^{\frac{1}{r}}$. Let $\mathcal{S}$ denote a sparse collection. For $1 \leq p, q < \infty$, the (p,q)*-sparse form* $\Lambda_{\mathcal{S},p,q}(f,g)$ is a bilinear form defined by

$$\Lambda_{\mathcal{S},p,q}(f,g) := \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,p} \langle g \rangle_{S,q}.$$

Weighted estimates for spherical maximal functions

Theorem [Lacey, J. D'Analyse Math., 2019]

For $n \geq 2$ and let F_n be the trapezium with vertexes $P_1 = (0,1), P_2 = (\frac{n-1}{n}, \frac{1}{n}), P_3 = (\frac{n-1}{n}, \frac{n-1}{n}), P_4 = (\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1})$. For all $(\frac{1}{r}, \frac{1}{s})$ in the interior of F_n, there holds

$$| < \mathcal{M}_{\mathbb{S}^{n-1}} f, g > | \leq C \sup_{\mathcal{S}} \Lambda_{\mathcal{S},r,s}(f,g). \tag{4}$$

Moreover, for $\frac{1}{r} + \frac{1}{s} > 1$ not in the closed set F_n, the inequality fails.

One of great advantages of sparse bounds: one can easily derive weighted inequalities for sparse operators.

Weighted estimates for spherical maximal functions

Continuity property [Lacey, J. D'Analyse Math., 2019]

Let F_n' be the closed convex hull of the four points $P_1' = (0,0)$, $P_2' = (\frac{n-1}{n}, \frac{n-1}{n})$, $P_3' = (\frac{n-1}{n}, \frac{1}{n})$ and $P_4' = (\frac{n^2-n}{n^2+1}, \frac{n-1}{n^2+1})$. For all $(\frac{1}{r}, \frac{1}{s})$ in the interior of F_n', we have for some $\eta = \eta(n, r, s) > 0$,

$$\| \sup_{1 \le t \le 2} |A_t f - \tau_y A_t f| \|_s \le C|y|^{\eta} \|f\|_r, \qquad |y| < 1. \tag{5}$$

Weighted estimates for spherical maximal functions

Local estimate [Schlag, 1997;Schlag-Sogge, 1997]

Let F'_n be the closed convex hull of the four points $P'_1 = (0,0)$, $P'_2 = (\frac{n-1}{n}, \frac{n-1}{n})$, $P'_3 = (\frac{n-1}{n}, \frac{1}{n})$ and $P'_4 = (\frac{n^2-n}{n^2+1}, \frac{n-1}{n^2+1})$. For all $(\frac{1}{r}, \frac{1}{s})$ in F'_n, we have

$$\| \sup_{1 \le t \le 2} |A_t f| \|_s \le C \|f\|_r. \tag{6}$$

Maximal operator $\mathcal{M}_{loc}$ related to hypersurfaces

Definition

Let S be a smooth hypersurface in $\mathbb{R}^n$ with a surface measure $d\mu$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ be a non-negative smooth function with compact support. Suppose that $\delta_t(x) = (t^{a_1}x_1, t^{a_2}x_2, \cdots, t^{a_n}x_n)$ is a family of dilations with $a_j > 0$. Then the associated local maximal operator $\mathcal{M}$ is given by

$$\mathcal{M}_{loc}f(x) := \sup_{t\in[1,2]} \left| \int_S f(x - \delta_t(y))\eta(y)d\mu(y) \right|, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

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For which p and q with $p < q$, $\mathcal{M}_{loc} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is a bounded operator?

Results for $\mathcal{M}_{loc}$ associated with isotropic dilations

- Schlag, 1997: Circle, combinatorial method, bounded if $(1/p, 1/q) \in$ interior of the closed triangle I, $n = 2$;

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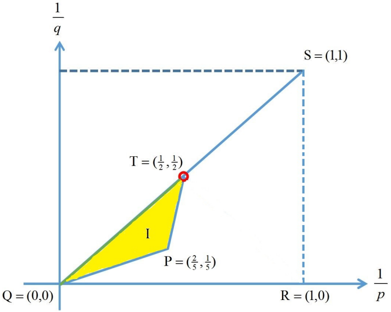
- Lee, 2002: Circle, local smoothing method, bounded if $(1/p, 1/q) \in I \setminus \{P, T\}$, $n = 2$.



