Local boundedness and Hölder continuity for the parabolic fractional p-Laplace equations *

Mengyao Ding[1], Chao Zhang[2]†, Shulin Zhou[1]

[1] School of Mathematical Sciences, Peking University, Beijing 100871, PR China

[2] School of Mathematics and Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, PR China

Abstract

In this paper, we study the boundedness and Hölder continuity of local weak solutions to the following nonhomogeneous equation

$$\partial_t u(x,t) + \text{P.V.} \int_{\mathbb{R}^N} K(x,y,t)|u(x,t)-u(y,t)|^{p-2}\big(u(x,t)-u(y,t)\big)dy = f(x,t,u)$$

in $Q_T = \Omega \times (0,T)$, where the symmetric kernel $K(x,y,t)$ has a generalized form of the fractional p-Laplace operator of s-order. We impose some structural conditions on the function f and use the De Giorgi-Nash-Moser iteration to establish the boundedness of local weak solutions in the *a priori* way. Based on the boundedness result, we also obtain Hölder continuity of bounded solutions in the superquadratic case. These results can be regarded as a counterpart to the elliptic case due to Di Castro, Kuusi and Palatucci (Ann. Inst. H. Poincaré Anal. Non Linéaire, 2016).

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1 Introduction

In this paper, we aim at investigating the local properties of the following integro-differential equations

$$\partial_t u(x,t) + Lu(x,t) = f(x,t,u) \tag{1.1}$$

in $Q_T = \Omega \times (0,T)$, where Ω is a bounded open domain in $\mathbb{R}^N$. Here the operator L is a nonlinear and nonlocal operator of fractional p-Laplace type, which is formally given by

$$Lu(x,t) = \text{P.V.} \int_{\mathbb{R}^N} K(x,y,t)|u(x,t)-u(y,t)|^{p-2}\big(u(x,t)-u(y,t)\big)dy, \tag{1.2}$$

where P.V. stands for the Cauchy principal value. The symmetric kernel K satisfies $K(x,y,t) = K(y,x,t)$ and

$$\frac{\Lambda^{-1}}{|x-y|^{N+sp}} \le K(x,y,t) \le \frac{\Lambda}{|x-y|^{N+sp}} \tag{1.3}$$

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†Corresponding author. E-Mail: myding@pku.edu.cn (M. Ding), czhangmath@hit.edu.cn (C. Zhang), szhou@math.pku.edu.cn (S. Zhou)

with $\Lambda \geq 1$ and $s \in (0,1)$ for any $x, y \in \mathbb{R}^N$ and $t \in (0,T)$. The source function f is assumed to satisfy

$$|f(x,t,u)| \leq c_0|u|^{\beta-1} + h(x,t) \tag{1.4}$$

for all $x \in \mathbb{R}^N$, $t \in (0,T)$ and $u \in \mathbb{R}$, where $\beta > 1$, $c_0 \geq 0$ and the nonnegative function h possesses certain integrability.

It is well-known that the operator L can be written in the divergence form. Denote

$$\mathcal{E}(u,v,t) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|u(x,t)-u(y,t)\right|^{p-2}\big(u(x,t)-u(y,t)\big)\big(v(x,t)-v(y,t)\big)K(x,y,t)dxdy.$$

Then it can be verified that

$$\int_{\mathbb{R}^N} Lu(x,t)v(x,t)dx = \mathcal{E}(u,v,t)$$

for suitable functions u, v.

Before stating the definition of weak solutions in this paper, we need to recall a tail space as below

$$L^q_\alpha\left(\mathbb{R}^N\right) := \left\{ v \in L^q_{\text{loc}}\left(\mathbb{R}^N\right) \,\Big|\, \int_{\mathbb{R}^N} \frac{|v(x)|^q}{1+|x|^{N+\alpha}}dx < +\infty \right\}, \quad q > 0 \text{ and } \alpha > 0.$$

Then a nonlocal tail of the supremum version is defined by

$$\begin{aligned} \operatorname{Tail}_\infty(v; x_0, r, I) &= \operatorname{Tail}_\infty(v; x_0, r, t_0 - T_1, t_0 + T_2) \\ &:= \operatorname*{ess\,sup}_{t\in I}\left(r^{sp}\int_{\mathbb{R}^N\setminus B_r(x_0)} \frac{|v(x,t)|^{p-1}}{|x-x_0|^{N+sp}}dx \right)^{\frac{1}{p-1}}, \end{aligned} \tag{1.5}$$

where $(x_0, t_0) \in \mathbb{R}^N \times (0,T)$ and the interval $I = [t_0 - T_1, t_0 + T_2] \subseteq (0,T)$. From these definitions, it is easy to deduce that $\operatorname{Tail}_\infty(v; x_0, r, I)$ is well-defined for any $v \in L^\infty(I, L^{p-1}_{sp}(\mathbb{R}^N))$. Now we present the definitions of the weak sub(super)-solutions to Eq. (1.1) as follows.

Definition 1.1. *Suppose that f satisfies (1.4) with $\beta \in (1, \max\{2, p(2s+N)/N\})$ and $h \in L^{\frac{\beta}{\beta-1}}_{\text{loc}}(Q_T)$. The function $u \in L^p\left(I; W^{s,p}_{\text{loc}}(\Omega)\right) \cap C(I; L^2_{\text{loc}}(\Omega)) \cap L^\infty(I, L^{p-1}_{sp}(\mathbb{R}^N))$ is a local weak sub(super)-solution to (1.1) if for any closed interval $I := [t_1, t_2] \subseteq (0,T)$, inequality*

$$\begin{aligned} &\int_\Omega u(x,t_2)\varphi(x,t_2)dx - \int_{t_1}^{t_2}\int_\Omega u(x,t)\partial_t\varphi(x,t)dxdt + \int_{t_1}^{t_2}\mathcal{E}(u,\varphi,t)dt \\ \leq (\geq) &\int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_{t_1}^{t_2}\int_\Omega f(x,t,u)\varphi(x,t)dxdt \end{aligned} \tag{1.6}$$

holds for every nonnegative function $\varphi \in L^p\left(I; W^{s,p}(\Omega)\right) \cap W^{1,2}(I; L^2(\Omega))$ with the property that φ has spatial support compactly contained in Ω.

Remark 1. *In Definition 1.1, we can invoke Lemma 2.3 to deduce that $u \in L^\beta(I, L^\beta_{\text{loc}}(\Omega))$ because of the fact $u \in L^p\left(I; W^{s,p}_{\text{loc}}(\Omega)\right) \cap C(I; L^2_{\text{loc}}(\Omega))$. This guarantees that the last integral in (1.6) makes sense.*

Definition 1.2. A function u is a local weak solution to (1.1) if and only if u is a local weak sub-solution and a local super-solution.

Before addressing our theorems for weak solutions to (1.1), we will introduce some related results provided by the existing literature in the coming subsection.

1.1 Overview of related literature

The integro-differential operator in (1.1) emphasizes the Lévy process which indicates the emergence of the jump diffusion. In the last decades, the study for the equations of this type has attracted extensive attentions not only in the field of pure mathematical analysis but also in the real world applications (see e.g. [23, 33, 11, 39, 20, 1, 21]). Consider the elliptic Dirichlet problem as below

$$\begin{cases} \text{P.V.} \displaystyle\int_{\mathbb{R}^N} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)dy = f(x,u) & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u(x) = g(x) & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{1.7}$$

Under the condition that K satisfies (1.3) and $f \equiv 0$, Di Castro, Kuusi and Palatucci [15] obtained the existence of weak solutions by constructing a variational functional, and then investigated the local boundedness and Hölder continuity of weak solutions by utilizing the De Giorgi-Nash-Moser theory. Based on the boundedness result, they also established a nonlocal Harnack inequality involving the negative part of the solution (the tail term) in [16]. Here it is noteworthy that without the global nonnegative assumption on the solutions, the classical Harnack inequality fails for the nonlocal elliptic operators, which was proved by Kassmann [25]. This fact indicates that the tail term exhibited in the Harnack inequality [16, Theorem 1.1] enters in a crucial way. For the equation involving a general source term $f(x,u)$, Cozzi [12] introduced the fractional De Giorgi classes and proved the local boundedness, Hölder continuity and Harnack inequality of weak solutions to (1.7).

Let us turn to the equation given by the following form

$$\text{P.V.} \int_{\mathbb{R}^N} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)dy = f(x) \quad \text{in } \Omega. \tag{1.8}$$

When $f(x) \equiv 0$ and $K(x,y) = |x-y|^{-N-sp}$, Brasco, Lindgren and Shikorra [4] obtained Hölder continuity with an explicit exponent condition for (1.8) in the superquadratic case. Before that, Lindgren [34] studied the Hölder estimate for (1.8) with the nonhomogenous term $f \in C^0$. The results given in [34] cover more general kernels than the one appearing in (1.8). In fact, there both p and s are allowed to vary with the space variables. In sharp contrast with what happens in the local variational setting, no regularity assumptions are imposed on $p(\cdot)$ and $s(\cdot)$ apart from boundedness and measurability. Analogous results have been obtained in [13], where operators of double phase type were studied and also, in this case, the constraints linking the various parameters of the problem are much weaker than those considered in the local variational setting. For nonlinear integro-differential equations involving measure data, Kuusi, Mingione and Sire [30] established Calderón-Zygmund type estimates, continuity and boundedness criteria via Wolff potentials. Meanwhile, it is worth mentioning that Sobolev regularity for fractional elliptic equations has also been performed in [6] and [31].

In the linear case that $p = 2$ and $K(x,y) = |x-y|^{-N-2s}$, the nonlocal operator L boils down to the well-known fractional Laplacian $(-\Delta)^s$. The regularities of weak and viscosity solutions to the corresponding equations have been extensively developed by Bass-Kassmann e.g. [2, 24, 26, 27] and Caffarelli-Silvestre e.g. [9, 10, 40, 41].

Next, we proceed to introduce some known results on the linear parabolic equation as below

$$\partial_t u(x,t) + \text{P.V.} \int_{\mathbb{R}^N} K(x,y,t)\big(u(x,t)-u(y,t)\big)dy = f(x,t) \quad \text{in } Q_T. \tag{1.9}$$

When $f(x,t) \equiv 0$, Caffarelli, Chan and Vasseur [7] studied the Cauchy problem (1.9) under the condition that the symmetric kernel $K(x,y,t)$ satisfies (1.3) with $p = 2$. It has been shown in [7] that (1.9) is solvable in the classical sense with any initial value $u_0 \in H^1(\mathbb{R}^N)$ and weak solutions are

Hölder continuous in $(t_0, T) \times \mathbb{R}^N$ with any $t_0 \in (0, T)$. Also for the homogeneous equation posed in the whole space, Bonforte, Sire and Vázquez [3] established a theory of solvability and regularity for the fractional Laplacian equation (1.9). More precisely, the authors utilized a convolution formula to obtain the existence and uniqueness of the very weak solution emanating from nonnegative measure μ_0. They also discussed the regularity (including the boundedness, Hölder continuity, Harnack inequality) and the behaviors (such as stability, self-similar property, asymptotic behavior) of these very weak solutions. By imposing conditions on $K(x, y, t)$ in the integral form, Felsinger and Kassmann [18] established a weak Harnack inequality for the nonnegative super-solution to (1.9) with $f \in L^\infty(Q_T)$, and also proved the local Hölder continuity for bounded weak solutions to (1.9) with $f \equiv 0$. Similar to the elliptic case, the Harnack inequality for the parabolic nonlocal operators is normally presented with the negative part of the solution. In [18], the global positivity assumption on the solution guarantees that the weak Harnack inequality can hold without adding any tail term. The regularity results exhibited in [18] were extended by Schwab and Kassmann [28] to the equation (1.9) with $a(x, y, t)d\mu(x, y)$ in place of $K(x, y, t)dxdy$, where μ is a measure, not necessarily absolutely continuous w.r.t. Lebesgue measure. Recently, Strömqvist [42] and Kim [29] investigated the Harnack inequality for the Cauchy problem and Dirichlet problem, respectively. In their results, the weak solutions do not need to be globally positive, but the tail terms are inevitably involved.

Finally, we turn to the general nonlinear and nonlocal parabolic equation. The theory for this part seems incomplete. Consider the problem

$$\partial_t u(x, t) + Lu(x, t) = 0 \quad \text{in } Q_T, \tag{1.10}$$

where L is associated with the kernel $K(x, y, t)$ as specified in (1.2). In [44], Vázquez provided the existence and uniqueness of strong solutions to (1.10) under the assumption that $u = 0$ in $\mathbb{R}^N \setminus \Omega$, and investigated the large-time behaviors of solutions by using a special separate variable solution $U(x, t) = t^{-1/(p-2)}F(x)$. Besides, the well-posedness for the equation (1.10) subject to the Dirichlet condition, Neumann condition or defined on $\mathbb{R}^N$ was discussed by Mazón, Rossi and Toledo [37], where they also studied the asymptotic behaviour of strong solutions. Recently, Strömqvist [43] investigated the problem (1.10) with $u = g$ in $\mathbb{R}^N \setminus \Omega$ and obtained the existence and local boundedness of weak solutions provided that $K(x, y, t)$ satisfies (1.3) with $p \geq 2$. Under the assumption that $L = (-\Delta_p)^s$, Brasco, Lindgren and Strömqvist [5] worked with the local weak solution of (1.10) and established the Hölder continuity with specific exponents for all $p \geq 2$. Very recently, a theory involving the fundamental solution and asymptotic behaviour for equation (1.10) posed in $\mathbb{R}^N$ was developed by Vázquez [45, 46] for the superquadratic and subquadratic case, respectively.

1.2 Statements of the main results and strategy of the proof

As far as we know, there is no theory yet for the nonlinear and nonlocal equation (1.10) with a nonzero source function. Even for the homogenous equation, the existing boundedness result only focused on the case $p \geq 2$. Thus, one purpose of this work is to find the conditions on f such that the local boundedness holds for the local weak solutions to (1.1) with all $p > 1$. Another motivation is to establish Hölder regularity for the equation with a general nonhomogeneous term. In order to simplify our presentation, we introduce some notation, which is needed later.

Notation. As usual, the domain $B_\rho(x)$ is a ball with radius $\rho > 0$ and center $x \in \mathbb{R}^N$, the parabolic cylinders are given by $Q_{\rho,r}(x, t) := B_\rho(x) \times (t - r, t + r)$, $Q_\rho(x, t) := Q_{\rho,\rho^{sp}}(x, t) = B_\rho(x) \times (t - \rho^{sp}, t + \rho^{sp})$ and $Q_\rho^-(x, t) := Q_{\rho,\rho^{sp}}^-(x, t) = B_\rho(x) \times (t - \rho^{sp}, t)$ with $r, \rho > 0$ and $(x, t) \in \mathbb{R}^N \times (0, T)$. These symbols can be simplified by writing $B_\rho = B_\rho(x)$, $Q_{\rho,r} = Q_{\rho,r}(x, t)$, $Q_\rho = Q_\rho(x, t)$ and $Q_\rho^- = Q_\rho^-(x, t)$

when there is no confusion. We also need define notation of the scaling domain: if $\tilde{B} = B_\rho(x)$ and $\tilde{Q} = B_\rho(x) \times (t - t_1, t + t_2)$, then we denote $\lambda\tilde{B} := B_{\lambda\rho}(x)$ and $\lambda\tilde{Q} := B_{\lambda\rho}(x) \times (t - \lambda t_1, t + \lambda t_2)$ with any $\lambda > 0$. For $g \in L^1(V)$, the mean average of g is given by

$$(g)_V := \fint_V g(x)dx := \frac{1}{|V|} \int_V g(x)dx.$$

We denote

$$a \vee b := \max\{a, b\}, \quad a_+ := \max\{a, 0\}, \quad a_- := -\min\{a, 0\}$$

and

$$J_p(a, b) = |a - b|^{p-2}(a - b)$$

for any $a, b \in \mathbb{R}$. The continuous measure μ in this work admits the presentation

$$d\mu = d\mu(x, y, t) = K(x, y, t)dxdy.$$

In the next four sections, we use C to denote a general positive constant which only depends on $s, p, \Lambda, \beta, c_0$ and N.

Now, we present the boundedness results in the *a priori* way.

Theorem 1. (**Local boundedness**) *Let $p \geq 2N/(2s + N)$ and u be a local weak sub-solution to (1.1). Assume that the nonhomogeneous function f satisfies (1.4), where*

$$\max\{p, 2\} \leq \beta < p\frac{2s + N}{N} \quad \text{and} \quad h^{\frac{\beta}{\beta-1}} \in L^{\hat{q}}_{\mathrm{loc}}(Q_T) \quad \text{with} \quad \hat{q} > \frac{N + sp}{sp}.$$

Let $(x_0, t_0) \in Q_T$, $R \in (0, 1)$ and $Q^-_R \equiv B_R(x_0) \times (t_0 - R^{sp}, t_0)$ such that $\overline{B}_R(x_0) \subseteq \Omega$ and $[t_0 - R^{sp}, t_0] \subseteq (0, T)$. Then we have

$$\operatorname*{ess\,sup}_{Q^-_{R/2}} u \leq \mathrm{Tail}_\infty(u_+; x_0, R/2, t_0 - R^{sp}, t_0) + C\left(\fint_{Q^-_R} u_+^\beta dxdt\right)^{\frac{sp}{N(p\kappa-\beta)}} \vee 1, \tag{1.11}$$

where $\kappa := 1 + 2s/N$ and $C > 0$ only depends on $s, p, \beta, \Lambda, N, c_0$ and h.

For the case $1 < p < 2N/(2s+N)$, we need assume that our weak sub-solution can be constructed as follow: there is a sequence of functions $\{u_k\}_{k\in\mathbb{N}^+}$ whose components are bounded sub-solutions of (1.1) such that

$$\|u_k\|_{L^\infty_{\mathrm{loc}}(0,T;L^{p-1}_{sp}(\mathbb{R}^N))} \leq C \tag{1.12}$$

and

$$u_k \to u \quad \text{in } L^m_{\mathrm{loc}}(Q_T) \quad \text{as } k \to \infty, \tag{1.13}$$

where the constant m is taken to satisfy $m > \max\{2, N(2 - p)/sp\}$.

Theorem 2. (**Local boundedness**) *Let $1 < p < 2N/(2s + N)$, $\kappa := 1 + 2s/N$ and $m > 2$ be such that $m > N(2-p)/sp$. Assume that $u \in L^m_{\mathrm{loc}}(Q_T)$ with the properties (1.12) and (1.13) is a local weak sub-solution to (1.1), where the nonhomogeneous function f satisfies (1.4) with*

$$1 < \beta \leq 2 \quad \text{and} \quad h \in L^\infty_{\mathrm{loc}}(Q_T).$$

Let $(x_0,t_0)\in Q_T$*,* $R\in(0,1)$ *and* $Q_R^-\equiv B_R(x_0)\times(t_0-R^{sp},t_0)$ *such that* $\overline{B}_R(x_0)\subseteq\Omega$ *and* $[t_0-R^{sp},t_0]\subseteq(0,T)$*. Then we have*

$$\begin{aligned}\operatorname*{ess\,sup}_{Q^-_{R/2}} u &\le \mathrm{Tail}_\infty(u_+;x_0,R/2,t_0-R^{sp},t_0)\\ &\quad + C\left(\fint_{Q_R^-} u_+^m\,dxdt\right)^{\frac{sp}{(sp+N)(m-\lambda_m)}}\vee\left(\fint_{Q_R^-} u_+^m\,dxdt\right)^{\frac{sp}{(sp+N)(m-2-\lambda_m)}},\end{aligned}\tag{1.14}$$

where $\lambda_m:=(m-p\kappa)N/(sp+N)$ *and* $C>0$ *only depends on* $s,p,\beta,m,\Lambda,N,c_0$ *and* h*.*

Based on the above boundedness result, we can further obtain Hölder continuity of weak solutions in the superquadratic case.

