Gamma conjecture I for del Pezzo surfaces

Changzheng Li

Sun Yat-sen University

joint with Jianxun Hu, Huazhong Ke and Tuo Yang

2022 Colloquium of Arithmetic, Harbin Institute of Technology

January 08, 2022

Riemann-Zeta function $\zeta(s) = \sum_{s=1}^{\infty} \frac{1}{n^s}$

Conjecture

The value $\zeta(s)$ is irrational for all $s \in \mathbb{Z}_{\geq 2}$.

- Euler's formula: $\zeta(2k) = C_{2k}(\pi)^{2k}$, where $C_{2k} \in \mathbb{Q}$, $k \in \mathbb{Z}_{>0}$.
- Apéry (1978): $\zeta(3)$ is irrational.
- Rivoal (2000): $\zeta(s)$ is irrational for infinitely many $s \in \mathbb{Z}_{>0}^{\mathrm{odd}}$.
- Zudilin (2001): one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

Gamma class

- Gamma function: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.
- Gamma class of a complex manifold X:

$$\hat{\Gamma}_X = \prod_{i=1}^{N} \Gamma(1 + x_i) \in H^{\mathrm{ev}}(X, \mathbb{R}), \quad \text{where} \quad c(TX) = \prod_{i=1}^{N}(1 + x_i)$$

$$= \exp\Big(- c_{\mathrm{eu}} c_1(X) + \sum_{k=2}^{\infty} (-1)^k (k-1)! \zeta(k) ch_k(TX) \Big)$$

$$\overset{e.g.}{=} \begin{cases} 1 - c_{\mathrm{eu}} c_1(X), & \text{if } \dim X = 1 \\ 1 - c_{\mathrm{eu}} c_1(X) + \zeta(2) ch_2(TX) + \frac{1}{2} c_{\mathrm{eu}}^2 c_1^2(X), & \text{if } \dim X = 2. \end{cases}$$

Conjectures

- Galkin-Golyshev-Iritani (Duke Math. J. 2016)

  X: Fano manifold.

  - Conjecture $\mathcal{O}$ $\leftrightsquigarrow$ Eigenvalue of $\widehat{c}_1$ on $QH^*(X)$.
  - Assume Conjecture $\mathcal{O}$ first.

    Gamma conjecture I ($\hat{\Gamma}_X$) $\leftrightsquigarrow$ Asymptotic expansion of J_X.
  - Assume the semisimplicity of quantum cohomology first.

    Gamma conjecture II $\leftrightsquigarrow$ Full exceptional collection in $D^b_{\mathrm{coh}}(X)$.
- Hosono (2006), Abouzaid-Ganatra-Iritani-Sheridan (2018)

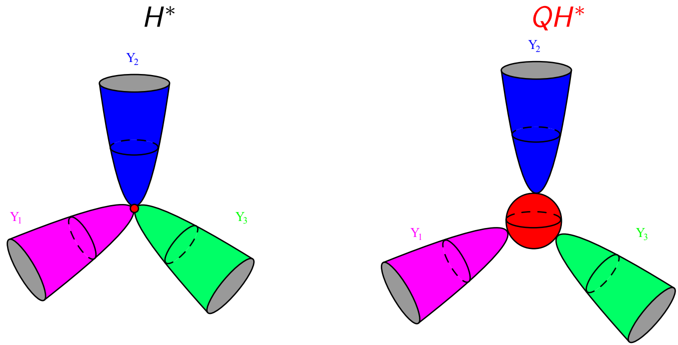
  Gamma conjectures for Calabi-Yau manifolds.

Fano manifold X

X: compact complex manifold with $-K_X$ ample.

- $\mathbb{P}^n$
- Smooth hypersurface $\{f = 0\}$ in $\mathbb{P}^n$ with $\deg f \leq n$
- Flag variety G/P
- complete classification for $\dim X \leq 3$
  1. $\mathbb{P}^1$
  2. del Pezzo surface
  3. 105 deformation families of Fano 3-folds

Quantum cohomology $QH^*(X)$



Fano X: (Gromov-Witten invariants are involved in "$\star$")

$QH^*(X) = (H^*(X) \otimes \mathbb{C}[q_1, \cdots, q_m], \star)$ where $m = \dim H_2(X)$

$$\alpha \star \beta = \alpha \cup \beta + \sum \text{terms of higher degree in } \mathbf{q}$$

Conjecture $\mathcal{O}$

Conjecture $\mathcal{O}$ (Galkin-Golyshev-Iritani)

Let X be a Fano manifold, and denote by Spec$(\hat{c}_1)$ the set of eigenvalues of the following linear operator $\hat{c}_1$ induced by quantum multiplication $\star$.

$$\hat{c}_1 : H^{\text{even}}(X, \mathbb{C}) \longrightarrow H^{\text{even}}(X, \mathbb{C});\ \beta \mapsto c_1(X) \star \beta|_{\mathbf{q}=\mathbf{1}}.$$

Then the following should hold.