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- Schlag-Sogge, 1997: spherical maximal estimate is bounded if $(1/p, 1/q) \in$ interior of quadrangle $\square$ with vertices $V_1 = (0,0)$, $V_2 = (\frac{n-1}{n}, \frac{n-1}{n})$, $V_3 = (\frac{n-1}{n}, \frac{1}{n})$ and $V_4 = (\frac{n(n-1)}{n^2+1}, \frac{n-1}{n^2+1})$, $n \geq 3$.

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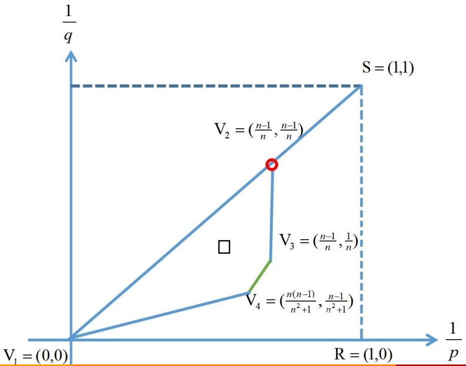
- Lee, 2002: spherical maximal estimate is bounded if $(1/p, 1/q) \in \Box \backslash \{V_2, V_3, V_4\}$.



$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane

Now we consider the $L^p \to L^q$ estimates for local maximal operators along curves of finite type d ($d \geq 2$) at the origin. For convenience, we define the following regions of boundedness exponents that will be referred to later on:

$$\Delta_0 := \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1\} \cup \{(0,0)\}. \tag{7}$$

$$\Delta_1 := \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0)\}; \tag{8}$$

$$\Delta_2 := \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{d+1}{p} - 1\} \cup \{(0,0)\}; \tag{9}$$

$$\Delta_3 := \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{2}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0), (1,1)\}. \tag{10}$$

$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane

Let $\phi \in C^\infty(I, \mathbb{R})$, where I is a bounded interval containing the origin, and

$$\phi(0) \neq 0;\ \phi^{(j)}(0) = 0,\ j = 1, 2, \cdots, m-1;\ \phi^{(m)}(0) \neq 0\ (m \geq 1). \quad (11)$$

$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane

Theorem 1 (Li-Wang-Zhai, arXiv, 2022)

Define the averaging operator

$$A_t f(y) := \int_{\mathbb{R}} f(y_1 - tx, y_2 - t(x^d \phi(x) + c))\eta(x)dx, \tag{12}$$

where η is supported in a sufficiently small neighborhood of the origin. Then we have the following results:

(1) when $c = 0$, for $(\frac{1}{p}, \frac{1}{q}) \in \Delta_1 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$;

(2) when $c \neq 0$, for $(\frac{1}{p}, \frac{1}{q}) \in \Delta_2 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{d+1}{p} - 1\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$.

$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane

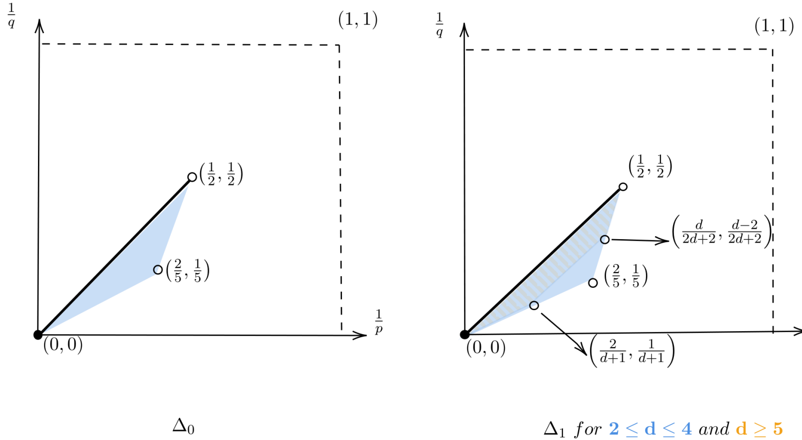
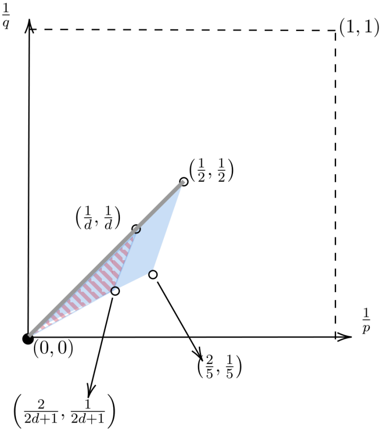