Theorem 3. (**Hölder continuity**) *Let* $p>2$ *and* u *be a local weak solution to* (1.1)*. Assume that the nonhomogeneous function* f *satisfies*

$$f(x,t,u)=h(x,t)\quad in\quad Q_T\times\mathbb{R}\quad with\quad h\in L^\infty_{\mathrm{loc}}(Q_T).$$

Let $(x_0,t_0)\in Q_T$*,* $R\in(0,1)$ *and* $Q_R\equiv B_R(x_0)\times(t_0-R^{sp},t_0+R^{sp})$ *with the property* $\overline{Q}_R\subseteq Q_T$*. Then there exists* $d\in(0,1)$ *such that for every* $\rho\in(0,R/2]$*,*

$$\operatorname*{ess\,osc}_{Q_{\rho,d\rho^{sp}}} u\le C\left(\frac{\rho}{R}\right)^\alpha\left[\mathrm{Tail}_\infty(u;x_0,R/2,t_0-R^{sp},t_0+R^{sp})+\left(\fint_{Q_R}|u|^p\,dxdt\right)^{\frac12}\vee 1\right],$$

where constants $\alpha\in(0,sp/(p-1))$ *and* $C\in[1,\infty)$ *only depend on* s,p,Λ,N *and* h*.*

Proposition 1.1. (**Hölder continuity**) *Let* $p>2$ *and* u *be a local weak solution to* (1.1)*. Assume that the nonhomogeneous function* f *satisfies* (1.4) *with*

$$1<\beta<p\frac{2s+N}{N}\quad and\quad h\in L^\infty_{\mathrm{loc}}(Q_T).$$

Let $(x_0,t_0)\in Q_T$*,* $R\in(0,1)$ *and* $Q_R\equiv B_R(x_0)\times(t_0-R^{sp},t_0+R^{sp})$ *with the property* $\overline{Q}_R\subseteq Q_T$*. Then there exists a constant* $\alpha\in(0,sp/(p-1))$ *such that* $u\in C^{\alpha,\frac{\alpha}{sp}}(\overline{Q}_{R/2})$*.*

Profile of this paper. This paper is organized as follows. Section 2 is used to collect Sobolev imbedding and Poincaré-type inequalities as essential ingredients in our proof. In Section 3, we follow the arguments provided in [15] to establish the Caccioppoli estimates for the nonlocal parabolic operators. Based on the Caccioppoli inequality, Section 4 is devoted to proving the boundedness results by using the De Giorgi-Nash-Moser iteration. Here, we remark that the requirements on parameters $\beta,\hat{q}$ and m in Theorems 1 and 2 are the nonlocal counterpart of those appearing in [17, Chapter V]. More precisely, let $\beta(s):=p(2s+N)/N$ be the upper bound condition on β, $\hat{q}(s):=(N+sp)/sp$ and $m(s):=N(2-p)/sp$ be lower bound conditions on q,m. When we take $s\to1^-$, it is obvious that $\beta(s)\to p(2+N)/N$, $\hat{q}(s)\to(N+p)/p$ and $m(s)\to N(2-p)/p$, where the limits are restrictions on corresponding exponents for the p-Laplace equation discussed in [17]. With the help of this boundedness result, we further consider Hölder continuity of weak solutions in Section 5. The idea of the proof in this part is motivated by [15], in which Hölder regularity was established for the elliptic counterpart. Although the existing arguments for elliptic equations can be adapted to parabolic ones, we have to perform more careful estimates and choose proper cylinders to solve the difficulties caused by the space-time anisotropy.

2 Sobolev & Poincaré inequalities

This section collects some imbedding inequalities as preliminary ingredients.

Lemma 2.1. *Let $s,\theta \in (0,1)$ and $1 \le p, p_2 < p_1 \le \infty$ satisfy*

$$s > \frac{N}{p} - \frac{N}{p_1}$$

and

$$\theta\left(s - \frac{N}{p} + \frac{N}{p_1}\right) + (1-\theta)\left(\frac{N}{p_1} - \frac{N}{p_2}\right) = 0.$$

Then there exists a constant $C > 0$ only depending on s, p, p_1, p_2 and N such that

$$\|f\|_{L^{p_1}(B_1)} \le C\|f\|^{\theta}_{W^{s,p}(B_1)}\|f\|^{1-\theta}_{L^{p_2}(B_1)} \tag{2.1}$$

for all $f \in W^{s,p}(B_1) \cap L^{p_2}(B_1)$.

Proof. By using the extension theorem [14, Theorem 5.4], we can find $\tilde{f} \in W^{s,p}(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N)$ such that

$$\tilde{f}\big|_{B_1} = f, \quad \|\tilde{f}\|_{W^{s,p}(\mathbb{R}^N)} \le C\|f\|_{W^{s,p}(B_1)} \quad \text{and} \quad \|\tilde{f}\|_{L^{p_2}(\mathbb{R}^N)} \le C\|f\|_{L^{p_2}(B_1)} \tag{2.2}$$

with $C > 0$ only depending on s, p, p_2 and N. The restrictions on the parameters s, θ, p, p_1, p_2 enable us to apply the interpolation inequality [35, Theorem 1] and obtain that

$$\|\tilde{f}\|_{\dot{B}^0_{p_1,1}(\mathbb{R}^N)} \le C\|\tilde{f}\|^{\theta}_{\dot{B}^s_{p,p}(\mathbb{R}^N)}\|\tilde{f}\|^{1-\theta}_{\dot{B}^0_{p_2,\infty}(\mathbb{R}^N)},$$

where $\dot{B}^{\lambda}_{q,r}$ denotes the homogeneous Besov space. Then it follows by the embeddings $\dot{B}^0_{p_1,1}(\mathbb{R}^N) \hookrightarrow L^{p_1}(\mathbb{R}^N)$, $L^{p_2}(\mathbb{R}^N) \hookrightarrow \dot{B}^0_{p_2,\infty}(\mathbb{R}^N)$ and $W^{s,p}(\mathbb{R}^N) \hookrightarrow \dot{B}^s_{p,p}(\mathbb{R}^N)$ that

$$\|\tilde{f}\|_{L^{p_1}(\mathbb{R}^N)} \le C\|\tilde{f}\|^{\theta}_{W^{s,p}(\mathbb{R}^N)}\|\tilde{f}\|^{1-\theta}_{L^{p_2}(\mathbb{R}^N)},$$

which along with (2.2) implies the claim. □

Lemma 2.2. (see [14, Theorem 6.7]) *Let $s \in (0,1)$ and $p \in [1,\infty)$ satisfy $sp < N$. Then for any $f \in W^{s,p}(B_1)$, we have*

$$\|f\|_{L^{\frac{pN}{N-sp}}(B_1)} \le C\|f\|_{W^{s,p}(B_1)}$$

with $C > 0$ only depending on s, p and N.

Lemma 2.3. *Let $t_2 > t_1 > 0$. Suppose $s \in (0,1)$ and $p \in [1,\infty)$. Then for any*

$$f \in L^p(t_1, t_2; W^{s,p}(B_r)) \cap L^{\infty}\left(t_1, t_2; L^2(B_r)\right),$$

we have

$$\int_{t_1}^{t_2} \fint_{B_r} |f(x,t)|^{p(1+\frac{2s}{N})}dxdt \le C\left(r^{sp}\int_{t_1}^{t_2}\int_{B_r}\fint_{B_r}\frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}}dxdydt + \int_{t_1}^{t_2}\fint_{B_r}|f(x,t)|^p dxdt\right)$$
$$\times\left(\operatorname*{ess\,sup}_{t_1<t<t_2}\fint_{B_r}|f(x,t)|^2dx\right)^{\frac{sp}{N}}, \tag{2.3}$$

where $C > 0$ only depends on s, p and N.

Proof. We prove the imbedding inequality with $r = 1$. For any $r > 0$, we get the desired inequality by using a scaling argument.

Case 1: $sp < N$**.** We have by Hölder's inequality that

$$\begin{aligned}\int_{t_1}^{t_2} \fint_{B_1} |f(x,t)|^{p(1+\frac{2s}{N})} dxdt &= \int_{t_1}^{t_2} \fint_{B_1} |f(x,t)|^{\frac{2sp}{N}} |f(x,t)|^p dxdt \\ &\le \int_{t_1}^{t_2} \left(\fint_{B_1} |f(x,t)|^2 dx \right)^{\frac{sp}{N}} \left(\fint_{B_1} |f(x,t)|^{\frac{pN}{N-sp}} dx \right)^{\frac{N-sp}{N}} dt \\ &\le \left(\operatorname*{ess\,sup}_{t_1<t<t_2} \fint_{B_1} |f(x,t)|^2 dx \right)^{\frac{sp}{N}} \int_{t_1}^{t_2} \left(\fint_{B_1} |f(x,t)|^{\frac{pN}{N-sp}} dx \right)^{\frac{N-sp}{N}} dt, \quad (2.4)\end{aligned}$$

which in conjunction with Lemma 2.2 gives us the desired estimate.

Case 2: $sp \ge N$**.** The condition $sp \ge N$ ensures that

$$s > \frac{N}{p} - \frac{N}{p(1+\frac{2s}{N})}, \quad p(1+\frac{2s}{N}) > 2$$

and

$$\theta \left(s - \frac{N}{p} + \frac{N}{p(1+\frac{2s}{N})} \right) + (1-\theta) \left(\frac{N}{p(1+\frac{2s}{N})} - \frac{N}{2} \right) = 0$$

with $\theta = \frac{N}{N+2s} \in (0,1)$. These allow us to utilize Lemma 2.1 and obtain

$$\|f\|_{L^{p(1+\frac{2s}{N})}(B_1)} \le C \|f\|_{W^{s,p}(B_1)}^{\frac{N}{N+2s}} \|f\|_{L^2(B_1)}^{\frac{2s}{N+2s}} \quad \text{for all } t \in (t_1,t_2),$$

namely,

$$\begin{aligned}\fint_{B_1} |f(x,t)|^{p(1+\frac{2s}{N})} dx \le & C \Big(\int_{B_1} \fint_{B_1} \frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}} dxdy + \fint_{B_1} |f(x,t)|^p dx \Big) \\ & \times \left(\fint_{B_1} |f(x,t)|^2 dx \right)^{\frac{sp}{N}} \quad \text{for all } t \in (t_1,t_2). \qquad (2.5)\end{aligned}$$

An integration w.r.t the time-variable to (2.5) yields that

$$\begin{aligned}\int_{t_1}^{t_2} \fint_{B_1} |f(x,t)|^{p(1+\frac{2s}{N})} dxdt \le & C \int_{t_1}^{t_2} \Big(\int_{B_1} \fint_{B_1} \frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}} dxdy + \fint_{B_1} |f(x,t)|^p dx \Big) \\ & \times \left(\fint_{B_1} |f(x,t)|^2 dx \right)^{\frac{sp}{N}} dt \\ \le & C \left(\int_{t_1}^{t_2} \int_{B_1} \fint_{B_1} \frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}} dxdydt + \int_{t_1}^{t_2} \fint_{B_1} |f(x,t)|^p dxdt \right) \\ & \times \left(\operatorname*{ess\,sup}_{t_1<t<t_2} \fint_{B_1} |f(x,t)|^2 dx \right)^{\frac{sp}{N}}. \qquad (2.6)\end{aligned}$$

Thus, we can conclude the proof by virtue of (2.4) and (2.6). □

Lemma 2.4. *Let $t_2 > t_1 > 0$. Suppose $s \in (0,1)$ and $p \in [1,\infty)$. Then for any*

$$f \in L^p(t_1,t_2;W^{s,p}(B_r)) \cap L^\infty(t_1,t_2;L^p(B_r)),$$

there holds that

$$\int_{t_1}^{t_2}\fint_{B_r}|f(x,t)|^{p(1+\frac{sp}{N})}dxdt \le C\left(r^{sp}\int_{t_1}^{t_2}\int_{B_r}\fint_{B_r}\frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}}dxdydt+\int_{t_1}^{t_2}\fint_{B_r}|f(x,t)|^p dxdt\right) \times\left(\operatorname*{ess\,sup}_{t_1<t<t_2}\fint_{B_r}|f(x,t)|^p dx\right)^{\frac{sp}{N}},$$

where $C>0$ only depends on s,p and N.

Proof. According to Lemma 2.1, we can see

$$\|f\|_{L^{p(1+\frac{sp}{N})}(B_1)} \le C\|f\|_{W^{s,p}(B_1)}^{\frac{N}{N+sp}}\|f\|_{L^p(B_1)}^{\frac{sp}{N+sp}} \tag{2.7}$$

for all $f\in W^{s,p}(B_1)$ with $C>0$ only depending on s,p and N. By using a scaling argument, we have from (2.7) that

$$\fint_{B_r}|f(x,t)|^{p(1+\frac{sp}{N})}dx \le C\Big(r^{sp}\int_{B_r}\fint_{B_r}\frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}}dxdy+\fint_{B_r}|f(x,t)|^p dx\Big) \times\Big(\fint_{B_r}|f(x,t)|^p dx\Big)^{\frac{sp}{N}} \quad \text{for all } t\in(t_1,t_2). \tag{2.8}$$

Integrating (2.8) w.r.t the time-variable gives that

$$\begin{aligned}\int_{t_1}^{t_2}\fint_{B_r}|f(x,t)|^{p(1+\frac{sp}{N})}dxdt \le& C\int_{t_1}^{t_2}\Big(r^{sp}\int_{B_r}\fint_{B_r}\frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}}dxdy+\fint_{B_r}|f(x,t)|^p dx\Big) \\ &\times\Big(\fint_{B_r}|f(x,t)|^p dx\Big)^{\frac{sp}{N}}dt \\ \le& C\left(r^{sp}\int_{t_1}^{t_2}\int_{B_r}\fint_{B_r}\frac{|f(x,t)-f(y,t)|^p}{|x-y|^{N+sp}}dxdydt+\int_{t_1}^{t_2}\fint_{B_r}|f(x,t)|^p dxdt\right) \\ &\times\left(\operatorname*{ess\,sup}_{t_1<t<t_2}\fint_{B_r}|f(x,t)|^p dx\right)^{\frac{sp}{N}},\end{aligned}$$

as desired. □

We end this section with a statement of a Poincaré-type inequality.

Lemma 2.5. (see [38, Formula (6.3)]) *Let $s\in(0,1)$ and $p\in[1,\infty)$. Then for any $f\in W^{s,p}(B_r)$, there holds that*

$$\fint_{B_r}|f(x)-(f)_{B_r}|^p\,dx \le Cr^{sp-N}\int_{B_r}\int_{B_r}\frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}dxdy$$

with $C>0$ only depending on s,p and N.

3 Fundamental estimates

This section is devoted to establishing the Caccioppoli estimates and logarithmic form estimates. We begin with a preliminary lemma which can be found in [15, Lemma 3.1].

Lemma 3.1. *Let $p\ge1$. For $a,b\in\mathbb{R}$ and $\varepsilon>0$, we have that*

$$|a|^p\le|b|^p+C_p\varepsilon|b|^p+(1+C_p\varepsilon)\,\varepsilon^{1-p}|a-b|^p$$

with $C_p:=(p-1)\Gamma(\max\{1,p-2\})$. Here Γ is the standard Gamma function.

Before giving our desired Caccioppoli estimates, we invoke the technique provided in [5, Section 3.2] to regularize test functions w.r.t the time-variable. Let the function $\zeta : \mathbb{R} \to \mathbb{R}$ be a nonnegative, even smooth function with compact support in $(-\frac{1}{2}, \frac{1}{2})$. For any $\varphi \in L^1((a,b))$, we define the convolution

$$\varphi^{\varepsilon}(t) := \frac{1}{\varepsilon}\int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} \zeta\Big(\frac{t-\tau}{\varepsilon}\Big)\varphi(\tau)d\tau = \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \zeta\Big(\frac{\sigma}{\varepsilon}\Big)\varphi(t-\sigma)d\sigma, \quad t \in (a,b), \tag{3.1}$$

where $0 < \varepsilon < \min\{b-t, t-a\}$. The properties of convolutions exhibited in the forthcoming lemma are necessary ingredients when we proceed the regularization procedure. The results in Lemma 3.2 can be immediately proved by applying fundamental inequalities and utilizing the property of ζ. Here we omit the details.

Lemma 3.2. *Let $s > 0$ and $p, q > 1$. Assume that $0 < T_1 < T_2$ and $0 < \varepsilon < \varepsilon_0 < \frac{T_2 - T_1}{2}$.*

(i) *If $\varphi \in C([T_1, T_2]; L^q(\Omega))$, then we have $\varphi^{\varepsilon}(\cdot, t)$ converges to $\varphi(\cdot, t)$ in $L^q(\Omega)$ for every $t \in (T_1 + \frac{\varepsilon_0}{2}, T_2 - \frac{\varepsilon_0}{2})$ as $\varepsilon \to 0$.*

(ii) *Suppose that $\varphi \in C([T_1, T_2]; L^q(\Omega))$. Then there holds that $\varphi^{\varepsilon}(\cdot, t + \frac{\varepsilon}{2})$ converges to $\varphi(\cdot, t)$ in $L^q(\Omega)$ for each $t \in (T_1, T_2 - \varepsilon_0)$ as $\varepsilon \to 0$.*

(iii) *Assume that $\varphi \in L^q(T_1, T_2; L^p(\Omega))$. Then there is $C > 0$ only depending on p, q such that*

$$\|\varphi^{\varepsilon}\|_{L^q(T_1+\frac{\varepsilon_0}{2}, T_2-\frac{\varepsilon_0}{2}; L^p(\Omega))} \le C \quad \textit{for any } \varepsilon \le \varepsilon_0.$$

(iv) *If $\varphi \in L^q(T_1, T_2; W^{s,p}(\Omega))$, then we can find $C > 0$ only depending on s, p, q such that*

$$\|\varphi^{\varepsilon}\|_{L^q(T_1+\frac{\varepsilon_0}{2}, T_2-\frac{\varepsilon_0}{2}; W^{s,p}(\Omega))} \le C \quad \textit{for all } \varepsilon \le \varepsilon_0.$$

Lemma 3.3. (**Caccioppoli estimates**) *Let $p > 1$ and u be a local sub-solution to (1.1). Suppose that f satisfies (1.4) with some $\beta > 1$ and $h \in L_{\mathrm{loc}}^{\frac{\beta}{\beta-1}}(Q_T)$. Let $x_0 \in \Omega$, $r > 0$, $B_r \equiv B_r(x_0)$ satisfying $\overline{B}_r \subseteq \Omega$ and $0 < \tau_1 < \tau_2$, $\ell > 0$ satisfying $[\tau_1 - \ell, \tau_2] \subseteq (0, T)$. For all nonnegative functions $\psi \in C_0^{\infty}(B_r)$ and $\eta \in C^{\infty}(\mathbb{R})$ such that $\eta(t) \equiv 0$ if $t \le \tau_1 - \ell$ and $\eta(t) \equiv 1$ if $t \ge \tau_1$, there exists a constant $C > 0$ only depending on $s, p, \beta, \Lambda, c_0$ and N such that*

$$\begin{aligned}
&\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}\int_{B_r} |w_+(x,t)\psi(x) - w_+(y,t)\psi(y)|^p \eta^2(t) d\mu dt + \operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_{B_r} w_+^2(x,t)\psi^p(x)dx \\
&\le C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}\int_{B_r} \max\{w_+(x,t), w_+(y,t)\}^p |\psi(x)-\psi(y)|^p \eta^2(t) d\mu dt \\
&\quad + C \operatorname*{ess\,sup}_{\substack{\tau_1-\ell<t<\tau_2 \\ x\in \operatorname{supp}\psi}} \int_{\mathbb{R}^N\setminus B_r} \frac{w_+^{p-1}(y,t)}{|x-y|^{N+sp}} dy \int_{\tau_1-\ell}^{\tau_2}\int_{B_r} w_+(x,t)\psi^p(x)\eta^2(t) dx dt \\
&\quad + C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r} \big(|u(x,t)|^{\beta} + h^{\frac{\beta}{\beta-1}}(x,t) + w_+^{\beta}(x,t)\big)\chi_{\{u\ge k\}}(x,t)\psi^p(x)\eta^2(t) dx dt \\
&\quad + C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r} w_+^2(x,t)\psi^p(x)\eta(t)|\partial_t\eta(t)| dx dt,
\end{aligned} \tag{3.2}$$

where $w := u - k$ with a level $k \in \mathbb{R}$.