(i) $\rho := \max_{\delta \in \mathrm{Spec}(\hat{c}_1)} |\delta|$ *belongs to Spec$(\hat{c}_1)$; mult.$\rho = 1$.*

(ii) $r := \max\{k \in \mathbb{Z} \mid \frac{c_1(X)}{k} \in H^2(X, \mathbb{Z})\}$. $\forall \delta \in$ *Spec$(\hat{c}_1)$ with* $|\delta| = \rho$,

$$(\delta/\rho)^r = 1.$$

Conjecture $\mathcal{O}$ holds if ...

Example

Conjecture $\mathcal{O}$ holds if $X = \mathbb{P}^1$.

- $H^*(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}x$, where $x = P.D.[\mathbb{P}^0] \in H^2(\mathbb{P}^1, \mathbb{Z})$.
- $c_1(\mathbb{P}^1) = 2x \quad \Longrightarrow \quad r(\mathbb{P}^1) = 2$.

Conjecture $\mathcal{O}$ holds if ...

Example

Conjecture $\mathcal{O}$ holds if $X = \mathbb{P}^1$.

- $H^*(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}x$, where $x = P.D.[\mathbb{P}^0] \in H^2(\mathbb{P}^1, \mathbb{Z})$.
- $c_1(\mathbb{P}^1) = 2x \quad \Longrightarrow \quad r(\mathbb{P}^1) = 2$.
- $QH^*(\mathbb{P}^1) = \mathbb{C}[x, q]/(x^2 - q)$
  - $QH^{\mathrm{ev}}(\mathbb{P}^1)|_{q=1} = \mathbb{C}[x]/(x^2 - 1) = \mathbb{C} \oplus \mathbb{C}x$
  - $$\hat{c}_1 \begin{pmatrix} 1 \\ x \end{pmatrix} = 2x \star \begin{pmatrix} 1 \\ x \end{pmatrix} \Big|_{q=1} = \begin{pmatrix} 2x \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}$$

$$\Longrightarrow \quad \mathrm{Spec}(\hat{c}_1) = \mathrm{Spec} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \{2, -2\}$$

Gamma conjecture I: Givental's J-function

Quantum connection on $H^{\mathrm{ev}}(X) \times \mathbb{P}^1 \to \mathbb{P}^1$: (for $\phi \in H^{2p}(X)$)

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}(c_1(X)\star_{\mathbf{q}=\mathbf{1}}) + \mu, \text{ where } \mu(\phi) = (p - \frac{\dim X}{2})\phi$$

Proposition

There exists a unique End($H^{\mathrm{ev}}(X)$)-valued power series $S(z) = id + S_1 z^{-1} + S_2 z^{-2} + \cdots$ *such that* $(z^A := \exp^{A \log z})$

$$\begin{cases} \nabla(S(z)z^{-\mu}z^{c_1(X)}\phi) = 0, & \forall \phi \in H^{\mathrm{ev}}(X) \\ T(z) = z^{\mu}S(z)z^{-\mu} \text{ is regular at } z = \infty \text{ and } T(\infty) = id. \end{cases}$$

- Givental's J-function (restricted at $c_1(X)\log t$ with $t = z^{-1}$):

$$J_X(t) = z^{\frac{\dim X}{2}}\left(S(z)z^{\mu}z^{c_1(X)}\right)^{-1}1.$$

Gamma conjectures

$$H^{\mathrm{ev}}(X) \longleftrightarrow \{s : \mathbb{R}_{>0} \to H^{\mathrm{ev}}(X) \mid \nabla s = 0\}$$
$$\alpha \mapsto (2\pi)^{\frac{-\dim X}{2}} S(z) z^{-\mu} z^{c_1(X)} \alpha.$$

- $\mathcal{A} := \left\{ s : \mathbb{R}_{>0} \to H^{\mathrm{ev}}(X) \,\middle|\, \nabla s = 0, \quad \|e^{\rho/z} s(z)\| = O(z^{-m}) \text{ as } z \to +0 (\exists m) \right\}.$