Figure 1: $\Delta_1 = \Delta_0$ for $2 \leq d \leq 4$ and $\Delta_1 \subsetneq \Delta_0$ for $d \geq 5$

$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane



Δ_2 *for* $\mathbf{d} = \mathbf{2}$ *and* $\mathbf{d} \geq \mathbf{3}$

$L^p \to L^q$ boundedness for$\mathcal{M}_{loc}$ in the plane

Theorem 2 (Li-Wang-Zhai, arXiv, 2022)

Define the averaging operator

$$A_t f(y) := \int_{\mathbb{R}} f(y_1 - t^{a_1}x, y_2 - t^{a_2}(x^d\phi(x) + c))\eta(x)dx, \quad da_1 \neq a_2, \ (13)$$

where η is supported in a sufficiently small neighborhood of the origin. Then we have the following results:

(1) when $c = 0$, for $(\frac{1}{p}, \frac{1}{q}) \in \Delta_1 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$;

(2) when $c \neq 0$, for $(\frac{1}{p}, \frac{1}{q}) \in \Delta_2 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{d+1}{p} - 1\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$.

$L^p \to L^q$ boundedness for$\mathcal{M}_{loc}$ in the plane

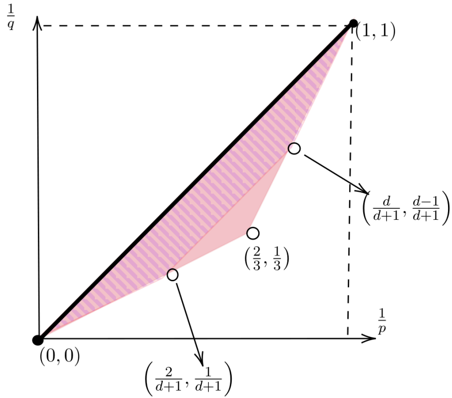
Theorem 3 (Li-Wang-Zhai, arXiv, 2022)

Define the averaging operator

$$A_t f(y) := \int_0^1 f(y_1 - tx, y_2 - t^d x^d) dx, \tag{14}$$

then for $(\frac{1}{p}, \frac{1}{q}) \in \Delta_3 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{2}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0),(1,1)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t||_{L^p \to L^q} \leq C_{p,q}$.

Results for $\mathcal{M}_{loc}$ associated with isotropic dilations



Δ_3 *for* $\mathbf{d=2}$ *and* $\mathbf{d \geq 3}$

$L^p \to L^q$ boundedness for$\mathcal{M}_{loc}$ in the plane

Theorem 4 (Li-Wang-Zhai, arXiv, 2022)

Define the averaging operator

$$A_t f(y) := \int_{\mathbb{R}} f(y_1 - tx, y_2 - t^d x^d \phi(x))\eta(x)dx, \tag{15}$$

where η is supported in a sufficiently small neighborhood of the origin. Then for
$(\frac{1}{p}, \frac{1}{q}) \in \Delta_1 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0)\}$,
there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$.