Remark 2. *If the source function $f(x,t,u) = h(x,t)$, then the third integral on the right-hand side in (3.2) can be replaced by $\int_{\tau_1-\ell}^{\tau_2}\int_{B_r} h(x,t)w_+(x,t)\psi^p(x)\eta^2(t)dxdt$.*

Proof. We begin this proof with regularizing the test function by invoking ideas introduced in [5, Lemma 3.3 and Appendix B]. With taking

$$0<\varepsilon<\frac{\varepsilon_0}{2}:=\frac{1}{4}\min\left\{\tau_1-\ell,T-\tau_2,\tau_2-\tau_1+\ell\right\}$$

and abbreviating

$$\hat{\tau}_1:=\tau_1-\ell-\varepsilon_0,\quad \hat{\tau}_2:=\tau_2+\varepsilon_0,$$

we arbitrarily choose $\varphi\in L^p(\hat{\tau}_1,\hat{\tau}_2;W^{s,p}(B_r))\cap W^{1,2}(\hat{\tau}_1,\hat{\tau}_2;L^2(B_r))$ whose spatial support is compactly contained in B_r, and then define $\varphi^\varepsilon(\cdot,t)$ according to (3.1). Now we choose φ^ε as the test function in (1.6) to obtain that

$$\begin{aligned}&\int_{B_r}u(x,t_2)\varphi^\varepsilon(x,t_2)dx-\int_{t_1}^{t_2}\int_{B_r}u(x,t)\partial_t\varphi^\varepsilon(x,t)dxdt+\int_{t_1}^{t_2}\mathcal{E}(u,\varphi^\varepsilon,t)dt\\&\le\int_{B_r}u(x,t_1)\varphi^\varepsilon(x,t_1)dx+\int_{t_1}^{t_2}\int_{B_r}f(x,t,u)\varphi^\varepsilon(x,t)dxdt,\end{aligned}\tag{3.3}$$

here we fix $t_1=\tau_1-\ell$ and let $t_2\in(\tau_1,\tau_2]$ be determined later. It is clear that $t_1-\varepsilon,t_2+\varepsilon\in(\hat{\tau}_1,\hat{\tau}_2)$ for any $\varepsilon\le\varepsilon_0/2$, which ensures any integral in (3.3) and all the terms below make sense. Then it follows by the elementary properties of convolutions and Fubini's Theorem that

$$\begin{aligned}-\int_{t_1}^{t_2}\int_{B_r}u(x,t)\partial_t\varphi^\varepsilon(x,t)dxdt&=-\int_{B_r}\int_{t_1}^{t_2}u(x,t)\left(\partial_t\varphi\right)^\varepsilon(x,t)dtdx\\&=-\int_{B_r}\int_{t_1}^{t_2}\frac{1}{\varepsilon}\int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}}u(x,t)\partial_\tau\varphi(x,\tau)\zeta\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau dtdx\\&=-\int_{B_r}\int_{t_1-\frac{\varepsilon}{2}}^{t_1+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon}\int_{t_1}^{\tau+\frac{\varepsilon}{2}}u(x,t)\zeta\Big(\frac{\tau-t}{\varepsilon}\Big)dt\right)\partial_\tau\varphi(x,\tau)d\tau dx\\&\quad-\int_{B_r}\int_{t_2-\frac{\varepsilon}{2}}^{t_2+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon}\int_{\tau-\frac{\varepsilon}{2}}^{t_2}u(x,t)\zeta\Big(\frac{\tau-t}{\varepsilon}\Big)dt\right)\partial_\tau\varphi(x,\tau)d\tau dx\\&\quad-\int_{B_r}\int_{t_1+\frac{\varepsilon}{2}}^{t_2-\frac{\varepsilon}{2}}u^\varepsilon(x,\tau)\partial_\tau\varphi(x,\tau)d\tau dx.\end{aligned}\tag{3.4}$$

With taking

$$\begin{aligned}\Sigma(\varepsilon):=&-\int_{B_r}\int_{t_1-\frac{\varepsilon}{2}}^{t_1+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon}\int_{t_1}^{\tau+\frac{\varepsilon}{2}}u(x,t)\zeta\Big(\frac{\tau-t}{\varepsilon}\Big)dt\right)\partial_\tau\varphi(x,\tau)d\tau dx\\&-\int_{B_r}\int_{t_2-\frac{\varepsilon}{2}}^{t_2+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon}\int_{\tau-\frac{\varepsilon}{2}}^{t_2}u(x,t)\zeta\Big(\frac{\tau-t}{\varepsilon}\Big)dt\right)\partial_\tau\varphi(x,\tau)d\tau dx,\end{aligned}\tag{3.5}$$

an integration by parts to the last integral of (3.4) infers that

$$\begin{aligned}-\int_{t_1}^{t_2}\int_{B_r}u(x,t)\partial_t\varphi^\varepsilon(x,t)dxdt=&\int_{t_1+\frac{\varepsilon}{2}}^{t_2-\frac{\varepsilon}{2}}\int_{B_r}\partial_t u^\varepsilon(x,t)\varphi(x,t)dxdt+\Sigma(\varepsilon)\\&-\int_{B_r}u^\varepsilon\Big(x,t_2-\frac{\varepsilon}{2}\Big)\varphi\Big(x,t_2-\frac{\varepsilon}{2}\Big)dx\\&+\int_{B_r}u^\varepsilon\Big(x,t_1+\frac{\varepsilon}{2}\Big)\varphi\Big(x,t_1+\frac{\varepsilon}{2}\Big)dx.\end{aligned}\tag{3.6}$$

Combining (3.3) and (3.6), we have

$$\int_{t_1+\frac{\varepsilon}{2}}^{t_2-\frac{\varepsilon}{2}}\int_{B_r}\partial_t u^\varepsilon(x,t)\varphi(x,t)dxdt+\int_{t_1}^{t_2}\mathcal{E}(u,\varphi^\varepsilon,t)dt+\Sigma(\varepsilon)$$
$$\leq-\int_{B_r}u(x,t_2)\varphi^\varepsilon(x,t_2)dx+\int_{B_r}u(x,t_1)\varphi^\varepsilon(x,t_1)dx+\int_{t_1}^{t_2}\int_{B_r}f(x,t,u)\varphi^\varepsilon(x,t)dxdt$$
$$+\int_{B_r}u^\varepsilon\Big(x,t_2-\frac{\varepsilon}{2}\Big)\varphi\Big(x,t_2-\frac{\varepsilon}{2}\Big)dx-\int_{B_r}u^\varepsilon\Big(x,t_1+\frac{\varepsilon}{2}\Big)\varphi\Big(x,t_1+\frac{\varepsilon}{2}\Big)dx. \tag{3.7}$$

Now abbreviate $v^\varepsilon(x,t):=(u^\varepsilon-k)_+(x,t)$ and choose $\varphi(x,t)=v^\varepsilon(x,t)\psi^p(x)\eta^2(t)$ in (3.7) to get

$$\underbrace{\int_{t_1+\frac{\varepsilon}{2}}^{t_2-\frac{\varepsilon}{2}}\int_{B_r}\partial_t u^\varepsilon(x,t)(v^\varepsilon\psi^p\eta^2)(x,t)dxdt}_{I_1^\varepsilon}+\overline{\Sigma}(\varepsilon)$$
$$+\underbrace{\frac{1}{2}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}J_p\big(w(x,t)-w(y,t)\big)\times\big((v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t)-(v^\varepsilon\psi^p\eta^2)^\varepsilon(y,t)\big)d\mu dt}_{I_2^\varepsilon}$$
$$+\underbrace{\int_{t_1}^{t_2}\int_{\mathbb{R}^N\setminus B_r}\int_{B_r}J_p\big(w(x,t)-w(y,t)\big)\times(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t)d\mu dt}_{I_3^\varepsilon}$$
$$\leq\underbrace{\int_{t_1}^{t_2}\int_{B_r}f(x,t,u)(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t)dxdt}_{I_4^\varepsilon}$$
$$+\underbrace{\int_{B_r}u^\varepsilon\Big(x,t_2-\frac{\varepsilon}{2}\Big)(v^\varepsilon\psi^p\eta^2)\Big(x,t_2-\frac{\varepsilon}{2}\Big)dx-\int_{B_r}u(x,t_2)(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t_2)dx}_{I_5^\varepsilon}$$
$$+\underbrace{\int_{B_r}u(x,t_1)(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t_1)dx-\int_{B_r}u^\varepsilon\Big(x,t_1+\frac{\varepsilon}{2}\Big)(v^\varepsilon\psi^p\eta^2)\Big(x,t_1+\frac{\varepsilon}{2}\Big)dx}_{I_6^\varepsilon}, \tag{3.8}$$

where the quantity $\overline{\Sigma}(\varepsilon)$ is as specified in (3.5) with φ selected as $v^\varepsilon(x,t)\psi^p(x)\eta^2(t)$.

Before proceeding for our desired estimates, we need to take $\varepsilon\to 0$ and find the limits of $I_1^\varepsilon,I_2^\varepsilon,\dots,I_6^\varepsilon$. Clearly, it can be obtained by integrating by parts that

$$\begin{aligned}I_1^\varepsilon&=\int_{t_1+\frac{\varepsilon}{2}}^{t_2-\frac{\varepsilon}{2}}\int_{B_r}\partial_t(u^\varepsilon(x,t)-k)_+(u^\varepsilon(x,t)-k)_+\psi^p(x)\eta^2(t)dxdt\\&=\frac{1}{2}\int_{t_1+\frac{\varepsilon}{2}}^{t_2-\frac{\varepsilon}{2}}\int_{B_r}\partial_t(u^\varepsilon(x,t)-k)_+^2\psi^p(x)\eta^2(t)dxdt\\&=\frac{1}{2}\int_{B_r}\Big(u^\varepsilon\Big(x,t_2-\frac{\varepsilon}{2}\Big)-k\Big)_+^2\psi^p(x)\eta^2\Big(t_2-\frac{\varepsilon}{2}\Big)dx\\&\quad-\frac{1}{2}\int_{B_r}\Big(u^\varepsilon\Big(x,t_1+\frac{\varepsilon}{2}\Big)-k\Big)_+^2\psi^p(x)\eta^2\Big(t_1+\frac{\varepsilon}{2}\Big)dx\\&\quad-\int_{t_1+\frac{\varepsilon}{2}}^{t_2-\frac{\varepsilon}{2}}\int_{B_r}(u^\varepsilon(x,t)-k)_+^2\psi^p(x)\eta(t)\partial_t\eta(t)dxdt.\end{aligned} \tag{3.9}$$

Due to the fact $u \in C\left([\hat{\tau}_1, \hat{\tau}_2]; L^2(\Omega)\right)$ and Lemma 3.2 (i)&(ii), we can see that

$$
\begin{aligned}
I_1^\varepsilon \longrightarrow & \frac{1}{2}\int_{B_r} w_+^2(x,t_2)\psi^p(x)dx - \frac{1}{2}\int_{B_r} w_+^2(x,t_1)\psi^p(x)dx \\
& - \int_{t_1}^{t_2}\int_{B_r} \eta(t)\partial_t\eta(t)\psi^p(x)w_+^2(x,t)dxdt \quad \text{as } \varepsilon \to 0,
\end{aligned}
\tag{3.10}
$$

and then we denote the limit of I_1^ε by I_1. Let us turn to the term I_2^ε. A simple calculation infers that

$$
\begin{aligned}
I_2^\varepsilon = & \frac{1}{2}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big) \times \big((v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t) - (w_+\psi^p\eta^2)(x,t)\big)d\mu dt \\
& + \frac{1}{2}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big) \times \big((w_+\psi^p\eta^2)(y,t) - (v^\varepsilon\psi^p\eta^2)^\varepsilon(y,t)\big)d\mu dt + I_2,
\end{aligned}
\tag{3.11}
$$

where we set

$$
I_2 = \frac{1}{2}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big) \times \big(w_+(x,t)\psi^p(x) - w_+(y,t)\psi^p(y)\big)\eta^2(t)d\mu dt. \tag{3.12}
$$

In light of Lemma 3.2 (iv), it can be obtained that $\{(v^\varepsilon\psi^p\eta^2)^\varepsilon\}_{\varepsilon\in(0,\frac{\varepsilon_0}{2})}$ admits an ε-independent bound in the space $L^p(t_1,t_2; W^{s,p}(B_r))$, which implies that

$$
\left\| \frac{(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t)}{|x-y|^{\frac{N}{p}+s}} \right\|_{L^p(t_1,t_2;L^p(B_r\times B_r))} \leq C.
$$

This combined with (1.3) gives that

$$
\left\| K^{\frac{1}{p}}(x,y,t)(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t) \right\|_{L^p(t_1,t_2;L^p(B_r\times B_r))} \leq C. \tag{3.13}
$$

Considering the convergence $(v^\varepsilon\psi^p\eta^2)^\varepsilon \to (u-k)_+\psi^p\eta^2$ a.e. in $B_r \times (t_1,t_2)$ and recalling the fact $(u-k)_+ = w_+$, we thus derive from (3.13) that as $\varepsilon \to 0$,

$$
\begin{aligned}
& K^{\frac{1}{p}}(x,y,t)(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t) \\
& \rightharpoonup K^{\frac{1}{p}}(x,y,t)w_+(x,t)\psi^p(x)\eta^2(t) \quad \text{in } L^p(t_1,t_2;L^p(B_r\times B_r)).
\end{aligned}
\tag{3.14}
$$

On the other hand, we have

$$
\begin{aligned}
& \int_{t_1}^{t_2}\int_{B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big) \times \big((v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t) - (w_+\psi^p\eta^2)(x,t)\big)d\mu dt \\
= & \int_{t_1}^{t_2}\int_{B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big)K^{1-\frac{1}{p}}(x,y,t) \\
& \qquad \times \big((v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t) - (w_+\psi^p\eta^2)(x,t)\big)K^{\frac{1}{p}}(x,y,t)dxdydt.
\end{aligned}
\tag{3.15}
$$

By utilizing (1.3), we can check

$$
J_p\big(w(x,t)-w(y,t)\big)K^{1-\frac{1}{p}}(x,y,t) \in L^{\frac{p}{p-1}}(t_1,t_2;L^{\frac{p}{p-1}}(B_r\times B_r)). \tag{3.16}
$$

Thus, a combination of (3.14)-(3.16) leads to the convergence that

$$
\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big) \times \big((v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t) - (w_+\psi^p\eta^2)(x,t)\big)d\mu dt \to 0 \tag{3.17}
$$

as $\varepsilon \to 0$. Similar arguments performed on the second integral on the right-hand side of (3.11) tells that

$$\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big)\times\big((w_+\psi^p\eta^2)(y,t)-(v^\varepsilon\psi^p\eta^2)^\varepsilon(y,t)\big)d\mu dt \to 0 \tag{3.18}$$

as $\varepsilon \to 0$. Hence, we can conclude from (3.11), (3.17) and (3.18) that

$$I_2^\varepsilon \to I_2 \quad \text{as } \varepsilon \to 0.$$

For the term I_3^ε, by utilizing similar arguments as those used for I_2^ε, we can derive that

$$I_3^\varepsilon \to \int_{t_1}^{t_2}\int_{\mathbb{R}^N\setminus B_r}\int_{B_r} J_p\big(w(x,t)-w(y,t)\big)\times(w_+\psi^p\eta^2)(x,t)d\mu dt =: I_3 \quad \text{as } \varepsilon \to 0. \tag{3.19}$$

The explicit reasoning of proving (3.19) can be found in [5, Section B, P44], thus we omits the details here. As for the term I_4^ε, the condition (1.4) and the assumption $h \in L_{\text{loc}}^{\frac{\beta}{\beta-1}}(Q_T)$ ensure that $f(u,x,t) \in L^{\frac{\beta}{\beta-1}}(t_1,t_2;L^{\frac{\beta}{\beta-1}}(B_r))$. Thanks to Lemma 3.2 (iii), the integrability $u \in L^\beta(\hat\tau_1,\hat\tau_2;L^\beta(B_r))$ yields that

$$\big\|(v^\varepsilon\psi^p\eta^2)^\varepsilon\big\|_{L^\beta(t_1,t_2;L^\beta(B_r))} \le C,$$

which implies that

$$(v^\varepsilon\psi^p\eta^2)^\varepsilon(x,t) \rightharpoonup w_+(x,t)\psi^p(x)\eta^2(t) \quad \text{in } L^\beta(t_1,t_2;L^\beta(B_r)) \quad \text{as } \varepsilon \to 0.$$

Hence, we obtain that

$$I_4^\varepsilon \to \int_{t_1}^{t_2}\int_{B_r} f(x,t,u)w_+(x,t)\psi^p(x)\eta^2(t)dxdt =: I_4 \quad \text{as } \varepsilon \to 0. \tag{3.20}$$

By employing the arguments used on [5, Lemma 3.3, Formula (3.6)], we also can verify $\overline{\Sigma}(\varepsilon) \to 0$ as $\varepsilon \to 0$. An application of Lemma 3.2 (ii) permits us to derive that $I_5^\varepsilon \to 0$ and $I_6^\varepsilon \to 0$ as $\varepsilon \to 0$. Finally, by virtue of the above convergence properties and (3.8), we obtain that

$$I_1 + I_2 + I_3 \le I_4. \tag{3.21}$$

The rest part as the last step of our proof is devoted to establishing the desired estimates of I_1, I_2, I_3 and I_4.

The estimate of I_1. Noticing the assumption that $\eta(t_1) = 0$ and $\eta(t) \equiv 1$ on the interval $[\tau_1,\tau_2]$, we directly have

$$I_1 = \frac{1}{2}\int_{B_r} w_+^2(x,t_2)\psi^p(x)dx - \int_{t_1}^{t_2}\int_{B_r}\eta(t)\partial_t\eta(t)\psi^p(x)w_+^2(x,t)dxdt \tag{3.22}$$

because of (3.10).

The estimate of I_2. The pointwise estimate used in the part is derived from [15, pp. 1285-1287]. For the sake of completeness, we give every detail here. The following arguments (3.23)-(3.27) are performed based on the assumption that $u(x,t) \ge u(y,t)$ with some $t \in (0,T)$. Otherwise, when the case $u(x,t) \le u(y,t)$ happens, the desired inequality (3.27) below can be obtained by exchanging the roles of x and y. Since $u(x,t) \ge u(y,t)$, it can be verified that

$$\big|w(x,t)-w(y,t)\big|^{p-2}\big(w(x,t)-w(y,t)\big)\big[w_+(x,t)\psi^p(x)-w_+(y,t)\psi^p(y)\big]$$

$$\geq \big(w(x,t)-w(y,t)\big)^{p-1}\big[w_+(x,t)\psi^p(x)-w_+(y,t)\psi^p(y)\big]. \tag{3.23}$$

When $w(x,t)\geq 0$ and $w(y,t)\geq 0$, we clearly have

$$\begin{aligned}&\big(w(x,t)-w(y,t)\big)^{p-1}\big[w_+(x,t)\psi^p(x)-w_+(y,t)\psi^p(y)\big]\\ &\geq \big(w_+(x,t)-w_+(y,t)\big)^{p-1}\big[w_+(x,t)\psi^p(x)-w_+(y,t)\psi^p(y)\big],\end{aligned} \tag{3.24}$$

where the two sides are actually equal in this case. If $w(x,t)\geq 0$ and $w(y,t)<0$, it can be verified that $\big(w(x,t)-w(y,t)\big)^{p-1}\geq w_+^{p-1}(x,t)=\big(w_+(x,t)-w_+(y,t)\big)^{p-1}$ which ensures that (3.24) still holds. When $w(x,t)<0$ and $w(y,t)<0$, the both sides of (3.23) equal zero. Thus, all of these guarantee that (3.24) is true whenever $u(x,t)\geq u(y,t)$.