Proposition

Assume conjecture $\mathcal{O}$ holds. Then

- $\dim \mathcal{A} = 1$, *so that* $\mathcal{A} = \mathbb{C}[S(z) z^{-\mu} z^{c_1(X)} A_X]$.
- $J_X(z) = C z^{\frac{\dim X}{2}} e^{\frac{\rho}{z}} (A_X + O(z))$

Gamma conjectures

- $\mathcal{A} := \{ s : \mathbb{R}_{>0} \to H^{\mathrm{ev}}(X) \,\Big|\, \nabla s{=}0, \quad \|e^{\rho/z} s(z)\|{=}O(z^{-m}) \text{ as } z{\to}{+}0 (\exists m) \}.$

Proposition

Assume conjecture $\mathcal{O}$ holds. Then $J_X(z){=}Cz^{\frac{\dim X}{2}} e^{\frac{\rho}{z}}(A_X + O(z))$

Gamma Conjectures (Galkin-Golyshev-Iritani)

- *Gamma Conjecture I:* $A_X = \hat{\Gamma}_X$ *(assuming Conj. $\mathcal{O}$)*
- *Gamma Conjecture II:* $A_{X,\delta} = \hat{\Gamma}_X Ch(E_\delta)$ *for some full exceptional collection* $\{E_\delta\}$ *of* $\mathcal{D}^b_{\mathrm{coh}}(X)$ *(assuming semi-simplicity of* $QH^{\mathrm{ev}}(X)$*)*

Here $Ch(E) := \sum_p (2\pi\sqrt{-1})^p ch_p(E)$.

In particular $Ch(\mathcal{O}_X) = 1$.

Conjecture $\mathcal{O}$ holds if ...

Theorem (Hu-Ke-L.-Yang)

Conjecture $\mathcal{O}$ holds if X is a del Pezzo surface.

- Flag varieties G/P of arbitrary Lie type: Cheong-L. 2017.
  - Complex Grassmannians: Rietsch 2003, Galkin-Golyshev 2006.
  - Lagrangian and orthogonal Grassmannians: Cheong 2017.
- Fano 3-folds:
  - Picard rank one: Golyshev-Zagier 2016.
  - Bott-Samelson varieties: Withrow 2018.
- Horospherical varieties: L.-Mihalcea-Shifler 2017; Bones-Fowler-Schneider-Shifler 2018
- Fano complete intersections in $\mathbb{P}^n$: Galkin-Iritani 2015; Sanda-Shamoto 2017; Ke 2018.

Gamma conjectures hold if ...

Theorem (Hu-Ke-L.-Yang)

Gamma conjecture I holds if X is a del Pezzo surface.

To my knowledge:

- Complex Grassmannians: Galkin-Golyshev-Iritani 2016.
- Fano 3-folds of Picard rank one: Golyshev-Zagier 2016.
- Fano complete intersections in $\mathbb{P}^n$ ($r \geq 2$): Sanda-Shamoto 2017.
- Toric Fano manifolds that satisfy conjecture $\mathcal{O}$: Galkin-Iritani.

Remark (Gamma conjecture II holds if)

- X is a complex Grassmannian (Galkin-Golyshev-Iritani).
- X is a toric Fano manifold (Fang-Zhou).
- X is a smooth quadrics in $\mathbb{P}^n$ (Hu-Ke).

Approach to Conjecture $\mathcal{O}$ in good cases

- Kaoru Ono: *Conjecture $\mathcal{O}$ (i) follows from Perron-Frobenius theorem, if $\hat{c}_1$ is represented by an irreducible non-negative matrix.*
  - This may happen in good cases, e.g. when $H^{\mathrm{ev}}(X)$ has a basis of "*positive*" algebraic cycles.
- Conjecture $\mathcal{O}$ (ii) would probably hold automatically if $r = 1$.

Perron-Frobenius theorem

Definition

A nonnegative matrix M is called reducible if the induced operator has a nontrivial invariant coordinate subspace, i.e., if

$$M = Q \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} Q^T \quad \text{for some permutation matrix } Q.$$

irreducible $=$ not reducible

Theorem (Perron(1907)-Frobenius(1912))

1. Every irreducible nonnegative matrix M has a real eigenvalue δ_0 of multiplicity one such that $\delta_0 \geq |\delta|$ for all $\delta \in \mathrm{Spec}(M)$.
2. All eigenvaules δ with $|\delta| = \delta_0$ are simple, and precisely the solutions of $\delta_0^h - \delta^h = 0$ for some $h \in \mathbb{Z}_{>0}$. $\left(\overset{\text{inv.by } e^{\frac{2\pi\sqrt{-1}}{r}}}{\Longrightarrow} r|h \right)$

Del Pezzo surfaces = 2-dim. Fano manifolds

$c_1(X_k) = -K_{X_k} = 3H - E_1 - \cdots - E_k$
Fano index $r = 1$ (since $\langle -K_{X_r}, E_1 \rangle = 1$).