$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane

Theorem 5 (Li-Wang-Zhai, arXiv, 2022)

For all $j > mk$, and p, q satisfying $\frac{1}{2p} < \frac{1}{q} \leq \frac{3}{5p}$, $\frac{3}{q} \leq 1 - \frac{1}{p}$, $\frac{1}{q} \geq \frac{1}{2} - \frac{1}{p}$, we have for some $\epsilon > 0$,

$$\left(\int_{\mathbb{R}^2}\int_{1/2}^{4} |\tilde{F}_j^k f(y,t)|^q dtdy\right)^{1/q} \leq C2^{\frac{mk}{2}(1-\frac{1}{p}-\frac{1}{q})}2^{(\frac{1}{2}-\frac{3}{q}+\frac{1}{p}+\epsilon)j}\|f\|_{L^p(\mathbb{R}^2)}, \tag{16}$$

where

$$\tilde{F}_j^k f(y,t) = \rho_1(y,t)\int_{\mathbb{R}^2} e^{i(\xi\cdot y - t^2\xi_2\tilde{\Phi}(s,\delta))}a(\xi,t)\rho_0(2^{-j}|\xi|)\tilde{\chi}(\frac{\xi_1}{\xi_2})\hat{f}(\xi)d\xi. \tag{17}$$

In fact, here $-t^2\xi_2\tilde{\Phi}(s,\delta)$ can be considered as a small perturbation of $\frac{\xi_1^2}{2\xi_2} + 2^{-km}t\frac{\xi_1^3}{\xi_2^2}$.

$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane

[Lee, JFA, 2006]

Let $\mathcal{F}$ be given by

$$\mathcal{F}f(z) = \int_{\mathbb{R}^2} e^{i\phi(z,\xi)} a(z,\xi)\rho_0(2^{-j}|\xi|)\hat{f}(\xi)d\xi, \quad z = (x,t). \tag{18}$$

Suppose a is a symbol of order zero, supp $a(\cdot,\xi)$ is contained in a fixed compact set and suppose that $\phi(z,\cdot)$ is a homogeneous function of degree one. For all $(z,\xi) \in$ supp a, ϕ satisfies

$$\text{rank } \partial^2_{z\xi}\phi = 2,$$

and

$$\text{rank } \partial^2_{\xi\xi}\langle \partial_z\phi, \theta\rangle = 1,$$

provided $\theta \in \mathcal{S}^2$ is the unique direction for which $\nabla_\xi\langle \partial_z\phi, \theta\rangle = 0$, also all non-zero eigenvalues of $\partial^2_{\xi\xi}\langle \partial_z\phi, \theta\rangle$ have the same sign.

$L^p \to L^q$ boundedness for $\mathcal{M}_{loc}$ in the plane

[Lee, JFA, 2006]

Then for $\frac{14}{3} \leq q \leq \infty$, $\frac{3}{q} \leq 1 - \frac{1}{p}$, $q \geq \frac{5p}{3}$,

$$\|\mathcal{F}f\|_{L^q} \leq C2^{j(\frac{1}{2}-\frac{3}{q}+\frac{1}{p}+\epsilon)}\|f\|_{L^p}. \tag{19}$$

Localized maximal functions associated with surfaces in $\mathbb{R}^3$

We first show $L^p \to L^q$ estimates for maximal functions related to hypersurfaces with at least one non-vanishing principal curvature when $2a_2 \neq a_3$.

Theorem 6 (Li-Wang-Zhai, arXiv, 2022)

Assume that $\Phi(x_1, x_2) \in C^\infty(\Omega)$ satisfies

$$\partial_2 \Phi(0,0) = 0, \qquad \partial_2^2 \Phi(0,0) \neq 0, \tag{20}$$

and $2a_2 \neq a_3$. Define the averaging operator by

$$A_t f(y) := \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, \Phi(x_1, x_2)))\eta(x)dx, \tag{21}$$

where η is supported in a sufficiently small neighborhood $U \subset \Omega$ of the origin. For $(\frac{1}{p}, \frac{1}{q}) \in \Delta_0 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t \in [1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$.

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Corollary 7 (Li-Wang-Zhai, arXiv, 2022)

Assume that $\Phi(x_1, x_2) \in C^\infty(\Omega)$ satisfies inequality (20). Define the averaging operator

$$A_t f(y) := \int_{\mathbb{R}^2} f(y - t(x_1, x_2, \Phi(x_1, x_2)))\eta(x)dx, \tag{22}$$

where η is supported in a sufficiently small neighborhood $U \subset \Omega$ of the origin. For $(\frac{1}{p}, \frac{1}{q}) \in \Delta_0 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$.