Now we further assume that $\psi(y)>\psi(x)$ and $w_+(x,t)>w_+(y,t)\geq 0$, and then continue to estimate the right-hand side of (3.24). It follows from Lemma 3.1 that

$$\begin{aligned}\psi^p(x)&\geq \psi^p(y)-C_p\varepsilon\psi^p(x)-(1+C_p\varepsilon)\,\varepsilon^{1-p}|\psi(x)-\psi(y)|^p\\ &\geq \psi^p(y)-C_p\varepsilon\psi^p(y)-(1+C_p\varepsilon)\,\varepsilon^{1-p}|\psi(x)-\psi(y)|^p\end{aligned} \tag{3.25}$$

for any $\varepsilon\in(0,1]$ with $C_p=(p-1)\Gamma(\max\{1,p-2\})$. We choose

$$\varepsilon=\frac{1}{\max\{1,2C_p\}}\cdot\frac{w_+(x,t)-w_+(y,t)}{w_+(x,t)}$$

in (3.25) to get that

$$\begin{aligned}&\big(w_+(x,t)-w_+(y,t)\big)^{p-1}w_+(x,t)\psi^p(x)\\ &\geq \big(w_+(x,t)-w_+(y,t)\big)^{p-1}w_+(x,t)\big(\max\{\psi(x),\psi(y)\}\big)^p\\ &\quad-\frac{1}{2}\big(w_+(x,t)-w_+(y,t)\big)^p\big(\max\{\psi(x),\psi(y)\}\big)^p\\ &\quad-C\big(\max\{w_+(x,t),w_+(y,t)\}\big)^p|\psi(x)-\psi(y)|^p.\end{aligned} \tag{3.26}$$

For the other cases $\psi(y)>\psi(x)$ with $w_+(x,t)=w_+(y,t)\geq 0$, or $\psi(y)\leq\psi(x)$, the above inequality (3.26) apparently holds. A combination of (3.23)–(3.26) shows that

$$\begin{aligned}&\big|w(x,t)-w(y,t)\big|^{p-2}\big(w(x,t)-w(y,t)\big)\big[w_+(x,t)\psi^p(x)-w_+(y,t)\psi^p(y)\big]\\ &\geq \big(w_+(x,t)-w_+(y,t)\big)^{p-1}\big[w_+(x,t)\psi^p(x)-w_+(y,t)\psi^p(y)\big]\\ &\geq \big(w_+(x,t)-w_+(y,t)\big)^{p-1}\big[w_+(x,t)(\max\{\psi(x),\psi(y)\})^p-w_+(y,t)\psi^p(y)\big]\\ &\quad-\frac{1}{2}\big(w_+(x,t)-w_+(y,t)\big)^p\big(\max\{\psi(x),\psi(y)\}\big)^p\\ &\quad-C\big(\max\{w_+(x,t),w_+(y,t)\}\big)^p|\psi(x)-\psi(y)|^p\\ &\geq \frac{1}{2}\big(w_+(x,t)-w_+(y,t)\big)^p\big(\max\{\psi(x),\psi(y)\}\big)^p\\ &\quad-C\big(\max\{w_+(x,t),w_+(y,t)\}\big)^p|\psi(x)-\psi(y)|^p\end{aligned} \tag{3.27}$$

whenever $w_+(x,t)\geq w_+(y,t)$. For the case that $w_+(y,t)>w_+(x,t)$ in the integrand, the above estimate can be obtained by interchanging the roles of x and y. Finally, we can derive from (3.12) and (3.27) that

$$I_2\geq\frac{1}{4}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\big|w_+(x,t)-w_+(y,t)\big|^p\big(\max\{\psi(x),\psi(y)\}\big)^p\eta^2(t)d\mu dt$$

$$- C\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\big(\max\{w_+(x,t),w_+(y,t)\}\big)^p|\psi(x)-\psi(y)|^p\eta^2(t)d\mu dt.$$

This combined with an elementary inequality

$$\begin{aligned}\big|w_+(x,t)\psi(x)-w_+(y,t)\psi(y)\big|^p \leq & 2^{p-1}\big|w_+(x,t)-w_+(y,t)\big|^p\big(\max\{\psi(x),\psi(y)\}\big)^p \\ & + 2^{p-1}\big(\max\{w_+(x,t),w_+(y,t)\}\big)^p|\psi(x)-\psi(y)|^p \end{aligned} \tag{3.28}$$

gives that

$$\begin{aligned} I_2 \geq & \frac{1}{2^{p+1}}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}|w_+(x,t)\psi(x)-w_+(y,t)\psi(y)|^p\eta^2(t)d\mu dt \\ & - C\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\big(\max\{w_+(x,t),w_+(y,t)\}\big)^p|\psi(x)-\psi(y)|^p\eta^2(t)d\mu dt. \end{aligned}$$

The estimate of I_3. When $w(x,t)>0$, it is easy to verify that

$$\begin{aligned} & \big|w(x,t)-w(y,t)\big|^{p-2}\big(w(x,t)-w(y,t)\big)w_+(x,t) \\ \geq & -\big(w(y,t)-w(x,t)\big)_+^{p-1}w_+(x,t) \\ \geq & -w_+^{p-1}(y,t)w_+(x,t). \end{aligned} \tag{3.29}$$

If $w(x,t)\leq 0$, then we can check that (3.29) is still valid because both sides of (3.29) are zero. It follows by (3.19) and (3.29) that

$$\begin{aligned} I_3 = & \int_{\tau_1-\ell}^{t_2}\int_{\mathbb{R}^N\setminus B_r}\int_{B_r}\big|w(x,t)-w(y,t)\big|^{p-2}\big(w(x,t)-w(y,t)\big)w_+(x,t)\psi^p(x)\eta^2(t)d\mu dt \\ \geq & -C\int_{\tau_1-\ell}^{t_2}\int_{B_r}\Big(\int_{\mathbb{R}^N\setminus B_r}\frac{w_+^{p-1}(y,t)}{|x-y|^{N+sp}}dy\Big)w_+(x,t)\psi^p(x)\eta^2(t)dxdt \\ \geq & -C\operatorname*{ess\,sup}_{\substack{\tau_1-\ell<t<t_2\\ x\in\operatorname{supp}\psi}}\int_{\mathbb{R}^N\setminus B_r}\frac{w_+^{p-1}(y,t)}{|x-y|^{N+sp}}dy\int_{\tau_1-\ell}^{t_2}\int_{B_r}w_+(x,t)\psi^p(x)\eta^2(t)dxdt. \end{aligned}$$

The estimate of I_4. By using the structural condition on f and Young's inequality, we have

$$\begin{aligned} f(x,t,u)w_+(x,t) & \leq c_0|u(x,t)|^{\beta-1}w_+(x,t)+h(x,t)w_+(x,t) \\ & \leq Cw_+^{\beta}(x,t)+C|u(x,t)|^{\beta}\chi_{\{u\geq k\}}(x,t)+Ch^{\frac{\beta}{\beta-1}}(x,t)\chi_{\{u\geq k\}}(x,t), \end{aligned}$$

which in conjunction with (3.20) directly gives the estimate of I_4 as below

$$\begin{aligned} I_4 \leq & C\int_{\tau_1-\ell}^{t_2}\int_{B_r}w_+^{\beta}(x,t)\psi^p(x)\eta^2(t)dxdt \\ & + C\int_{\tau_1-\ell}^{t_2}\int_{B_r}|u(x,t)|^{\beta}\chi_{\{u\geq k\}}(x,t)\psi^p(x)\eta^2(t)dxdt \\ & + C\int_{\tau_1-\ell}^{t_2}\int_{B_r}h^{\frac{\beta}{\beta-1}}(x,t)\chi_{\{u\geq k\}}(x,t)\psi^p(x)\eta^2(t)dxdt. \end{aligned}$$

Hence, we can conclude from the estimates of I_1-I_4 and (3.21) that

$$\int_{\tau_1}^{t_2}\int_{B_r}\int_{B_r}|w_+(x,t)\psi(x)-w_+(y,t)\psi(y)|^p\eta^2(t)d\mu dt+\int_{B_r}w_+^2(x,t_2)\psi^p(x)dx$$

$$
\begin{aligned}
&\leq C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}\int_{B_r}\max\left\{w_+(x,t),w_+(y,t)\right\}^p|\psi(x)-\psi(y)|^p\eta^2(t)d\mu dt\\
&\quad+C\operatorname*{ess\,sup}_{\substack{\tau_1-\ell<t<\tau_2\\ x\in\operatorname{supp}\psi}}\int_{\mathbb{R}^N\setminus B_r}\frac{w_+^{p-1}(y,t)}{|x-y|^{N+sp}}dy\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}w_+(x,t)\psi^p(x)\eta^2(t)dxdt\\
&\quad+C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}\left(|u(x,t)|^\beta+|h(x,t)|^{\frac{\beta}{\beta-1}}+w_+^\beta(x,t)\right)\chi_{\{u\geq k\}}(x,t)\psi^p(x)\eta^2(t)dxdt\\
&\quad+C\int_{\tau_1-\ell}^{\tau_2}\int_{B_r}w_+^2(x,t)\psi^p(x)\eta(t)|\partial_t\eta(t)|dxdt. \qquad (3.30)
\end{aligned}
$$

In (3.30), we separately take $t_2=\tau_2$ and $t_2\in(\tau_1,\tau_2]$ with the property

$$
\int_{B_r}w_+^2\left(x,t_2\right)\psi^p(x)dx=\operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_{B_r}w_+^2(x,t)\psi^p(x)dx
$$

to obtain the desired estimate exhibited in (3.2). □

Lemma 3.4. *Let $p>1$. Then there exists a constant $0<\bar{C}_p<\min\{(p-1)/2,1/2\}$ only depending on p such that for all $s\in(0,1)$,*

$$
(1-s)^p\left(\frac{s^{1-p}-1}{1-s}-\bar{C}_p\right)\geq\left(\frac{p-1}{8p}\log\frac{1}{s}\right)^p. \qquad (3.31)
$$

Proof. We divide the proof into two cases.

Case 1: $s<1/2$. Let $h(z)=z-\log(z^p+1)/(2p)$ defined on $[1,\infty)$. The fact $h'(z)\geq 1/2$ ensures that $h(z)\geq h(1)>0$ for all $z\geq 1$. Thus we have

$$
z>\frac{1}{2p}\log(z^p+1)=\frac{p-1}{2p}\log\left((z^p+1)^{\frac{1}{p-1}}\right),\quad\forall\, z\geq 1.
$$

With the translation $t=(z^p+1)^{\frac{1}{p-1}}$ for $t\geq 2$, the above inequality directly yields that

$$
(t^{p-1}-1)^{\frac{1}{p}}>\frac{p-1}{2p}\log t,\quad\forall\, t\geq 2.
$$

This allows us to find a positive constant $\bar{C}_p<\min\{(p-1)/2,1/2\}$ such that

$$
\left(t^{p-1}-1-\bar{C}_p\right)^{\frac{1}{p}}\geq\frac{p-1}{4p}\log t,\quad\forall\, t\geq 2. \qquad (3.32)
$$

Now we take $s=1/t$ with $s\in(0,1/2)$ and derive from (3.32) that

$$
\begin{aligned}
(1-s)^p\left(\frac{s^{1-p}-1}{1-s}-\bar{C}_p\right)&\geq\frac{1}{2^p}\left(s^{1-p}-1-\bar{C}_p\right)\\
&\geq\left(\frac{p-1}{8p}\log\frac{1}{s}\right)^p,\quad\forall\, s<\frac{1}{2}. \qquad (3.33)
\end{aligned}
$$

We arrive at the claim for the case $s<1/2$.

Case 2: $s\geq 1/2$. Now, consider the function $s\mapsto g(s)$ given by

$$
g(s):=\frac{s^{1-p}-1}{1-s}=\frac{p-1}{1-s}\int_s^1\tau^{-p}\mathrm{d}\tau,\quad\forall\, s\in\left[\frac{1}{2},1\right).
$$

Since the integrand is a decreasing function, it can be deduced that $g(s)$ also decreases w.r.t s. Thus we have

$$g(s) \geq p-1, \quad \forall\, s \in \big[\frac{1}{2}, 1\big),$$

which directly tells that

$$\frac{s^{1-p}-1}{1-s} - \bar{C}_p \geq \frac{p-1}{2}, \quad \forall\, s \in \big[\frac{1}{2}, 1\big), \tag{3.34}$$

where we used the assumption $\bar{C}_p \leq (p-1)/2$. Moreover, we set $k(s) := 2(1-s) - \log\left(\frac{1}{s}\right)$ on $[1/2, 1]$, and then verify that $k'(s) = -2 + 1/s \leq 0$ for $s \in [1/2, 1]$. Thus, there holds that $k(s) \geq k(1) = 0$ for all $s \in [1/2, 1]$. This results in the estimate

$$(1-s)^p > \left(\frac{1}{2}\log\frac{1}{s}\right)^p, \quad \forall\, s \in \big[\frac{1}{2}, 1\big). \tag{3.35}$$

A combination of (3.34) and (3.35) gives that

$$(1-s)^p\Big(\frac{s^{1-p}-1}{1-s} - \bar{C}_p\Big) > \frac{p-1}{2^{p+1}}\Big(\log\frac{1}{s}\Big)^p, \quad \forall\, s \in \big[\frac{1}{2}, 1\big). \tag{3.36}$$

Hence, we complete the proof by virtue of (3.33) and (3.36). □

What follows is the Logarithmic estimate for the parabolic nonlocal equation. The elliptic version can be found in [15, Lemma 3.1]. Here, in the technical level, it is necessary to impose the condition $p > 2$ for controlling the term I_1 below by a desired form $Cr^N d^{2-p}$. As a consequence, this restriction prevents an extension of Theorem 3 to the subquadratic case.

Lemma 3.5. (**Logarithmic Lemma**) *Let $p > 2$ and u be a local solution to (1.1). Assume that $f(x,t,u) = h(x,t)$ in $Q_T \times \mathbb{R}$ and $h \in L^\infty_{\mathrm{loc}}(Q_T)$. Let $(x_0, t_0) \in Q_T$, $T_0 > 0$, $0 < r \leq R/2$ and $\tilde{Q} \equiv B_R(x_0) \times (t_0 - 2T_0, t_0 + 2T_0)$ such that $\overline{B}_R(x_0) \subseteq \Omega$ and $[t_0 - 2T_0, t_0 + 2T_0] \subseteq (0, T)$. Assume that $u \in L^\infty(\tilde{Q})$ and $u \geq 0$ in $\tilde{Q}$. Then the following estimate holds for $B_r \equiv B_r(x_0)$ and any $d > 0$,*

$$\begin{aligned}
&\int_{t_0-T_0}^{t_0+T_0}\int_{B_r}\int_{B_r}\left|\log\left(\frac{d+u(x,t)}{d+u(y,t)}\right)\right|^p d\mu dt \\
&\leq CT_0 r^{N-sp} d^{1-p}\left(\frac{r}{R}\right)^{sp}\left[\mathrm{Tail}_\infty(u; x_0, R, t_0 - 2T_0, t_0 + 2T_0)\right]^{p-1} \\
&\quad + CT_0 r^{N-sp} + Cr^N d^{2-p} + CT_0 r^N d^{1-p},
\end{aligned} \tag{3.37}$$

where $C > 0$ only depends on s, p, Λ, N and h.

Proof. The first step of the proof should be the regularization procedure, which can be performed by straightforward adaptation of standard reasonings used in Lemma 3.3. In order to avoid repeating the arguments, we omit this part. Let d be a positive constant and $\psi \in C_0^\infty\left(B_{3r/2}\right)$ be such that $0 \leq \psi \leq 1$, $|\nabla\psi| < Cr^{-1}$ in B_{2r} and $\psi \equiv 1$ in B_r. Let $\eta \in C_0^\infty(t_0 - 2T_0, t_0 + 2T_0)$ be such that $0 \leq \eta \leq 1$, $|\partial_t \eta| < CT_0^{-1}$ in $(t_0 - 2T_0, t_0 + 2T_0)$ and $\eta \equiv 1$ in $(t_0 - T_0, t_0 + T_0)$. The test function φ in (1.6) is given by

$$\varphi(x,t) = \big(u(x,t) + d\big)^{1-p}\psi^p(x)\eta^2(t).$$

This test function is well-defined since $u \geq 0$ in the supports of ψ and η. We deduce from (1.6) that

$$0 = -\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}} \partial_t\big(\big(u(x,t)+d\big)^{1-p}\eta^2(t)\big)\psi^p(x)u(x,t)\,dxdt$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\int_{B_{2r}}|u(x,t)-u(y,t)|^{p-2}\big(u(x,t)-u(y,t)\big)\\
&\qquad\qquad\times\left[\frac{\psi^p(x)}{\big(u(x,t)+d\big)^{p-1}}-\frac{\psi^p(y)}{(u(y,t)+d)^{p-1}}\right]\eta^2(t)d\mu dt\\
&+\int_{t_0-2T_0}^{t_0+2T_0}\int_{\mathbb{R}^N\setminus B_{2r}}\int_{B_{2r}}|u(x,t)-u(y,t)|^{p-2}\frac{u(x,t)-u(y,t)}{(u(x,t)+d)^{p-1}}\psi^p(x)\eta^2(t)d\mu dt\\
&-\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}f(x,t,u)\big(u(x,t)+d\big)^{1-p}\psi^p(x)\eta^2(t)dxdt\\
=:&\, I_1+I_2+I_3+I_4. \qquad (3.38)
\end{aligned}
$$

The estimate of I_1. By integrating by parts, we obtain

$$
\begin{aligned}
I_1&=\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\big(u(x,t)+d\big)^{1-p}\eta^2(t)\psi^p(x)\partial_t u(x,t)dxdt\\
&=\frac{1}{2-p}\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\partial_t\big(u(x,t)+d\big)^{2-p}\psi^p(x)\eta^2(t)dxdt\\
&\le C\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\big(u(x,t)+d\big)^{2-p}\psi^p(x)\eta(t)|\eta_t(t)|dxdt\\
&\le Cr^N d^{2-p}. \qquad (3.39)
\end{aligned}
$$

The estimate of I_2. As performed in Lemma 3.3, we first consider the time point $t\in(t_0-2T_0,t_0+2T_0)$ with the property that $u(x,t)>u(y,t)$. The assumption $u(y,t)\ge 0$ in supp $\psi\times(t_0-2T_0,t_0+2T_0)$ ensures that $(u(x,t)-u(y,t))/(u(x,t)+d)\in(0,1)$. With $\delta\in(0,1)$ to be determined later, we choose $a=\psi(x), b=\psi(y)$ and

$$
\varepsilon=\delta\frac{u(x,t)-u(y,t)}{u(x,t)+d}\in(0,1)
$$

in Lemma 3.1, and then deduce from Lemma 3.1 that

$$
\begin{aligned}
\psi^p(x)\le&\psi^p(y)+C_p\delta\frac{u(x,t)-u(y,t)}{u(x,t)+d}\psi^p(y)\\
&+\left[1+C_p\delta\frac{u(x,t)-u(y,t)}{u(x,t)+d}\right]\left[\delta\frac{u(x,t)-u(y,t)}{u(x,t)+d}\right]^{1-p}|\psi(x)-\psi(y)|^p\\
\le&\psi^p(y)+C_p\delta\frac{u(x,t)-u(y,t)}{u(x,t)+d}\psi^p(y)+(1+C_p)\left[\delta\frac{u(x,t)-u(y,t)}{u(x,t)+d}\right]^{1-p}|\psi(x)-\psi(y)|^p, \quad (3.40)
\end{aligned}
$$

where $C_p=(p-1)\Gamma(\max\{1,p-2\})$. It can be obtained by (3.40) that

$$
\begin{aligned}
&|u(x,t)-u(y,t)|^{p-2}\big(u(x,t)-u(y,t)\big)\left[\frac{\psi^p(x)}{\big(u(x,t)+d\big)^{p-1}}-\frac{\psi^p(y)}{\big(u(y,t)+d\big)^{p-1}}\right]\\
\le&\frac{\big(u(x,t)-u(y,t)\big)^{p-1}}{\big(u(x,t)+d\big)^{p-1}}\psi^p(y)\left[1+C_p\delta\frac{u(x,t)-u(y,t)}{u(x,t)+d}-\left(\frac{u(x,t)+d}{u(y,t)+d}\right)^{p-1}\right]\\
&+(C_p+1)\delta^{1-p}|\psi(x)-\psi(y)|^p\\
=&\left[\frac{u(x,t)-u(y,t)}{u(x,t)+d}\right]^p\psi^p(y)\left[\frac{1-\left(\frac{u(y,t)+d}{u(x,t)+d}\right)^{1-p}}{1-\frac{u(y,t)+d}{u(x,t)+d}}+C_p\delta\right]+(C_p+1)\delta^{1-p}|\psi(x)-\psi(y)|^p. \qquad (3.41)
\end{aligned}
$$

Let $\bar{C}_p > 0$ be as given in Lemma 3.4. It follows by choosing $s = \big(u(y,t)+d\big)/\big(u(x,t)+d\big)$ in Lemma 3.4 that

$$\left[\frac{u(x,t)-u(y,t)}{u(x,t)+d}\right]^p \left[\frac{1-\left(\frac{u(y,t)+d}{u(x,t)+d}\right)^{1-p}}{1-\frac{u(y,t)+d}{u(x,t)+d}} + \bar{C}_p\right] \le -\Big(\frac{p-1}{8p}\Big)^p \left[\log\left(\frac{u(x,t)+d}{u(y,t)+d}\right)\right]^p. \tag{3.42}$$