- Good: $QH^*(X_k)$ is well studied.
- NOT good: $\hat{c}_1$ is unlikely given by a nonnegative matrix.

Key point for Conjecture $\mathcal{O}$ for del Pezzo surfaces

Theorem (Generalized Perron-Frobenius Theorem)

Suppose that a real matrix $M = (m_{ij})$ satisfies the following properties:

1. $\sum_j m_{ij} > 0$ for any i;
2. M^k is an irreducible nonnegative matrix for some k.

Then M has a real eigenvalue δ_0 of multiplicity one such that $\delta_0 \geq |\delta|$ for all $\delta \in \mathrm{Spec}(M)$.

Remark

Generalized Perron-Frobenius Theorem is also applicable if

- X is the Bott-Samelson resolution of $F\ell_3$.
- X is the blowup of $\mathbb{P}^4$ at a point.

Approach to Gamma conjecture I by Galkin-Iritani

- Gamma conjecture I holds for a toric Fano manifold if the mirror Landau-Ginzburg potential f satisfies the B-analogue of conjecture $\mathcal{O}$ (together with the coincidence with the conifold point of $f|_{\text{realpart}}$).
- Gamma conjecture I is compatible with the quantum Lefschetz principle (whenever it is applicable), since so is Givental's J-function.

Del Pezzo surfaces = 2-dim. Fano manifolds

Either $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ or one of the following:

X_1: degree $(1,1)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$;

X_2: complete intersection of divisors of degree $(1,0,1)$ and $(0,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$;

X_3: complete intersection of two divisors of degree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$;

X_4: complete intersection of four hyperplanes in Grassmannian $G(2,5) \subset \mathbb{P}^9$ (embedded by Plücker);

X_5: complete intersection of two quadrics in $\mathbb{P}^4$;

X_6: cubic surface in $\mathbb{P}^3$;

X_7: hypersurface of degree 4 in $\mathbb{P}(2,1,1,1)$;

X_8: hypersurface of degree 6 in $\mathbb{P}(3,2,1,1)$.

Key ingredient for Γ-conjecture I for X_7, X_8

Theorem

Conjecture $\mathcal{O}$ and Gamma conjecture I hold for weighted projective spaces $\mathbb{P}(1, w_1, \cdots, w_N)$.

Apéry's proof of irrationality of $\zeta(3)$

Proposition

A real number ξ is irrational, if there exist $\delta > 0$ and $\{\frac{p_n}{q_n}\}_n$ such that

$$|\xi - \frac{p_n}{q_n}| < \frac{1}{q_n^{1+\delta}}$$

where $p_n, q_n \in \mathbb{Z}$ with $(p_n, q_n) = 1$, $\frac{p_n}{q_n} \neq \xi$ and $\frac{p_n}{q_n} \neq \frac{p_m}{q_m}$ for all $n \neq m$

- Apéry considered the solutions $\{a_n\}, \{b_n\}$ to the recurrence

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n-1)^3 u_{n-2} = 0$$

with the initial values $a_0 = 1, a_1 = 5, b_0 = 0$ and $b_1 = 1$. He showed

$$|\xi - \frac{6b_n}{a_n}| = o(a_n^{-2})$$

Reinterpretation of Apéry's proof

- Quantum differential equation of X: a diff. equ. $L(D)(f(t)) = 0$ such that $J_X(t)$ is a solution, where $D = t\frac{d}{dt}$.
- Golyshev: let X be the Fano 3-fold by a section of $OG(5, 10)$ by a codimension 7 plane. Then

$$L(D) := D^3 - t(1 + 2D)(17D^2 + 17D + 5) + t^2(D + 1)^3$$

is a quantum differential equation.

Let $A(t) = \sum_n a_n t^n$ and $B(t) = \sum_n b_n t^n$, then

$$L(D)A(t) = 0, \qquad (D - 1)L(D)B(t) = 0.$$

Thank you!!...