We note that in [Schlag, 1997; Schlag-Sogge, 1997], the "cinematic curvature" condition in $\mathbb{R}^3$ is required to establish $L^p \to L^q$ estimates for the local maximal functions. While in this corollary, we just need the "cinematic curvature" condition in $\mathbb{R}^2$.

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Theorem 8 (Li-Wang-Zhai, arXiv, 2022)

Assume that $\Phi(x_1, x_2) \in C^{\infty}(\Omega)$ satisfies $\Phi(0,0) \neq 0$ and $da_2 \neq a_3$, $d \geq 2$. The associated averaging operator is defined by

$$A_t f(y) := \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, c + x_2^d \Phi(x_1, x_2)))\eta(x)dx, \qquad (23)$$

where η is supported in a sufficiently small neighborhood U of the origin. Then we have the following results:
(1) when $c = 0$, for
$(\frac{1}{p}, \frac{1}{q}) \in \Delta_1 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$;
(2) when $c \neq 0$, for
$(\frac{1}{p}, \frac{1}{q}) \in \Delta_2 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{d+1}{p} - 1\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t| \|_{L^p \to L^q} \leq C_{p,q}$.

Localized maximal functions associated with surfaces in $\mathbb{R}^3$

Theorem 9 (Li-Wang-Zhai, arXiv, 2022)

Let $\phi \in C^\infty(I)$, where I is a bounded interval containing the origin. Define the averaging operator by

$$A_t f(y) := \int_{\mathbb{R}^2} f(y - \delta_t(x_1, x_2, x_2^d \phi(x_2)))\eta(x)dx, \tag{24}$$

where η is supported in a sufficiently small neighborhood U of the origin. Assume that ϕ satisfies (11), and $da_2 = a_3$. Then for $(\frac{1}{p}, \frac{1}{q}) \in \Delta_1 = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\} \cup \{(0,0)\}$, there exists a constant $C_{p,q} > 0$ such that $\| sup_{t\in[1,2]} |A_t|\|_{L^p \to L^q} \leq C_{p,q}$.

Continuity lemma

Theorem 10 [Li-Wang-Zhai, arXiv, 2022]

There exists a real number $\epsilon > 0$ such that for any $z \in \mathbb{R}^n$, there holds

$$\left\| \sup_{t\in[1,2]} \left| A_t f(y+z) - A_t f(y) \right| \right\|_{L^q(\mathbb{R}^n)} \lesssim |z|^{\epsilon} \|f\|_{L^p(\mathbb{R}^n)}, \tag{25}$$

provided that

(1) $n = 2$, A_t is defined by (13) with $c = 0$ or by (15), and
$(\frac{1}{p}, \frac{1}{q}) \in \Delta_1 \backslash \{(0,0)\} = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\}$;
(2) $n = 3$, A_t is defined by: (23) with $c = 0$ or by (24), and
$(\frac{1}{p}, \frac{1}{q}) \in \Delta_1 \backslash \{(0,0)\} = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{3}{p} - 1, \frac{1}{q} > \frac{1}{p} - \frac{1}{d+1}\}$;
(3) $n = 2$, A_t is defined by (13) with $c \neq 0$; $n = 3$, A_t is defined by (23) with $c \neq 0$, and
$(\frac{1}{p}, \frac{1}{q}) \in \Delta_2 \backslash \{(0,0)\} = \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{2p} < \frac{1}{q} \leq \frac{1}{p}, \frac{1}{q} > \frac{d+1}{p} - 1\}$;

Continuity lemma

Theorem 10 [Li-Wang-Zhai, arXiv, 2022]

(4) $n=2$, A_t is defined by (14), $(\frac{1}{p},\frac{1}{q})\in\Delta_3\backslash\{(0,0),(1,1)\}=\{(\frac{1}{p},\frac{1}{q}):\frac{1}{2p}<\frac{1}{q}\leq\frac{1}{p},\frac{1}{q}>\frac{2}{p}-1,\frac{1}{q}>\frac{1}{p}-\frac{1}{d+1}\}$;
(5) $n=3$, A_t is defined by (21),
$(\frac{1}{p},\frac{1}{q})\in\Delta_0\backslash\{(0,0)\}=\{(\frac{1}{p},\frac{1}{q}):\frac{1}{2p}<\frac{1}{q}\leq\frac{1}{p},\frac{1}{q}>\frac{3}{p}-1\}$.