With taking $\delta = \bar{C}_p/C_p$, we derive from (3.41) and (3.42) that

$$\begin{aligned}
&|u(x,t)-u(y,t)|^{p-2}\big(u(x,t)-u(y,t)\big)\left[\frac{\psi^p(x)}{\big(u(x,t)+d\big)^{p-1}} - \frac{\psi^p(y)}{\big(u(y,t)+d\big)^{p-1}}\right] \\
&\le -\Big(\frac{p-1}{8p}\Big)^p \psi^p(y)\left|\log\left(\frac{u(x,t)+d}{u(y,t)+d}\right)\right|^p + C|\psi(x)-\psi(y)|^p
\end{aligned} \tag{3.43}$$

whenever $u(x,t) > u(y,t)$. For the case $u(x,t) = u(y,t)$, the above estimate holds trivially. If $u(y,t) > u(x,t)$, then (3.43) can be proved by exchanging the roles of x and y. Finally, we have

$$\begin{aligned}
I_2 \le &-C\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\int_{B_{2r}} \left|\log\left(\frac{u(x,t)+d}{u(y,t)+d}\right)\right|^p \psi^p(y)\eta^2(t)d\mu dt \\
&+C\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\int_{B_{2r}} |\psi(x)-\psi(y)|^p\eta^2(t)d\mu dt,
\end{aligned} \tag{3.44}$$

where the last term can be estimated by utilizing the assumption (1.3) on K,

$$\begin{aligned}
&\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\int_{B_{2r}} |\psi(x)-\psi(y)|^p\eta^2(t)d\mu dt \\
&\le Cr^{-p}\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}\int_{B_{2r}} \frac{1}{|x-y|^{N-p+sp}}dxdydt \\
&\le CT_0 r^{N-sp}.
\end{aligned} \tag{3.45}$$

Observe that $\eta \equiv 1$ on $(t_0 - T_0, t_0 + T_0)$ and $\psi \equiv 1$ in B_r. A combination of (3.44) and (3.45) shows that

$$I_2 \le -C\int_{t_0-T_0}^{t_0+T_0}\int_{B_r}\int_{B_r} \left|\log\left(\frac{u(x,t)+d}{u(y,t)+d}\right)\right|^p dxdydt + CT_0 r^{N-sp}. \tag{3.46}$$

The estimate of I_3. Recall that $u(y,t) \ge 0$ for $(y,t) \in B_R \times (t_0 - 2T_0, t_0 + 2T_0)$. Then it follows that for $y \in B_R$,

$$\frac{\big(u(x,t)-u(y,t)\big)_+^{p-1}}{\big(d+u(x,t)\big)^{p-1}} \le 1 \quad \text{with any } x \in B_{2r},\ t \in (t_0-2T_0, t_0+2T_0).$$

Moreover, for $y \in \mathbb{R}^N \backslash B_R$, we can see

$$\big(u(x,t)-u(y,t)\big)_+^{p-1} \le 2^{p-1}\left[u^{p-1}(x,t) + \big(u(y,t)\big)_-^{p-1}\right] \quad \text{with any } x \in B_{2r},\ t \in (t_0-2T_0, t_0+2T_0).$$

Thus it can be obtained that

$$I_3 \le \int_{t_0-2T_0}^{t_0+2T_0}\int_{B_R\backslash B_{2r}}\int_{B_{2r}} \frac{\big(u(x,t)-u(y,t)\big)_+^{p-1}}{\big(d+u(x,t)\big)^{p-1}}\psi^p(x)d\mu dt$$

$$
\begin{aligned}
&+\int_{t_0-2T_0}^{t_0+2T_0}\int_{\mathbb{R}^N\setminus B_R}\int_{B_{2r}}\frac{\big(u(x,t)-u(y,t)\big)_+^{p-1}}{\big(d+u(x,t)\big)^{p-1}}\psi^p(x)d\mu dt\\
&\le C\int_{t_0-2T_0}^{t_0+2T_0}\int_{\mathbb{R}^N\setminus B_{2r}}\int_{B_{2r}}\psi^p(x)d\mu dt\\
&\quad + Cd^{1-p}\int_{t_0-2T_0}^{t_0+2T_0}\int_{\mathbb{R}^N\setminus B_R}\int_{B_{2r}}\big(u(y,t)\big)_-^{p-1}\psi^p(x)d\mu dt. \qquad (3.47)
\end{aligned}
$$

Applying the assumption (1.3) on K and noticing supp $\psi\subseteq B_{3r/2}$, we have

$$
\begin{aligned}
&\int_{t_0-2T_0}^{t_0+2T_0}\int_{\mathbb{R}^N\setminus B_{2r}}\int_{B_{2r}}\psi^p(x)d\mu dt\\
&\le CT_0\sup_{x\in B_{3r/2}} r^N\int_{\mathbb{R}^N\setminus B_{2r}}\frac{1}{|x-y|^{N+sp}}dydt\le CT_0r^{N-sp}. \qquad (3.48)
\end{aligned}
$$

Since there holds

$$
\frac{|y-x_0|}{|y-x|}\le 1+\frac{|x-x_0|}{|x-y|}\le 1+\frac{3r/2}{R-3r/2}\le 4 \quad \text{for any } x\in B_{3r/2} \text{ and } y\in\mathbb{R}^N\setminus B_R,
$$

we can see

$$
\begin{aligned}
&\int_{t_0-2T_0}^{t_0+2T_0}\int_{\mathbb{R}^N\setminus B_R}\int_{B_{2r}}\big(u(y,t)\big)_-^{p-1}\psi^p(x)d\mu dt\\
&\le CT_0\,|B_{2r}|\operatorname*{ess\,sup}_{t\in(t_0-2T_0,t_0+2T_0)}\int_{\mathbb{R}^N\setminus B_R}\frac{\big(u(y,t)\big)_-^{p-1}}{|y-x_0|^{N+sp}}dy\\
&\le \frac{CT_0r^N}{R^{sp}}\big[\operatorname{Tail}_\infty(u;x_0,R,t_0-2T_0,t_0+2T_0)\big]^{p-1}. \qquad (3.49)
\end{aligned}
$$

Substituting (3.48) and (3.49) into (3.47) implies that

$$
I_3\le CT_0r^{N-sp}+\frac{CT_0r^N}{R^{sp}}d^{1-p}\big[\operatorname{Tail}_\infty(u;x_0,R,t_0-2T_0,t_0+2T_0)\big]^{p-1}. \qquad (3.50)
$$

The estimate of I_4. Noticing that $f(x,t,u)=h(x,t)$ in $Q_T\times\mathbb{R}$ and $h\in L^\infty_{\text{loc}}(Q_T)$, we immediately have

$$
I_4=-\int_{t_0-2T_0}^{t_0+2T_0}\int_{B_{2r}}h(x,t)\big(u(x,t)+d\big)^{1-p}\psi^p(x)\eta^2(t)dxdt\le CT_0r^Nd^{1-p}. \qquad (3.51)
$$

Together with (3.39), (3.46) and (3.50), this guarantees the claim. □

Corollary 1. *Let $p>2$ and u be a local solution to the problem (1.1). Assume that $f(x,t,u)=h(x,t)$ in $Q_T\times\mathbb{R}$ and $h\in L^\infty_{\text{loc}}(Q_T)$. Let $(x_0,t_0)\in Q_T$, $T_0>0$, $0<r\le R/2$ and $\tilde{Q}\equiv B_R(x_0)\times(t_0-2T_0,t_0+2T_0)$ such that $\overline{B}_R(x_0)\subseteq\Omega$ and $[t_0-2T_0,t_0+2T_0]\subseteq(0,T)$. Assume that $u\in L^\infty(\tilde{Q})$ and $u\ge 0$ in $\tilde{Q}$. Let $a,d>0,b>1$ and*

$$
v:=\min\Big\{\big(\log(a+d)-\log(u+d)\big)_+,\log b\Big\}.
$$

Then the following estimate holds for $B_r\equiv B_r(x_0)$,

$$
\int_{t_0-T_0}^{t_0+T_0}⨍_{B_r}|v(x,t)-(v)_{B_r}(t)|^p\,dxdt
$$

$$\leq CT_0 d^{1-p}\left(\frac{r}{R}\right)^{sp}\left[\,\mathrm{Tail}_\infty\left(u;x_0,R,t_0-2T_0,t_0+2T_0\right)\right]^{p-1}$$
$$+CT_0+Cr^{sp}d^{2-p}+CT_0r^{sp}d^{1-p}, \tag{3.52}$$

where $C>0$ *depends only on* s,p,Λ,N *and* h.

Proof. The fractional Poincaré inequality exhibited in Lemma 2.5 and the assumption (1.3) on K indicate that

$$\begin{aligned}\fint_{B_r}|v(x,t)-(v)_{B_r}(t)|^p\,dx &\leq Cr^{sp-N}\int_{B_r}\int_{B_r}\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+sp}}dxdy\\ &\leq Cr^{sp-N}\int_{B_r}\int_{B_r}K(x,y,t)|v(x,t)-v(y,t)|^p dxdy\end{aligned} \tag{3.53}$$

for all $t\in(t_0-T_0,t_0+T_0)$. An integration of (3.53) w.r.t the time variable leads to

$$\begin{aligned}&\int_{t_0-T_0}^{t_0+T_0}\fint_{B_r}|v(x,t)-(v)_{B_r}(t)|^p\,dxdt\\ \leq\ & Cr^{sp-N}\int_{t_0-T_0}^{t_0+T_0}\int_{B_r}\int_{B_r}K(x,y,t)|v(x,t)-v(y,t)|^p dxdydt\end{aligned} \tag{3.54}$$

with $C=C(s,p,\Lambda,N)>0$. Noticing that v is a truncation function of a constant and $\log(u+d)$, we have

$$\begin{aligned}&\int_{t_0-T_0}^{t_0+T_0}\int_{B_r}\int_{B_r}K(x,y,t)|v(x,t)-v(y,t)|^p dxdydt\\ \leq &\int_{t_0-T_0}^{t_0+T_0}\int_{B_r}\int_{B_r}K(x,y,t)\left|\log\left(\frac{u(y,t)+d}{u(x,t)+d}\right)\right|^p dxdydt.\end{aligned} \tag{3.55}$$

Now we apply Lemma 3.5 to estimate the right-hand side of (3.55), and then immediately arrive at the desired result by a combination of (3.54) and (3.55). □

4 Local boundedness

This section is devoted to obtaining the local boundedness of weak solutions to (1.1).

4.1 Recursive inequalities

In this subsection, we give the recursive inequalities for the cases $p\geq 2N/(N+2s)$ and $1<p<2N/(N+2s)$, respectively. Before this, some preparations need to be performed as below. Let $(x_0,t_0)\in Q_T$, $r>0$ and $Q_r^-\equiv B_r(x_0)\times(t_0-r^{sp},t_0)$ such that $\overline{B}_r(x_0)\subseteq\Omega$ and $[t_0-r^{sp},t_0]\subseteq(0,T)$. Take decreasing sequences

$$r_0:=r,\quad r_j:=\sigma r+2^{-j}(1-\sigma)r,\quad \tilde{r}_j:=\frac{r_j+r_{j+1}}{2},\quad j=0,1,2,\dots \tag{4.1}$$

with some $\sigma\in[1/2,1)$. Then, set the domains

$$Q_j^-:=B_j\times\Gamma_j:=B_{r_j}(x_0)\times(t_0-r_j^{sp},t_0),\quad j=0,1,2,\dots, \tag{4.2}$$
$$\tilde{Q}_j^-:=\tilde{B}_j\times\tilde{\Gamma}_j:=B_{\tilde{r}_j}(x_0)\times(t_0-\tilde{r}_j^{sp},t_0),\quad j=0,1,2,\dots. \tag{4.3}$$

For
$$\tilde{k} \geq \frac{\operatorname{Tail}_\infty\left(u_+; x_0, \sigma r, t_0 - r^{sp}, t_0\right)}{2},$$
we choose sequences of increasing levels as
$$k_j := \left(1 - 2^{-j}\right) \tilde{k}, \quad \tilde{k}_j := \frac{k_{j+1} + k_j}{2}, \quad j = 0, 1, 2, \ldots. \tag{4.4}$$
Let
$$w_j := (u - k_j)_+, \quad \tilde{w}_j := (u - \tilde{k}_j)_+, \quad j = 0, 1, 2, \ldots. \tag{4.5}$$
In the two coming lemmas, we deal with the Caccioppoli inequality written for the function $\tilde{w}_j$ over the domain Q_j^-. To this end, the cut-off functions $\psi_j \in C_0^\infty(\tilde{B}_j)$ and $\eta_j \in C_0^\infty(\tilde{\Gamma}_j)$ are taken to satisfy the conditions as follows:
$$0 \leq \psi_j \leq 1, \quad |\nabla \psi_j| \leq \frac{C 2^j}{(1-\sigma) r} \text{ in } \tilde{B}_j, \quad \psi_j \equiv 1 \text{ in } B_{j+1}$$
and
$$0 \leq \eta_j \leq 1, \quad |\partial_t \eta_j| \leq \frac{C 2^{spj}}{(1-\sigma)^{sp} r^{sp}} \text{ in } \tilde{\Gamma}_j, \quad \eta_j \equiv 1 \text{ in } \Gamma_{j+1}.$$

Lemma 4.1. *Let $p > 1$ and u be a local sub-solution to (1.1). Suppose that f satisfies (1.4), where*
$$\beta > 1 \quad \text{and} \quad h^{\frac{\beta}{\beta-1}} \in L^{\hat{q}}_{\mathrm{loc}}(Q_T) \quad \text{with} \quad \hat{q} > \frac{N+sp}{sp}.$$
Let $(x_0, t_0) \in Q_T$, $r \in (0,1)$ and $Q_r^- \equiv B_r(x_0) \times (t_0 - r^{sp}, t_0)$ such that $\overline{B}_r(x_0) \subseteq \Omega$ and $[t_0 - r^{sp}, t_0] \subseteq (0, T)$. Assume that q is a parameter with the property $q \geq \max\{p, 2, \beta\}$. Let $B_j, \tilde{B}_j, \Gamma_j, \tilde{\Gamma}_j$ be given in (4.2)-(4.3) and $\tilde{w}_j, w_j$ be defined in (4.5). Then we have for all $j \in \mathbb{N}$ that
$$\begin{aligned}
&\int_{\Gamma_{j+1}} \int_{B_{j+1}} \fint_{B_{j+1}} \frac{|\tilde{w}_j(x,t) - \tilde{w}_j(y,t)|^p}{|x-y|^{N+sp}} dx dy dt + \operatorname*{ess\,sup}_{t \in \Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^2(x,t) dx \\
&\leq \frac{C}{r^{sp}} \left[\frac{1}{\sigma^{sp}(1-\sigma)^{N+sp}} + \frac{1}{(1-\sigma)^p} \right] \left[\frac{2^{(sp+q-2)j}}{\tilde{k}^{q-2}} + \frac{2^{(N+sp+q-1)j}}{\tilde{k}^{q-p}} \right] \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t) dx dt \\
&\quad + \frac{C 2^{qj}}{\tilde{k}^{q-\beta}} \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t) dx dt + \frac{C 2^{\frac{qNj}{sp+N}(1+\frac{sp\kappa_0}{N})}}{\tilde{k}^{\frac{qN}{sp+N}(1+\frac{sp\kappa_0}{N})}} \left(\int_{\Gamma_j} \fint_{B_j} w_j^q(x,t) dx dt \right)^{\frac{N}{sp+N}(1+\frac{sp\kappa_0}{N})},
\end{aligned} \tag{4.6}$$
where $\kappa := 1 + 2s/N$ and $\kappa_0 := 1 - (sp+N)/(sp\hat{q}) \in (0,1]$ and $C > 0$ only depends on $s, p, \beta, \Lambda, N, c_0$ and h.

Proof. By simple calculations, there holds that $1/\hat{q} = (1-\kappa_0)sp/(sp+N)$. Before estimating the forthcoming integral terms, we first show that for any $0 \leq \tau < q$,
$$\begin{aligned}
(u - k_j)_+^q &\geq (u - k_j)_+^q \chi_{\{u \geq \tilde{k}_j\}}(x,t) \\
&\geq (\tilde{k}_j - k_j)^{q-\tau} (u - k_j)_+^\tau \chi_{\{u \geq \tilde{k}_j\}}(x,t) \\
&\geq C \tilde{k}^{q-\tau} 2^{-(q-\tau)j} (u - k_j)_+^\tau \chi_{\{u \geq \tilde{k}_j\}}(x,t) \\
&\geq C \tilde{k}^{q-\tau} 2^{-(q-\tau)j} (u - \tilde{k}_j)_+^\tau \quad \text{in } Q_T.
\end{aligned}$$

This directly tells that

$$\tilde{w}_j^{\tau}(x,t) \le \frac{C2^{(q-\tau)j}}{\tilde{k}^{q-\tau}} w_j^q(x,t) \quad \text{in } Q_T. \tag{4.7}$$

Now we choose $r = r_j, \tau_2 = t_0, \tau_1 = t_0 - r_{j+1}^{sp}$ and $\ell = \tilde{r}_j^{sp} - r_{j+1}^{sp}$ in Lemma 3.3 to get

$$\begin{aligned}
&\int_{\tilde{\Gamma}_j}\int_{B_j}⨍_{B_j} \frac{|\tilde{w}_j(x,t)\psi_j(x) - \tilde{w}_j(y,t)\psi_j(y)|^p}{|x-y|^{N+sp}} \eta_j^2(t)\,dxdydt + \operatorname*{ess\,sup}_{t\in\Gamma_{j+1}} ⨍_{B_j} \tilde{w}_j^2(x,t)\psi_j^p(x)\,dx \\
&\le C\int_{\tilde{\Gamma}_j}\int_{B_j}⨍_{B_j} \max\{\tilde{w}_j(x,t),\tilde{w}_j(y,t)\}^p\,|\psi_j(x)-\psi_j(y)|^p\eta_j^2(t)\,d\mu dt \\
&\quad + C\operatorname*{ess\,sup}_{\substack{t\in\tilde{\Gamma}_j\\ x\in\operatorname{supp}\psi_j}} \int_{\mathbb{R}^N\setminus B_j} \frac{\tilde{w}_j^{p-1}(y,t)}{|x-y|^{N+sp}}dy \int_{\tilde{\Gamma}_j}⨍_{B_j} \tilde{w}_j(x,t)\psi_j^p(x)\eta_j^2(t)\,dxdt \\
&\quad + C\int_{\tilde{\Gamma}_j}⨍_{B_j} \left(u^{\beta}(x,t)\chi_{\{u\ge\tilde{k}_j\}}(x,t) + \tilde{w}_j^{\beta}(x,t)\right)\psi_j^p(x)\eta_j^2(t)\,dxdt \\
&\quad + C\int_{\tilde{\Gamma}_j}⨍_{B_j} h^{\frac{\beta}{\beta-1}}(x,t)\chi_{\{u\ge\tilde{k}_j\}}(x,t)\psi_j^p(x)\eta_j^2(t)\,dxdt \\
&\quad + C\int_{\tilde{\Gamma}_j}⨍_{B_j} \tilde{w}_j^2(x,t)\psi_j^p(x)\eta_j(t)|\partial_t\eta_j(t)|\,dxdt \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{4.8}$$

The estimate of I_1. Based on the assumption on ψ_j and (4.7), we have

$$\begin{aligned}
I_1 &= \int_{\tilde{\Gamma}_j}\int_{B_j}⨍_{B_j} \max\{\tilde{w}_j(x,t),\tilde{w}_j(y,t)\}^p|\psi_j(x)-\psi_j(y)|^p\eta_j^2(t)\,d\mu dt \\
&\le \frac{C2^{pj}}{(1-\sigma)^p r^p} \sup_{x\in B_j}\int_{B_j} \frac{1}{|x-y|^{N+sp-p}}dy \int_{\tilde{\Gamma}_j}⨍_{B_j} \tilde{w}_j^p(x,t)\,dxdt \\
&\le \frac{C2^{pj}}{(1-\sigma)^p r^{sp}} \int_{\tilde{\Gamma}_j}⨍_{B_j} \tilde{w}_j^p(x,t)\,dxdt \\
&\le \frac{C2^{qj}}{\tilde{k}^{q-p}(1-\sigma)^p r^{sp}} \int_{\Gamma_j}⨍_{B_j} w_j^q(x,t)\,dxdt.
\end{aligned} \tag{4.9}$$