Sparse domination

Let $\eta : \mathbb{R}^k \to \mathbb{R}$ be a smooth function supported on a sufficiently small neighborhood $U \subseteq \Omega$ of the origin. We define the average associated to the hypersurface parametrized by $\vec{\Phi} := (\Phi_i)_{1 \leq i \leq n}$ as

$$A_t^{\vec{\Phi}} f(y) := \int_{\mathbb{R}^k} f\left(y - \delta_t\left(\Phi_1(x), \ldots, \Phi_n(x)\right)\right) \eta(x) dx.$$

The corresponding maximal function can then be defined as

$$\mathcal{M}_{\vec{\Phi}} f(y) := \sup_{t>0} |A_t^{\vec{\Phi}} f(y)|. \tag{26}$$

δ-cubes with dyadic size

Definition [Li-Wang-Zhai, arXiv, 2022]

Let $\mathcal{R}$ denote the collection of all axes-parallel hyperrectangles. Then the collection of δ-cubes with dyadic size is defined as

$$\mathcal{Q}^\delta := \{Q \in \mathcal{R} : l_1(Q) = 2^{\lceil kb_1 \rceil}, \dots, l_n(Q) = 2^{\lceil kb_n \rceil}, \text{ for some } k \in \mathbb{Z}\},$$

where $l_j(Q)$ denote the j-th side-length of Q and $b_j \geq 1$ for $1 \leq j \leq n$.

Sparse form

Definition [Li-Wang-Zhai, arXiv, 2022]

1. We say that a collection of δ-cubes $\mathcal{S}$ is *sparse* if there are sets $\{E_S \subseteq S : S \in \mathcal{S}\}$ such that they are pairwise disjoint and $|E_S| > \frac{1}{4}|S|$ for all $S \in \mathcal{S}$.
2. For any δ-cube Q and $1 \leq r < \infty$, define the r-average of a function f on Q by $\langle f \rangle_{Q,r} := \left(\frac{1}{|Q|}\int_Q |f(x)|^r dx\right)^{\frac{1}{r}}$. Let $\mathcal{S}$ denote a sparse collection. For $1 \leq p, q < \infty$, the (p,q)-*sparse form* $\Lambda_{\mathcal{S},p,q}(f,g)$ is a bilinear form defined by

$$\Lambda_{\mathcal{S},p,q}(f,g) := \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,p} \langle g \rangle_{S,q}.$$

Sparse domination

Theorem 11 [Li-Wang-Zhai, arXiv, 2022]

Let $\mathcal{M}_{\vec{\Phi}}$ denote the maximal function defined in (26). Suppose that the corresponding local operator defined by

$$\tilde{\mathcal{M}}_{\vec{\Phi}} f := \sup_{1 \le t \le 2} |A_t^{\vec{\Phi}} f|, \tag{27}$$

satisfies the local continuity property in the range $\mathcal{L}_n$. Then for all bounded compactly supported functions f, g and for any $(\frac{1}{p}, \frac{1}{q'}) \in \mathcal{L}'_n$, there exists a constant $C < \infty$ such that

$$|\langle \mathcal{M}_{\vec{\Phi}} f, g \rangle| \le C \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,q'}(f, g),$$

where the supreme is taken over all possible sparse collections of δ-cubes.

Weighted estimates

Definition [Li-Wang-Zhai, arXiv, 2022]

1. A weight ω is a poistive function defined on $\mathbb{R}^n$ equipped with the Lebesgue measure and the metric defined by

$$\rho_\delta(x,y) := \max_{1\le i\le n} |x_i - y_i|^{\frac{1}{b_i}}.$$

We usually denote by $\omega(E) := \int_E w(x)dx$ and $\|f\|_{L^p_\omega} := \left(\int |f(x)|^p w(x)dx\right)^{\frac{1}{p}}$.

2. A weight $w \in \mathcal{A}_p$ (Muckenhoupt class) for $1 < p < \infty$ if

$$[\omega]_{\mathcal{A}_p} := \sup_{Q\in\mathcal{Q}^\delta} \langle\omega\rangle_Q \langle\omega^{1-p'}\rangle_Q^{p-1} < \infty.$$

3. A weight $w \in RH_p$ (reverse Hölder class) for $1 < p < \infty$ if

$$[\omega]_{RH_p} := \sup \langle\omega\rangle_Q^{-1}\langle\omega\rangle_{Q,p} < \infty.$$

Weighted estimates

Definition [Li-Wang-Zhai, arXiv, 2022]

Suppose that $\mathcal{S}$ is a sparse collection of δ-cubes. For any $p < r < q$ with $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{L}_n$ and weight $\omega \in A_{\frac{r}{p}} \cap RH_{(\frac{q}{r})'}$,

$$\Lambda_{\mathcal{S},p,q'}(f,g) \lesssim \left([\omega]_{A_{\frac{r}{p}}} [\omega]_{RH_{(\frac{q}{r})'}} \right)^{\alpha} \|f\|_{L^r(\omega)} \|g\|_{L^{r'}(\omega^{1-r'})},$$

for

$$\alpha := \max\left(\frac{1}{r-p}, \frac{q-1}{q-r} \right). \tag{28}$$

Weighted estimates

Theorem 12 [Li-Wang-Zhai, arXiv, 2022]

Suppose that the maximal operator $\mathcal{M}_{\vec{\Phi}}$ defined in (26) satisfies the sparse bound described in Theorem 11. Then for any $p < r < q$ with $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{L}_n$ and weight $\omega \in A_{\frac{r}{p}} \cap RH_{(\frac{q}{r})'}$ defined on δ-cubes,

$$\|\mathcal{M}_{\vec{\Phi}}\|_{L^r(\omega)\to L^r(\omega)} \lesssim \left([\omega]_{A_{\frac{r}{p}}}[\omega]_{RH_{(\frac{q}{r})'}}\right)^{\alpha}, \tag{29}$$

for α specified in (28).

Weighted estimates

Theorem 12 [Li-Wang-Zhai, arXiv, 2022]

Let ω be a weight defined on δ-cubes such that $\omega \in A_{\frac{r}{p}} \cap RH_{(\frac{q}{r})'}$. Define the global maximal operator

$$\mathcal{M}f(y) := \sup_{t>0} |A_t f(y)|.$$

(1) When A_t is defined in terms of (13) with $c = 0$ or by (15), then for $\mathcal{M}$ and any $p < r < q$ with $(\frac{1}{p}, \frac{1}{q}) \in \Delta_1$, (29) holds true.
(2) When A_t is defined in terms of (23) with $c = 0$ or by (24), then for $\mathcal{M}$ and any $p < r < q$ with $(\frac{1}{p}, \frac{1}{q}) \in \Delta_1$, (29) holds true.
(3) When A_t is defined in terms of (13) with $c \neq 0$ or by (23) with $c \neq 0$, then for $\mathcal{M}$ and any $p < r < q$ with $(\frac{1}{p}, \frac{1}{q}) \in \Delta_2$, (29) holds true.

Some areas not covered

Theorem 12 [Li-Wang-Zhai, arXiv, 2022]

(4) When A_t is defined in terms of (14), then for $\mathcal{M}$ and any $p < r < q$ with $(\frac{1}{p}, \frac{1}{q}) \in \Delta_3$, (29) holds true.
(5) When A_t is defined in terms of (21), then for $\mathcal{M}$ and any $p < r < q$ with $(\frac{1}{p}, \frac{1}{q}) \in \Delta_0$, (29) holds true.

THANKS FOR YOUR ATTENTION!