The estimate of I_2. For the term I_2, there holds that

$$\int_{\tilde{\Gamma}_j}⨍_{B_j} \tilde{w}_j(x,t)\psi_j^p(x)\eta_j^2(t)\,dxdt \le \frac{C2^{(q-1)j}}{\tilde{k}^{q-1}} \int_{\Gamma_j}⨍_{B_j} w_j^q(x,t)\,dxdt \tag{4.10}$$

because of (4.7). Notice that

$$\frac{|y-x_0|}{|y-x|} \le 1 + \frac{|x-x_0|}{|x-y|} \le 1 + \frac{\tilde{r}_j}{r_j-\tilde{r}_j} \le 4 + \frac{2^{j+2}\sigma}{1-\sigma} \quad \text{for any } x\in\operatorname{supp}\psi_j \text{ and } y\in\mathbb{R}^N\setminus B_j.$$

Thus we obtain that

$$\operatorname*{ess\,sup}_{\substack{t\in\tilde{\Gamma}_j\\ x\in\operatorname{supp}\psi_j}} \int_{\mathbb{R}^N\setminus B_j} \frac{\tilde{w}_j^{p-1}(y,t)}{|x-y|^{N+sp}}dy$$

$$
\begin{aligned}
&\leq \frac{C2^{(N+sp)j}}{(1-\sigma)^{N+sp}} \operatorname*{ess\,sup}_{t\in\tilde{\Gamma}_j} \int_{\mathbb{R}^N\setminus B_j} \frac{\tilde{w}_j^{p-1}(y,t)}{|x_0-y|^{N+sp}}dy \\
&\leq \frac{C2^{(N+sp)j}}{(1-\sigma)^{N+sp}} \operatorname*{ess\,sup}_{t\in\tilde{\Gamma}_j} \int_{\mathbb{R}^N\setminus B_{\sigma r}} \frac{\tilde{w}_0^{p-1}(y,t)}{|x_0-y|^{N+sp}}dy \\
&\leq \frac{C2^{(N+sp)j}}{r^{sp}\sigma^{sp}(1-\sigma)^{N+sp}} \big[\,\mathrm{Tail}_\infty\,(u_+;x_0,\sigma r,t_0-r^{sp},t_0)\,\big]^{p-1}. \qquad (4.11)
\end{aligned}
$$

Recalling the choice of $\tilde{k}$, we derive from (4.10) and (4.11) that

$$
\begin{aligned}
I_2 = &\operatorname*{ess\,sup}_{\substack{t\in\tilde{\Gamma}_j\\ x\in\operatorname{supp}\psi_j}} \int_{\mathbb{R}^N\setminus B_j} \frac{\tilde{w}_j^{p-1}(y,t)}{|x-y|^{N+sp}}dy \int_{\tilde{\Gamma}_j}⨍_{\tilde{B}_j} \tilde{w}_j(x,t)\psi_j^p(x)\eta_j^2(t)dxdt \\
&\leq \frac{C2^{(N+sp+q-1)j}}{r^{sp}\sigma^{sp}(1-\sigma)^{N+sp}\tilde{k}^{q-p}} \int_{\Gamma_j}⨍_{B_j} w_j^q(x,t)dxdt. \qquad (4.12)
\end{aligned}
$$

The estimate of I_3. It is easy to check that

$$
\begin{aligned}
(u-k_j)_+^\beta &\geq (u-k_j)_+^\beta \chi_{\{u\geq\tilde{k}_j\}}(x,t) \\
&\geq u^\beta(x,t)\Big(1-\frac{k_j}{\tilde{k}_j}\Big)^\beta \chi_{\{u\geq\tilde{k}_j\}}(x,t) \\
&\geq C2^{-\beta j}u^\beta(x,t)\chi_{\{u\geq\tilde{k}_j\}}(x,t) \quad \text{in } Q_T
\end{aligned}
$$

and

$$
\begin{aligned}
(u-k_j)_+^{q-\beta} &\geq (u-k_j)_+^{q-\beta}\chi_{\{u\geq\tilde{k}_j\}}(x,t) \\
&\geq C\tilde{k}^{q-\beta}2^{-(q-\beta)j}\chi_{\{u\geq\tilde{k}_j\}}(x,t) \quad \text{in } Q_T,
\end{aligned}
$$

which ensure that

$$
u^\beta(x,t)\chi_{\{u\geq\tilde{k}_j\}}(x,t) \leq \frac{C2^{qj}}{\tilde{k}^{q-\beta}} w_j^q(x,t). \qquad (4.13)
$$

Hence, it follows from (4.13) that

$$
\begin{aligned}
I_3 &\leq C\int_{\tilde{\Gamma}_j}⨍_{B_j} u^\beta(x,t)\chi_{\{u\geq\tilde{k}_j\}}(x,t)\psi_j^p(x)\eta_j^2(t)dxdt \\
&\leq \frac{C2^{qj}}{\tilde{k}^{q-\beta}} \int_{\Gamma_j}⨍_{B_j} w_j^q(x,t)dxdt, \qquad (4.14)
\end{aligned}
$$

where we also utilized the fact that $\tilde{w}_j^\beta(x,t) \leq u^\beta(x,t)\chi_{\{u\geq\tilde{k}_j\}}$ due to $\tilde{k}\geq 0$.

The estimate of I_4. By (4.7) and the Hölder inequality, we have

$$
\begin{aligned}
I_4 &= \int_{\tilde{\Gamma}_j}⨍_{B_j} h^{\frac{\beta}{\beta-1}}(x,t)\chi_{\{u\geq\tilde{k}_j\}}(x,t)\psi_j^p(x)\eta_j^2(t)dxdt \\
&\leq \|h^{\frac{\beta}{\beta-1}}\|_{L^{\hat{q}}(Q_T)}\Big(\int_{\Gamma_j}⨍_{B_j}\chi_{\{u\geq\tilde{k}_j\}}dxdt\Big)^{\frac{\hat{q}-1}{\hat{q}}} \\
&\leq \frac{C2^{\frac{qNj}{sp+N}(1+\frac{sp\kappa_0}{N})}}{\tilde{k}^{\frac{qN}{sp+N}(1+\frac{sp\kappa_0}{N})}}\Big(\int_{\Gamma_j}⨍_{B_j} w_j^q(x,t)dxdt\Big)^{\frac{N}{sp+N}(1+\frac{sp\kappa_0}{N})}. \qquad (4.15)
\end{aligned}
$$

The estimate of I_5. Still by using (4.7), we can see

$$\begin{aligned} I_5 &= \int_{\tilde{\Gamma}_j} \fint_{B_j} \tilde{w}_j^2(x,t)\psi_j^p(x)\eta_j(t)|\partial_t\eta_j(t)|dxdt \\ &\le \frac{C2^{spj}}{(1-\sigma)^{sp}r^{sp}} \int_{\Gamma_j} \fint_{B_j} \tilde{w}_j^2(x,t)dxdt \\ &\le \frac{C2^{(sp+q-2)j}}{(1-\sigma)^{sp}r^{sp}\tilde{k}^{q-2}} \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)dxdt. \end{aligned} \tag{4.16}$$

Based on the facts that $\psi_j \equiv 1$ in B_{j+1} and $\eta_j \equiv 1$ and Γ_{j+1}, we can conclude from (4.8), (4.9), (4.12) and (4.14)-(4.16) that

$$\begin{aligned} &\int_{\Gamma_{j+1}} \int_{B_{j+1}} \fint_{B_{j+1}} \frac{|\tilde{w}_j(x,t)-\tilde{w}_j(y,t)|^p}{|x-y|^{N+sp}} dxdydt + \operatorname*{ess\,sup}_{t\in\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^2(x,t)dx \\ &\le \frac{C}{r^{sp}}\left[\frac{1}{\sigma^{sp}(1-\sigma)^{N+sp}} + \frac{1}{(1-\sigma)^p}\right]\left[\frac{2^{(sp+q-2)j}}{\tilde{k}^{q-2}} + \frac{2^{(N+sp+q-1)j}}{\tilde{k}^{q-p}}\right] \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)dxdt \\ &\quad + \frac{C2^{qj}}{\tilde{k}^{q-\beta}} \int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)dxdt + \frac{C2^{\frac{qNj}{sp+N}(1+\frac{sp\kappa_0}{N})}}{\tilde{k}^{\frac{qN}{sp+N}(1+\frac{sp\kappa_0}{N})}} \Big(\int_{\Gamma_j} \fint_{B_j} w_j^q(x,t)dxdt\Big)^{\frac{N}{sp+N}(1+\frac{sp\kappa_0}{N})}, \end{aligned} \tag{4.17}$$

as desired. □

Lemma 4.2. *Let $p \ge 2N/(N+2s)$ and u be a local sub-solution to (1.1). Suppose that f satisfies (1.4), where*

$$\max\{p,2\} \le \beta < p\frac{2s+N}{N} \quad \text{and} \quad h^{\frac{\beta}{\beta-1}} \in L^{\hat{q}}_{\mathrm{loc}}(Q_T) \quad \text{with} \quad \hat{q} > \frac{N+sp}{sp}.$$

Let $(x_0,t_0) \in Q_T$, $r \in (0,1)$ and $Q_r^- \equiv B_r(x_0) \times (t_0 - r^{sp}, t_0)$ such that $\overline{B}_r(x_0) \subseteq \Omega$ and $[t_0 - r^{sp}, t_0] \subseteq (0,T)$. Let the notations $B_j, \tilde{B}_j, \Gamma_j, \tilde{\Gamma}_j$ and $\tilde{w}_j, w_j$ be given in (4.2), (4.3) and (4.5). Then we have for all $j \in \mathbb{N}$ that

$$\begin{aligned} \int_{\Gamma_{j+1}} \fint_{B_{j+1}} w_{j+1}^\beta(x,t)dxdt &\le \frac{C2^{bj}}{r^{\frac{s^2p\beta}{N\kappa}}}\left[\frac{1}{\sigma^{\frac{s\beta(N+sp)}{\kappa N}}(1-\sigma)^{\frac{\beta(N+sp)^2}{p\kappa N}}} + \frac{1}{(1-\sigma)^{\frac{\beta(N+sp)}{\kappa N}}}\right] \\ &\quad \times \left[\frac{1}{\tilde{k}^{\frac{\beta}{\kappa}(\frac{s\beta}{N}+\frac{2s}{N}-\frac{sp}{N})}} + \frac{1}{\tilde{k}^{\frac{\beta}{\kappa}(\frac{s\beta}{N}+1-\frac{2}{p})}}\right]\Big(\int_{\Gamma_j} \fint_{B_j} w_j^\beta(x,t)dxdt\Big)^{1+\frac{s\beta}{N\kappa}} \\ &\quad + \frac{C2^{bj}}{\tilde{k}^{\beta(1-\frac{\beta}{p\kappa})}}\Big(\int_{\Gamma_j} \fint_{B_j} w_j^\beta(x,t)dxdt\Big)^{1+\frac{s\beta}{N\kappa}} \\ &\quad + \frac{C2^{bj}}{\tilde{k}^{\beta(1+\frac{s\kappa_0\beta}{N\kappa})}}\Big(\int_{\Gamma_j} \fint_{B_j} w_j^\beta(x,t)dxdt\Big)^{1+\frac{s\kappa_0\beta}{N\kappa}}, \end{aligned} \tag{4.18}$$

where $b := (1+sp/N)(N+sp+\beta)$, $\kappa := 1+2s/N$, $\kappa_0 := 1-(sp+N)/(sp\hat{q}) \in (0,1]$ and $C>0$ only depends on s,p,β,Λ,N,c_0 and h.

Proof. Since $\beta < p\kappa$, we have by the Hölder inequality that

$$\begin{aligned} &\int_{\Gamma_{j+1}} \fint_{B_{j+1}} w_{j+1}^\beta(x,t)dxdt \le \int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^\beta(x,t)dxdt \\ &\le \Big(\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)dxdt\Big)^{\frac{\beta}{p\kappa}}\Big(\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \chi_{\{u\ge\tilde{k}_j\}}(x,t)dxdt\Big)^{1-\frac{\beta}{p\kappa}}. \end{aligned} \tag{4.19}$$

Thanks to the estimate given in (4.7), there holds

$$\int_{\Gamma_{j+1}} ⨍_{B_{j+1}} \chi_{\{u\geq \tilde{k}_j\}}(x,t)dxdt \leq \frac{C2^{\beta j}}{\tilde{k}^\beta} \int_{\Gamma_j} ⨍_{B_j} w_j^\beta(x,t)dxdt \tag{4.20}$$

and

$$\int_{\Gamma_{j+1}} ⨍_{B_{j+1}} \tilde{w}_j^p(x,t)dxdt \leq \frac{C2^{(\beta-p)j}}{\tilde{k}^{\beta-p}} \int_{\Gamma_j} ⨍_{B_j} w_j^\beta(x,t)dxdt. \tag{4.21}$$

By using (4.21) and applying Lemma 2.3, Lemma 4.1 with $q=\beta$, we can see

$$\begin{aligned}
\int_{\Gamma_{j+1}} ⨍_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)dxdt &\leq C\Bigg(r^{sp}\int_{\Gamma_{j+1}}\int_{B_{j+1}} ⨍_{B_{j+1}} \frac{|\tilde{w}_j(x,t)-\tilde{w}_j(y,t)|^p}{|x-y|^{N+sp}}dxdydt \\
&\quad + \int_{\Gamma_{j+1}} ⨍_{B_{j+1}} \tilde{w}_j^p(x,t)dxdt\Bigg)\times\Bigg(\operatorname*{ess\,sup}_{t\in\Gamma_{j+1}} ⨍_{B_{j+1}} \tilde{w}_j^2(x,t)dx\Bigg)^{\frac{sp}{N}} \\
&\leq \frac{C2^{bj}}{r^{\frac{s^2p^2}{N}}}\left[\frac{1}{\sigma^{\frac{sp(N+sp)}{N}}(1-\sigma)^{\frac{(N+sp)^2}{N}}} + \frac{1}{(1-\sigma)^{\frac{p(N+sp)}{N}}}\right] \\
&\quad \times\left(\frac{1}{\tilde{k}^{\beta-2}}+\frac{1}{\tilde{k}^{\beta-p}}\right)^{1+\frac{sp}{N}}\Big(\int_{\Gamma_j} ⨍_{B_j} w_j^\beta(x,t)dxdt\Big)^{1+\frac{sp}{N}} \\
&\quad + C2^{bj}\Big(\int_{\Gamma_j} ⨍_{B_j} w_j^\beta(x,t)dxdt\Big)^{1+\frac{sp}{N}} \\
&\quad + \frac{C2^{bj}}{\tilde{k}^{\beta(1+\frac{sp\kappa_0}{N})}}\Big(\int_{\Gamma_j} ⨍_{B_j} w_j^\beta(x,t)dxdt\Big)^{1+\frac{sp\kappa_0}{N}}
\end{aligned} \tag{4.22}$$

with $b=(1+sp/N)(N+sp+\beta)$. Substituting (4.22) and (4.20) into (4.19), we can arrive at the claim. □

Lemma 4.3. *Let* $1<p<2N/(N+2s)$ *and* $u\in L^\infty_{\text{loc}}(Q_T)$ *be a local sub-solution to* (1.1). *Let* $\kappa := 1+2s/N$ *and* $m>2$ *satisfy* $m>N(2-p)/sp$. *Suppose that* f *satisfies* (1.4) *with*

$$1<\beta\leq 2 \quad \text{and} \quad h\in L^\infty_{\text{loc}}(Q_T).$$

Let $(x_0,t_0)\in Q_T$, $r\in(0,1)$ *and* $Q_r^-\equiv B_r(x_0)\times(t_0-r^{sp},t_0)$ *such that* $\overline{B}_r(x_0)\subseteq\Omega$ *and* $[t_0-r^{sp},t_0]\subseteq(0,T)$. *Assume that the notations* $B_j,\tilde{B}_j,\Gamma_j,\tilde{\Gamma}_j$ *and* $w_j,\tilde{w}_j$ *are given in* (4.2), (4.3) *and* (4.5). *Then we have for all* $j\in\mathbb{N}$ *that*

$$\begin{aligned}
\int_{\Gamma_{j+1}} ⨍_{B_{j+1}} w_{j+1}^m(x,t)dxdt &\leq \frac{C2^{bj}}{r^{\frac{s^2p^2}{N}}}\left[\frac{1}{\sigma^{\frac{sp(N+sp)}{N}}(1-\sigma)^{\frac{(N+sp)^2}{N}}} + \frac{1}{(1-\sigma)^{\frac{p(N+sp)}{N}}}\right] \\
&\quad \times\left(\frac{1}{\tilde{k}^{m-2}}+\frac{1}{\tilde{k}^{m-p}}\right)^{1+\frac{sp}{N}}\|\tilde{w}_j\|^{m-p\kappa}_{L^\infty(Q_{j+1}^-)}\Big(\int_{\Gamma_j} ⨍_{B_j} w_j^m(x,t)dxdt\Big)^{1+\frac{sp}{N}} \\
&\quad + \frac{C2^{bj}}{\tilde{k}^{m(1+\frac{sp}{N})}}\|\tilde{w}_j\|^{m-p\kappa}_{L^\infty(Q_{j+1}^-)}\Big(\int_{\Gamma_j} ⨍_{B_j} w_j^m(x,t)dxdt\Big)^{1+\frac{sp}{N}},
\end{aligned} \tag{4.23}$$

where $b:=(1+sp/N)(N+sp+\beta)$ *and* $C>0$ *only depends on* $s,p,\beta,m,\Lambda,N,c_0$ *and* h.

Proof. Let m be such that the assumption holds. Without loss of generalization, we suppose $\beta = 2$ to perform the proof. Apparently, there holds that

$$\int_{\Gamma_{j+1}} \fint_{B_{j+1}} w_{j+1}^m(x,t)dxdt \le \int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^m(x,t)dxdt$$
$$\le \|\tilde{w}_j\|^{m-p\kappa}_{L^\infty(Q^-_{j+1})} \int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)dxdt. \tag{4.24}$$

Based on (4.7), we can find that

$$\int_{\Gamma_j} \fint_{B_j} \tilde{w}_j^p(x,t)dxdt \le \frac{C2^{(m-p)j}}{\tilde{k}^{m-p}} \int_{\Gamma_j} \fint_{B_j} w_j^m(x,t)dxdt. \tag{4.25}$$

Observe that $h \in L^\infty_{\rm loc}(Q_T)$ entails $\kappa_0 = 1$ in Lemma 4.1. By using (4.25), Lemma 2.3 and Lemma 4.1 with $q = m$ and $\beta = 2$, there holds that

$$\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^{p\kappa}(x,t)dxdt \le C\Bigg(r^{sp} \int_{\Gamma_{j+1}} \int_{B_{j+1}} \fint_{B_{j+1}} \frac{|\tilde{w}_j(x,t) - \tilde{w}_j(y,t)|^p}{|x-y|^{N+sp}} dxdydt$$
$$+ \int_{\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^p(x,t)dxdt\Bigg) \times \Bigg(\operatorname*{ess\,sup}_{t\in\Gamma_{j+1}} \fint_{B_{j+1}} \tilde{w}_j^2(x,t)dx\Bigg)^{\frac{sp}{N}}$$
$$\le \frac{C2^{bj}}{r^{\frac{s^2p^2}{N}}} \left[\frac{1}{\sigma^{\frac{sp(N+sp)}{N}}(1-\sigma)^{\frac{(N+sp)^2}{N}}} + \frac{1}{(1-\sigma)^{\frac{p(N+sp)}{N}}}\right]$$
$$\times \left(\frac{1}{\tilde{k}^{m-2}} + \frac{1}{\tilde{k}^{m-p}}\right)^{1+\frac{sp}{N}} \left(\int_{\Gamma_j} \fint_{B_j} w_j^m(x,t)dxdt\right)^{1+\frac{sp}{N}}$$
$$+ \frac{C2^{bj}}{\tilde{k}^{m(1+\frac{sp}{N})}} \left(\int_{\Gamma_j} \fint_{B_j} w_j^m(x,t)dxdt\right)^{1+\frac{sp}{N}} \tag{4.26}$$

with b as given in the statement of lemma. Finally, we complete the proof by substituting (4.26) into (4.24). □

4.2 Proof of boundedness results

We start the subsection with a classical technical lemma. The particular case of $\delta_2 = \delta_1$ in the next lemma can be found in [32, Chapter II, Lemma 5.6] and [17, Chapter I, Lemma 4.1].

Lemma 4.4. ([19, Lemma 4.3]) *Let $\{Y_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{j+1} \le Kb^j(Y_j^{1+\delta_1} + Y_j^{1+\delta_2}), \quad j = 0, 1, 2, \dots,$$

where $K > 0, b > 1$ and $\delta_2 \ge \delta_1 > 0$ are given numbers. If

$$Y_0 \le \min\left\{1, (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}\right\}$$

or

$$Y_0 \le \min\left\{(2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}, (2K)^{-\frac{1}{\delta_2}} b^{-\frac{1}{\delta_1\delta_2} - \frac{\delta_2-\delta_1}{\delta_2^2}}\right\},$$

then $Y_j \le 1$ for some $j \in \mathbb{N}$. Moreover,

$$Y_j \le \min\left\{1, (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}} b^{-\frac{j}{\delta_1}}\right\} \quad \text{for all } j \ge j_0,$$

where j_0 is the smallest $j \in \mathbb{N} \cup \{0\}$ satisfying $Y_j \le 1$. In particular, Y_j converges to zero as $j \to \infty$.

Now we are ready to give the proof of boundedness results.

Proof of Theorem 1. Let the assumption of Theorem 1 hold. Now we take $r = R$, $\sigma = 1/2$ in (4.1), and then set

$$Y_j = \int_{\Gamma_j} \fint_{B_j} (u - k_j)_+^\beta \, dxdt, \quad j = 0, 1, 2, \ldots.$$

Further assuming $\tilde{k} \ge 1$ and recalling $r < 1$, we can deduce from Lemma 4.2 that

$$\begin{aligned}
\frac{Y_{j+1}}{r^{sp}} &\le \frac{C 2^{bj} Y_j^{1+\frac{s\beta}{N\kappa}}}{r^{sp(1+\frac{s\beta}{N\kappa})} \tilde{k}^{\frac{\beta}{\kappa}(\frac{s\beta}{N}+\frac{2s}{N}-\frac{sp}{N})}} + \frac{C 2^{bj} Y_j^{1+\frac{s\beta}{N\kappa}}}{r^{sp(1+\frac{s\beta}{N\kappa})} \tilde{k}^{\frac{\beta}{\kappa}(\frac{s\beta}{N}+1-\frac{2}{p})}} \\
&\quad + \frac{C 2^{bj} Y_j^{1+\frac{s\beta}{N\kappa}}}{r^{sp} \tilde{k}^{\beta(1-\frac{\beta}{p\kappa})}} + \frac{C 2^{bj} Y_j^{1+\frac{s\kappa_0\beta}{N\kappa}}}{r^{sp} \tilde{k}^{\beta(1+\frac{s\kappa_0\beta}{N\kappa})}} \\
&\le \frac{C 2^{bj}}{\tilde{k}^{\beta(1-\frac{\beta}{p\kappa})}} \left(\frac{Y_j}{r^{sp}}\right)^{1+\frac{s\beta}{N\kappa}} + \frac{C 2^{bj}}{\tilde{k}^{\beta(1-\frac{\beta}{p\kappa})}} \left(\frac{Y_j}{r^{sp}}\right)^{1+\frac{s\kappa_0\beta}{N\kappa}},
\end{aligned} \tag{4.27}$$

where $b = (1 + sp/N)(N + sp + \beta)$, $\kappa = 1 + 2s/N$, $\kappa_0 = 1 - (sp + N)/(sp\hat{q}) \in (0, 1]$ and C only depends on $s, p, \beta, N, \Lambda, c_0$ and h. With setting $W_j = Y_j/r^{sp}$ for any $j \in \mathbb{N}$, the estimate (4.27) indicates that

$$W_{j+1} \le \frac{C 2^{bj}}{\tilde{k}^{\beta(1-\frac{\beta}{p\kappa})}} \left(W_j^{1+\frac{s\beta}{N\kappa}} + W_j^{1+\frac{s\kappa_0\beta}{N\kappa}}\right). \tag{4.28}$$

Let $\tilde{k}$ be chosen to satisfy that

$$\tilde{k} \ge \max\left\{\mathrm{Tail}_\infty\left(u_+; x_0, R/2, t_0 - R^{sp}, t_0\right), C\left(\fint_{t_0-R^{sp}}^{t_0} \fint_{B_R} u_+^\beta dxdt\right)^{\frac{sp}{N(p\kappa-\beta)}} \vee 1\right\}, \tag{4.29}$$

where the large constant C only depends on s, p, β, h, c_0 and N. This along with Lemma 4.4 guarantees that $W_j \to 0$ as $j \to \infty$. Thus, we can derive that

$$\operatorname*{ess\,sup}_{Q_{R/2}^-} u \le \mathrm{Tail}_\infty\left(u_+; x_0, R/2, t_0 - R^{sp}, t_0\right) + C\left(\fint_{Q_R^-} u_+^\beta dxdt\right)^{\frac{sp}{N(p\kappa-\beta)}} \vee 1, \tag{4.30}$$

as intended □

Proof of Theorem 2. Let the assumptions required in Theorem 2 hold. On the basis of the conditions (1.12) and (1.13), we may assume that u is qualitatively locally bounded, which can be achieved by working with a suitable approximation procedure: All the arguments performed below are reasonable when we replace u with u_k, due to the fact that approximation sub-solutions u_k are bounded ones. Thus, the estimate (4.36) below holds with u replaced by u_k. This together with (1.12) and (1.13) results in a k-independent bound of u_k in L^∞, which combined with the a.e. convergence of u_k ensures that u is qualitatively locally bounded.

Now let us define that $R_0 = R/2$ and $R_n = R/2 + \sum_{i=1}^{n} 2^{-i-1}R$ with $n \in \mathbb{N}^+$. We set the domains $Q_n^- = B_{R_n}(x_0) \times (t_0 - R_n^{sp}, t_0)$ and the quantities

$$M_n = \operatorname*{ess\,sup}_{Q_n^-} u_+, \quad n = 0, 1, 2, 3, \ldots.$$

For any chosen $n \in \mathbb{N}$, we choose $r = R_{n+1}$ and $\sigma r = R_n$ in (4.1), and thus

$$\sigma = \frac{1/2 + \sum_{i=1}^{n} 2^{-i-1}}{1/2 + \sum_{i=1}^{n+1} 2^{-i-1}} \geq \frac{1}{2}.$$

With r_j taken in (4.1), we set

$$Y_j = \fint_{\Gamma_j} \fint_{B_j} (u - k_j)_+^m \, dxdt, \quad j = 0, 1, 2, \ldots.$$

According to Lemma 4.3, we can see

$$\begin{aligned}
Y_{j+1} \leq & \frac{C2^{bj}}{R_{n+1}^{\frac{s^2p^2}{N}}} \|u_+\|_{L^\infty(Q_{n+1}^-)}^{m-p\kappa} \left[\frac{1}{\sigma^{\frac{sp(N+sp)}{N}} (1-\sigma)^{\frac{(N+sp)^2}{N}}} + \frac{1}{(1-\sigma)^{\frac{p(N+sp)}{N}}} \right] \\
& \times \left(\frac{1}{\tilde{k}^{m-2}} + \frac{1}{\tilde{k}^{m-p}} \right)^{1+\frac{sp}{N}} Y_j^{1+\frac{sp}{N}} + \frac{C2^{bj}}{\tilde{k}^{m(1+\frac{sp}{N})}} \|u_+\|_{L^\infty(Q_{n+1}^-)}^{m-p\kappa} Y_j^{1+\frac{sp}{N}} \\
\leq & \frac{C2^{bj+dn}}{R_{n+1}^{\frac{s^2p^2}{N}} \tilde{k}^{(m-2)(1+\frac{sp}{N})}} M_{n+1}^{m-p\kappa} Y_j^{1+\frac{sp}{N}} + \frac{C2^{bj+dn}}{R_{n+1}^{\frac{s^2p^2}{N}} \tilde{k}^{m(1+\frac{sp}{N})}} M_{n+1}^{m-p\kappa} Y_j^{1+\frac{sp}{N}},
\end{aligned} \tag{4.31}$$

where we used the fact $p < 2$. By taking $W_j = Y_j / R_n^{sp}$, $d = \max\{(sp+N)^2/N, p(sp+N)/N\}$ and $b = (sp + \beta + N)(1 + sp/N)$, we derive from (4.31) that

$$W_{j+1} \leq C2^{bj+dn} \Big(\frac{1}{\tilde{k}^{(m-2)(1+\frac{sp}{N})}} + \frac{1}{\tilde{k}^{m(1+\frac{sp}{N})}} \Big) M_{n+1}^{m-p\kappa} W_j^{1+\frac{sp}{N}}.$$

This in conjunction with Lemma 4.4 indicates that $Y_j \to 0$ as $j \to \infty$, provided that

$$W_0 \leq C2^{-\frac{dnN}{sp} - \frac{bN^2}{s^2p^2}} M_{n+1}^{-\frac{N(m-p\kappa)}{sp}} \Big(\frac{1}{\tilde{k}^{m-2}} + \frac{1}{\tilde{k}^m} \Big)^{-\frac{sp+N}{sp}}.$$

In order to make sure the above inequality holds, we take

$$\begin{aligned}
\tilde{k} \geq \max \Bigg\{ & C2^{\frac{dnN}{(m-2)(sp+N)}} \Big(\fint_{t_0 - R_{n+1}^{sp}}^{t_0} \fint_{B_{R_{n+1}}} u_+^m \, dxdt \Big)^{\frac{sp}{(m-2)(sp+N)}} M_{n+1}^{\frac{N(m-p\kappa)}{(m-2)(sp+N)}}, \\
& C2^{\frac{dnN}{m(sp+N)}} \Big(\fint_{t_0 - R_{n+1}^{sp}}^{t_0} \fint_{B_{R_{n+1}}} u_+^m \, dxdt \Big)^{\frac{sp}{m(sp+N)}} M_{n+1}^{\frac{N(m-p\kappa)}{m(sp+N)}}, \\
& \frac{\mathrm{Tail}_\infty \left(u_+; x_0, R_n, t_0 - R_{n+1}^{sp}, t_0 \right)}{2} \Bigg\}
\end{aligned}$$

with C only depending on s, p, β, m, N, c_0 and h. With this choice, we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{Q_{R_n}^-} u_+ \leq & \frac{\mathrm{Tail}_\infty \left(u_+; x_0, R_n, t_0 - R_{n+1}^{sp}, t_0 \right)}{2} \\
& + C2^{\frac{dnN}{(m-2)(sp+N)}} \Big(\fint_{t_0 - R_{n+1}^{sp}}^{t_0} \fint_{B_{R_{n+1}}} u_+^m \, dxdt \Big)^{\frac{sp}{(m-2)(sp+N)}} M_{n+1}^{\frac{N(m-p\kappa)}{(m-2)(sp+N)}}
\end{aligned}$$

$$+ C2^{\frac{dnN}{m(sp+N)}} \left(\fint_{t_0 - R_{n+1}^{sp}}^{t_0} \fint_{B_{R_{n+1}}} u_+^m dxdt \right)^{\frac{sp}{m(sp+N)}} M_{n+1}^{\frac{N(m-p\kappa)}{m(sp+N)}}. \tag{4.32}$$

An application of Young's inequality to (4.32) implies

$$\begin{aligned} M_n \leq & \eta M_{n+1} + \frac{\mathrm{Tail}_\infty (u_+; x_0, R/2, t_0 - R^{sp}, t_0)}{2} \\ & + C2^{\frac{dnN}{(sp+N)(m-2-\lambda_m)}} \eta^{-\frac{\lambda_m}{m-2-\lambda_m}} \left(\fint_{t_0 - R^{sp}}^{t_0} \fint_{B_R} u_+^m dxdt \right)^{\frac{sp}{(sp+N)(m-2-\lambda_m)}} \\ & + C2^{\frac{dnN}{(sp+N)(m-\lambda_m)}} \eta^{-\frac{\lambda_m}{m-\lambda_m}} \left(\fint_{t_0 - R^{sp}}^{t_0} \fint_{B_R} u_+^m dxdt \right)^{\frac{sp}{(sp+N)(m-\lambda_m)}}, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{4.33}$$

where $\lambda_m = (m - p\kappa)N/(sp + N)$. In order to clarify the iteration clearly, let us abbreviate

$$A_1 := 2^{\frac{dN}{(sp+N)(m-2-\lambda_m)}}, \quad A_2 := 2^{\frac{dN}{(sp+N)(m-\lambda_m)}},$$

$$B_0 := \frac{\mathrm{Tail}_\infty (u_+; x_0, R/2, t_0 - R^{sp}, t_0)}{2},$$

$$B_1 := C\eta^{-\frac{\lambda_m}{m-2-\lambda_m}} \left(\fint_{t_0 - R^{sp}}^{t_0} \fint_{B_R} u_+^m dxdt \right)^{\frac{sp}{(sp+N)(m-2-\lambda_m)}}$$

and

$$B_2 := C\eta^{-\frac{\lambda_m}{m-\lambda_m}} \left(\fint_{t_0 - R^{sp}}^{t_0} \fint_{B_R} u_+^m dxdt \right)^{\frac{sp}{(sp+N)(m-\lambda_m)}},$$

where C is as specified in the right-hand side of (4.33). These definitions along with (4.33) tell that

$$M_n \leq \eta M_{n+1} + B_0 + A_1^n B_1 + A_2^n B_2, \quad n = 0, 1, 2, \ldots. \tag{4.34}$$

Now we first claim that

$$M_0 \leq \eta^{n+1} M_{n+1} + B_1 \sum_{i=0}^{n} (\eta A_1)^i + B_2 \sum_{i=0}^{n} (\eta A_2)^i + B_0 \sum_{i=0}^{n} \eta^i, \quad n = 0, 1, 2, \ldots, \tag{4.35}$$

which is obviously true for $n = 0$ thanks to a direct application of (4.34). To verify (4.35) for any $n \geq 0$, we assume this inequality holds for some $k \in \mathbb{N}$, then by using (4.34) with $n = k + 1$, we have

$$\begin{aligned} M_0 & \leq \eta^{k+1} M_{k+1} + B_1 \sum_{i=0}^{k} (\eta A_1)^i + B_2 \sum_{i=0}^{k} (\eta A_2)^i + B_0 \sum_{i=0}^{k} \eta^i \\ & \leq \eta^{k+1} \left(\eta M_{k+2} + A_1^{k+1} B_1 + A_2^{k+1} B_2 + B_0 \right) + B_1 \sum_{i=0}^{k} (\eta A_1)^i + B_2 \sum_{i=0}^{k} (\eta A_2)^i + B_0 \sum_{i=0}^{k} \eta^i \\ & = \eta^{k+2} M_{k+2} + B_1 \sum_{i=0}^{k+1} (\eta A_1)^i + B_2 \sum_{i=0}^{k+1} (\eta A_2)^i + B_0 \sum_{i=0}^{k+1} \eta^i, \end{aligned}$$

which clearly yields that (4.35) holds for $n = k + 1$. In conjunction with an induction argument, this guarantees the claimed inequality (4.35). Inserting our definitions of A_1, A_2, B_1, B_2 and B_0 to (4.35) shows that

$$M_0 \leq \eta^{n+1} M_{n+1} + C\eta^{-\frac{\lambda_m}{m-2-\lambda_m}} \left(\fint_{t_0 - R^{sp}}^{t_0} \fint_{B_R} u_+^m dxdt \right)^{\frac{sp}{(sp+N)(m-2-\lambda_m)}} \sum_{i=0}^{n} \left(2^{\frac{dN}{(sp+N)(m-2-\lambda_m)}} \eta \right)^i$$

$$+ C\eta^{-\frac{\lambda_m}{m-\lambda_m}} \left(\fint_{t_0-R^{sp}}^{t_0} \fint_{B_R} u_+^m dxdt \right)^{\frac{sp}{(sp+N)(m-\lambda_m)}} \sum_{i=0}^{n} (2^{\frac{dN}{(sp+N)(m-\lambda_m)}} \eta)^i$$

$$+ \frac{\mathrm{Tail}_\infty (u_+; x_0, R/2, t_0 - R^{sp}, t_0)}{2} \sum_{i=0}^{n} \eta^i, \quad n = 0, 1, 2, \ldots.$$

We choose $\eta = 1/2^{\frac{dN}{(sp+N)(m-2-\lambda_m)}+1}$ to deduce that the sum on the right-hand side can be majorised by a convergent series, and then take $n \to \infty$ to obtain

$$\operatorname*{ess\,sup}_{Q_{R/2}^-} u \le \mathrm{Tail}_\infty (u_+; x_0, R/2, t_0 - R^{sp}, t_0) + C\Big(\fint_{Q_R^-} u_+^m dxdt \Big)^{\frac{sp}{(sp+N)(m-\lambda_m)}} \vee \Big(\fint_{Q_R^-} u_+^m dxdt \Big)^{\frac{sp}{(sp+N)(m-2-\lambda_m)}}. \tag{4.36}$$

The proof is complete. □

5 Local Hölder continuity

This section is devoted to exhibiting the Hölder continuity of weak solutions to (1.1) based on the boundedness of weak solutions when $p > 2$. The proof of the crucial lemma, Lemma 5.1 is performed by using the argument provided in [36, Lemma 2.107] and [15, Lemma 5.1]. Different from the elliptic case, the appearance of the time-variable requires us to borrow the ideas from [17, Chapter III] and work with cylinders whose dimensions are suitably rescaled to reflect the degeneracy exhibited by the equation.

We first find a small constant $\sigma_* > 0$ only depending on p and s such that $\sigma^{sp/(p-1)} \le 1/4$ for all $\sigma \in (0, \sigma_*]$. Assume that $(\bar{x}_0, \bar{t}_0) \in Q_T$ and $r \in (0, R)$ for some $R \in (0, 1)$ satisfy $\overline{B}_R(\bar{x}_0) \subseteq \Omega$ and $[\bar{t}_0 - R^{sp}, \bar{t}_0 + R^{sp}] \subseteq (0, T)$. Now let us take a decreasing radii

$$r_j := \frac{\sigma^j r}{2}, \quad j = 0, 1, 2 \ldots \tag{5.1}$$

with $\sigma < \min\{\sigma_*, 1/4\}$ to be determined later, and denote

$$M := C \left[\mathrm{Tail}_\infty (u; \bar{x}_0, r/2, \bar{t}_0 - r^{sp}, \bar{t}_0 + r^{sp}) + \Big(\fint_{Q_r} |u|^p dxdt \Big)^{\frac{1}{2}} \vee 1 \right] \tag{5.2}$$

with some $C > 0$ only depending on s, p, Λ, N and h. Under the condition that $f(x, t, u) = h(x, t)$ in $Q_T \times \mathbb{R}$ with $h \in L^\infty_{\mathrm{loc}}(Q_T)$, Theorem 1 enables us to find a sufficiently large constant $C \ge 1$ in (5.2) ensuring the $L^\infty(Q_{r/2})$-norm of u can be controlled by $M/2$. Let $\alpha < sp/(p-1)$ be a positive constant to be chosen later. Then we define

$$\omega (r_0) = \omega (r/2) := M, \quad \omega (r_j) := \left(\frac{r_j}{r_0} \right)^\alpha \omega (r_0), \quad j = 1, 2, 3 \ldots$$

and

$$d_j := \begin{cases} [\varepsilon \sigma^{(j-1)\alpha} M]^{2-p} & \text{if } j \ge 1, \\ 1 & \text{if } j = 0, \end{cases}$$

where

$$\varepsilon = \sigma^{\frac{sp}{p-1} - \alpha}.$$

With taking

$$B_j := B_{r_j}(\bar{x}_0) \text{ and } t_j := d_j r_j^{sp},$$

we shall use an iteration argument to study the oscillation of weak solutions over the domains

$$Q_j := Q_{r_j,t_j}(\bar{x}_0,\bar{t}_0) = B_j \times (\bar{t}_0 - t_j, \bar{t}_0 + t_j). \tag{5.3}$$

It follows by simple calculations that

$$\frac{1}{d_{j+1}} = [\varepsilon\omega(r_j)]^{p-2} \quad \text{for all } j \geq 0. \tag{5.4}$$

Besides, the restriction $\sigma \leq \sigma_*$ ensures that

$$4\big(\sigma^{\frac{sp}{p-1}-\alpha}\big)^{2-p} r_1^{sp} \leq r_0^{sp} \text{ and } 4\sigma^{\alpha(2-p)} r_{j+1}^{sp} \leq r_j^{sp} \quad \text{for all } j \geq 1,$$

which along with the definitions of d_j and t_j imply that

$$4t_{j+1} \leq t_j \quad \text{for all } j \geq 0. \tag{5.5}$$

Lemma 5.1. *Let $p > 2$ and u be a local solution to (1.1). Assume that $f(x,t,u) = h(x,t)$ in $Q_T \times \mathbb{R}$ with $h \in L^\infty_{\text{loc}}(Q_T)$. Let $(\bar{x}_0,\bar{t}_0) \in Q_T$, $0 < r \leq R$ with some $R \in (0,1)$ and $Q_R \equiv B(\bar{x}_0) \times (\bar{t}_0 - R^{sp}, \bar{t}_0 + R^{sp})$ with the property $\overline{Q}_R \subseteq Q_T$. Suppose that Q_j and $\omega(r_j)$ are introduced as above. Then we have*

$$\operatorname*{ess\,osc}_{Q_j} u \leq \omega(r_j) \quad \textit{for all } j = 0,1,2,\ldots. \tag{5.6}$$

Proof. The claim is proved by using an induction argument. Based on Theorem 1, the choice of $\omega(r_0)$ ensures that the assertion (5.6) holds for $j=0$. Now, we suppose that (5.6) is true for all $i \in \{0,\ldots j\}$ with some $j \geq 0$, and then aim at proving it for $i = j+1$. Apparently, one of the following two assertions

$$\frac{\big|2Q_{j+1} \cap \big\{u \geq \operatorname{ess\,inf}_{Q_j} u + \omega(r_j)/2\big\}\big|}{|2Q_{j+1}|} \geq \frac{1}{2} \tag{5.7}$$

or

$$\frac{\big|2Q_{j+1} \cap \big\{u \leq \operatorname{ess\,inf}_{Q_j} u + \omega(r_j)/2\big\}\big|}{|2Q_{j+1}|} \geq \frac{1}{2} \tag{5.8}$$

must hold. We set $u_j := u - \operatorname*{ess\,inf}_{Q_j} u$ for the case (5.7), or take $u_j := \omega(r_j) - (u - \operatorname*{ess\,inf}_{Q_j} u)$ for the case (5.8). In all cases, we can deduce from (5.7), (5.8) and the definitions of u_j that

$$\frac{|2Q_{j+1} \cap \{u_j \geq \omega(r_j)/2\}|}{|2Q_{j+1}|} \geq \frac{1}{2} \tag{5.9}$$

and

$$0 \leq \operatorname*{ess\,inf}_{Q_i} u_j \leq \operatorname*{ess\,sup}_{Q_i} u_j \leq 2\omega(r_i) \quad \text{for all } \ i = 0,\ldots,j. \tag{5.10}$$

Besides, u_j is a local weak solution to the equation (1.1) apparently.

We first prove that

$$\big[\operatorname{Tail}_\infty(u_j;\bar{x}_0,r_j,\bar{t}_0 - t_j,\bar{t}_0 + t_j)\big]^{p-1} \leq C\sigma^{-\alpha(p-1)}[\omega(r_j)]^{p-1} \tag{5.11}$$

under the induction assumption. It is obvious the claim trivially holds for $j=0$. For $j \geq 1$, we have

$$
\begin{aligned}
&\left[\operatorname{Tail}_{\infty}\left(u_j ; \bar{x}_0, r_j, \bar{t}_0-t_j, \bar{t}_0+t_j\right)\right]^{p-1} \\
= & r_j^{sp} \underset{t \in\left(\bar{t}_0-t_j, \bar{t}_0+t_j\right)}{\operatorname{ess\,sup}} \sum_{i=1}^j \int_{B_{i-1} \backslash B_i} \frac{\left|u_j(x, t)\right|^{p-1}}{\left|x-\bar{x}_0\right|^{N+sp}} d x \\
& +r_j^{sp} \underset{t \in\left(\bar{t}_0-t_j, \bar{t}_0+t_j\right)}{\operatorname{ess\,sup}} \int_{\mathbb{R}^N \backslash B_0} \frac{\left|u_j(x, t)\right|^{p-1}}{\left|x-\bar{x}_0\right|^{N+sp}} d x \\
\leq & r_j^{sp} \sum_{i=1}^j\left(\underset{Q_{i-1}}{\operatorname{ess\,sup}}\, u_j\right)^{p-1} \int_{\mathbb{R}^N \backslash B_i} \frac{1}{\left|x-\bar{x}_0\right|^{N+sp}} d x \\
& +r_j^{sp} \underset{t \in\left(\bar{t}_0-t_j, \bar{t}_0+t_j\right)}{\operatorname{ess\,sup}} \int_{\mathbb{R}^N \backslash B_0} \frac{\left|u_j(x, t)\right|^{p-1}}{\left|x-\bar{x}_0\right|^{N+sp}} d x .
\end{aligned} \tag{5.12}
$$

It can be obtained by (5.10) that

$$
\left(\underset{Q_{i-1}}{\operatorname{ess\,sup}}\, u_j\right)^{p-1} \int_{\mathbb{R}^N \backslash B_i} \frac{1}{\left|x-\bar{x}_0\right|^{N+sp}} d x \leq C r_i^{-sp}\left[\omega\left(r_{i-1}\right)\right]^{p-1} . \tag{5.13}
$$

In light of (5.2) and (5.10), the definition of u_j infers that

$$
\begin{aligned}
\underset{t \in\left(\bar{t}_0-t_j, \bar{t}_0+t_j\right)}{\operatorname{ess\,sup}} \int_{\mathbb{R}^N \backslash B_0} \frac{\left|u_j(x, t)\right|^{p-1}}{\left|x-\bar{x}_0\right|^{N+sp}} d x \leq & \underset{t \in\left(\bar{t}_0-t_j, \bar{t}_0+t_j\right)}{\operatorname{ess\,sup}} \int_{\mathbb{R}^N \backslash B_0} \frac{|u(x, t)|^{p-1}}{\left|x-\bar{x}_0\right|^{N+sp}} d x \\
& +r_0^{-sp} \underset{Q_0}{\operatorname{ess\,sup}}\,|u|^{p-1}+r_0^{-sp}\left[\omega\left(r_0\right)\right]^{p-1} \\
\leq & C r_1^{-sp}\left[\omega\left(r_0\right)\right]^{p-1} .
\end{aligned} \tag{5.14}
$$

We derive from (5.12)-(5.14) that

$$
\left[\operatorname{Tail}_{\infty}\left(u_j ; \bar{x}_0, r_j, \bar{t}_0-t_j, \bar{t}_0+t_j\right)\right]^{p-1} \leq C \sum_{i=1}^j\left(\frac{r_j}{r_i}\right)^{sp}\left[\omega\left(r_{i-1}\right)\right]^{p-1}, \tag{5.15}
$$

where the right-hand side can be estimated as below

$$
\begin{aligned}
\sum_{i=1}^j\left(\frac{r_j}{r_i}\right)^{sp}\left[\omega\left(r_{i-1}\right)\right]^{p-1} & =\left[\omega\left(r_0\right)\right]^{p-1}\left(\frac{r_j}{r_0}\right)^{\alpha(p-1)} \sum_{i=1}^j\left(\frac{r_{i-1}}{r_i}\right)^{\alpha(p-1)}\left(\frac{r_j}{r_i}\right)^{sp-\alpha(p-1)} \\
& =\left[\omega\left(r_j\right)\right]^{p-1} \sigma^{-\alpha(p-1)} \sum_{i=0}^{j-1} \sigma^{i(sp-\alpha(p-1))} \\
& \leq\left[\omega\left(r_j\right)\right]^{p-1} \frac{\sigma^{-\alpha(p-1)}}{1-\sigma^{sp-\alpha(p-1)}} \\
& \leq \frac{4^{sp-\alpha(p-1)}}{(sp-\alpha(p-1)) \log 4} \sigma^{-\alpha(p-1)}\left[\omega\left(r_j\right)\right]^{p-1}
\end{aligned} \tag{5.16}
$$

because of $\sigma \leq 1/4$ and $\alpha<sp/(p-1)$. Hence, (5.11) is proved with C depending only on s, p, N and the difference of $sp/(p-1)$ and α.

Next, let v be given as follows

$$
v:=\min \left\{\left[\log \left(\frac{\omega\left(r_j\right)/2+d}{u_j+d}\right)\right]_{+}, k\right\} \quad \text{with some } k>0. \tag{5.17}
$$

By taking $a=\omega\left(r_j\right)/2$ and $b=\exp(k)$ in Corollary 1, we can see

$$
\begin{aligned}
&\int_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}} \fint_{2B_{j+1}} \left|v(x,t)-(v)_{2B_{j+1}}(t)\right|^p dxdt \\
&\leq C t_{j+1} d^{1-p}\left(\frac{r_{j+1}}{r_j}\right)^{sp}\left[\operatorname{Tail}_{\infty}\left(u_j;\bar{x}_0,r_j,\bar{t}_0-4t_{j+1},\bar{t}_0+4t_{j+1}\right)\right]^{p-1} \\
&\quad + C t_{j+1} + C d^{2-p} r_{j+1}^{sp} + C t_{j+1} d^{1-p} r_{j+1}^{sp}.
\end{aligned}
\tag{5.18}
$$

Since $4t_{j+1}\leq t_j$, we can insert (5.11) into (5.18) to get

$$
\begin{aligned}
&\int_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}} \fint_{2B_{j+1}} \left|v(x,t)-(v)_{2B_{j+1}}(t)\right|^p dxdt \\
&\leq C t_{j+1} d^{1-p}\left[\varepsilon\omega\left(r_j\right)\right]^{p-1} + C t_{j+1} + C d^{2-p} r_{j+1}^{sp} + C t_{j+1} d^{1-p} r_{j+1}^{sp}.
\end{aligned}
\tag{5.19}
$$

By choosing $d=\varepsilon\omega\left(r_j\right)$, utilizing (5.4) and recalling $\alpha<sp/(p-1)$, it can be verified that

$$d^{2-p}=d_{j+1}$$

and

$$
\begin{aligned}
d^{1-p} &= \left[\omega(r_0)\right]^{1-p}\sigma^{-sp+\alpha(p-1)}\sigma^{j(1-p)\alpha} \\
&\leq \left[\omega(r_0)\right]^{1-p}\sigma^{-(j+1)sp} \\
&\leq r_{j+1}^{-sp},
\end{aligned}
\tag{5.20}
$$

where we used $\omega(r_0)\geq 1$ and $r<1$ in the last line. Hence, for the function v given in (5.17) with $d=\varepsilon\omega\left(r_j\right)$, we deduce from (5.19) that

$$
\fint_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}} \fint_{2B_{j+1}} \left|v(x,t)-(v)_{2B_{j+1}}(t)\right| dxdt \leq C,
\tag{5.21}
$$

where the constant C depends on s,p,Λ,N,h and the difference of $sp/(p-1)$ and α.

In view of (5.9), we obtain

$$
\begin{aligned}
k &= \frac{1}{\left|2Q_{j+1}\cap\left\{u_j\geq\omega\left(r_j\right)/2\right\}\right|}\iint_{2Q_{j+1}\cap\left\{u_j\geq\omega\left(r_j\right)/2\right\}} k\,dxdt \\
&= \frac{1}{\left|2Q_{j+1}\cap\left\{u_j\geq\omega\left(r_j\right)/2\right\}\right|}\iint_{2Q_{j+1}\cap\{v=0\}} k\,dxdt \\
&\leq \frac{2}{\left|2Q_{j+1}\right|}\iint_{2Q_{j+1}}(k-v)\,dxdt = 2\left[k-(v)_{2Q_{j+1}}\right].
\end{aligned}
\tag{5.22}
$$

It follows by integrating the above inequality over the set $2Q_{j+1}\cap\{v=k\}$ that

$$
\begin{aligned}
\frac{\left|2Q_{j+1}\cap\{v=k\}\right|}{\left|2Q_{j+1}\right|}k &\leq \frac{2}{\left|2Q_{j+1}\right|}\iint_{2Q_{j+1}\cap\{v=k\}}\left[k-(v)_{2Q_{j+1}}\right]dxdt \\
&= \frac{2}{\left|2Q_{j+1}\right|}\iint_{2Q_{j+1}\cap\{v=k\}}\left[k-\frac{1}{4t_{j+1}}\int_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}}(v)_{2B_{j+1}}(\tau)d\tau\right]dxdt \\
&= \frac{1}{2t_{j+1}\left|2Q_{j+1}\right|}\iint_{2Q_{j+1}\cap\{v=k\}}\int_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}}\left[k-(v)_{2B_{j+1}}(\tau)\right]d\tau dxdt
\end{aligned}
$$

$$\leq \frac{1}{2t_{j+1}|2Q_{j+1}|}\int_{\bar{t}_0-2t_{j+1}}^{\bar{t}_0+2t_{j+1}}\iint_{2Q_{j+1}}\left|v-(v)_{2B_{j+1}}(\tau)\right|d\tau dxdt \leq C, \tag{5.23}$$

where we used the estimate (5.21) in the last line. Let us take

$$k = \log\left(\frac{\omega(r_j)/2+\varepsilon\omega(r_j)}{3\varepsilon\omega(r_j)}\right), \tag{5.24}$$

which directly results in the observation that

$$k = \log\left(\frac{1/2+\varepsilon}{3\varepsilon}\right) \approx \log\left(\frac{1}{\varepsilon}\right), \tag{5.25}$$

where we take ε small enough to ensure the positivity of k. By virtue of (5.17) and (5.24), we can verify that $2Q_{j+1} \cap \{v = k\} = 2Q_{j+1} \cap \{u_j \leq 2\varepsilon\omega(r_j)\}$. This combined with (5.23) leads to the estimate

$$\frac{|2Q_{j+1}\cap\{u_j \leq 2\varepsilon\omega(r_j)\}|}{|2Q_{j+1}|} = \frac{|2Q_{j+1}\cap\{v=k\}|}{|2Q_{j+1}|} \leq \frac{C}{k}. \tag{5.26}$$

Recalling $\varepsilon = \sigma^{\frac{sp}{p-1}-\alpha}$ and utilizing (5.26) with (5.25), we have

$$\frac{|2Q_{j+1}\cap\{u_j \leq 2\varepsilon\omega(r_j)\}|}{|2Q_{j+1}|} \leq \frac{C^*}{\log\left(\frac{1}{\sigma}\right)}, \tag{5.27}$$

where $C^* > 0$ depends on s, p, Λ, N and the difference of $sp/(p-1)$ and α.

Based on the preparations (5.11) and (5.27), we can start a suitable iteration to deduce the desired oscillation reduction over the domain Q_{j+1}. With $j \in \mathbb{N}$ fixed, for each $i = 0, 1, 2, \ldots$, we set

$$\begin{aligned}
&\varrho_i = r_{j+1} + 2^{-i}r_{j+1}, \quad \tilde{\varrho}_i := \frac{\varrho_i+\varrho_{i+1}}{2},\\
&\theta_i := t_{j+1} + 2^{-i}t_{j+1}, \quad \tilde{\theta}_i := \frac{\theta_i+\theta_{i+1}}{2},\\
&Q^i := B^i \times \Gamma_i := B_{\varrho_i}(\bar{x}_0) \times (\bar{t}_0-\theta_i, \bar{t}_0+\theta_i),\\
&\tilde{Q}^i := \tilde{B}^i \times \tilde{\Gamma}_i := B_{\tilde{\varrho}_i}(\bar{x}_0) \times (\bar{t}_0-\tilde{\theta}_i, \bar{t}_0+\tilde{\theta}_i).
\end{aligned} \tag{5.28}$$

Then the corresponding cut-off functions $\psi_i \in C_0^\infty(\tilde{B}^i)$ and $\eta_i \in C_0^\infty(\tilde{\Gamma}_i)$ are taken to satisfy

$$0 \leq \psi_i \leq 1, \ \ |\nabla\psi_i| \leq c2^i r_{j+1}^{-1} \text{ in } \tilde{B}^i, \ \ \psi_i \equiv 1 \text{ in } B^{i+1}$$

and

$$0 \leq \eta_i \leq 1, \ \ |\partial_t\eta_i| \leq c2^i t_{j+1}^{-1} \text{ in } \tilde{\Gamma}_i, \ \ \eta_i \equiv 1 \text{ in } \Gamma_{i+1}.$$

Let us define

$$k_i := \left(1+2^{-i}\right)\varepsilon\omega(r_j), \quad v_i := (k_i-u_j)_+ \tag{5.29}$$

and

$$A_i := \frac{|Q_i \cap \{u_j \leq k_i\}|}{|Q_i|}.$$

Thus it can be seen that $Q^0 = 2Q_{j+1}$ and

$$A_0 := \frac{|2Q_{j+1}\cap\{u_j \leq 2\varepsilon\omega(r_j)\}|}{|2Q_{j+1}|} \leq \frac{C^*}{\log\left(\frac{1}{\sigma}\right)} \tag{5.30}$$

due to (5.27). By taking $\ell=\theta_i-\theta_{i+1}$, $\tau_1=\bar{t}_0-\theta_{i+1}$ and $\tau_2=\bar{t}_0+\theta_{i+1}$ in Lemma 3.3 (see Remark 2), we have

$$\begin{aligned}
&\int_{\Gamma_{i+1}}\int_{B^i}\fint_{B^i}\frac{|v_i(x,t)\psi_i(x)-v_i(y,t)\psi_i(y)|^p}{|x-y|^{N+sp}}dxdydt+\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}\fint_{B^i}v_i^2(x,t)\psi_i^p(x)dx\\
&\le C\int_{\Gamma_i}\int_{B^i}\fint_{B^i}\max\{v_i(x,t),v_i(y,t)\}^p|\psi_i(x)-\psi_i(y)|^p\eta_i^2(t)d\mu dt\\
&\quad+C\operatorname*{ess\,sup}_{\substack{t\in\Gamma_i\\ x\in\operatorname{supp}\psi_i}}\int_{\mathbb{R}^N\setminus B^i}\frac{v_i^{p-1}(y,t)}{|x-y|^{N+sp}}dy\int_{\Gamma_i}\fint_{B^i}v_i(x,t)\psi_i^p(x)\eta_i^2(t)dxdt\\
&\quad+C\int_{\Gamma_i}\fint_{B^i}h(x,t)v_i(x,t)\psi_i^p(x)\eta_i^2(t)dxdt\\
&\quad+C\int_{\Gamma_i}\fint_{B^i}v_i^2(x,t)\psi_i^p(x)\eta_i(t)|\partial_t\eta_i(t)|dxdt. \qquad (5.31)
\end{aligned}$$

We estimate the terms on the right-hand side, respectively. It follows from the definitions of v_i and k_i that

$$\begin{aligned}
&\int_{\Gamma_i}\int_{B^i}\fint_{B^i}\max\{v_i(x,t),v_i(y,t)\}^p|\psi_i(x)-\psi_i(y)|^p\eta^2(t)d\mu dt\\
&\le C2^{pi}r_{j+1}^{-p}k_i^p\sup_{x\in B^i}\int_{B^i}\frac{1}{|x-y|^{N+sp-p}}dy\int_{\Gamma_i}\fint_{B^i}\chi_{\{u_j\le k_i\}}(x,t)dxdt\\
&\le C2^{pi}r_{j+1}^{-sp}\left[\varepsilon\omega\left(r_j\right)\right]^p\int_{\Gamma_i}\fint_{B^i}\chi_{\{u_j\le k_i\}}(x,t)dxdt \qquad (5.32)
\end{aligned}$$

and

$$\int_{\Gamma_i}\fint_{B^i}v_i(x,t)\psi_i^p(x)dxdt\le C\left[\varepsilon\omega\left(r_j\right)\right]\int_{\Gamma_i}\fint_{B^i}\chi_{\{u_j\le k_i\}}(x,t)dxdt. \qquad (5.33)$$

Besides, for $y\in\mathbb{R}^N/B^i$ and $x\in\tilde{B}^i$, we have

$$|y-\bar{x}_0|\le|x-y|\Big(1+\frac{|x-\bar{x}_0|}{|y-x|}\Big)\le|x-y|\Big(1+\frac{\varrho_i}{\varrho_i-\tilde{\varrho}_i}\Big)\le C2^i|x-y|,$$

which directly tells that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\substack{t\in\Gamma_i\\ x\in\operatorname{supp}\psi_i}}\int_{\mathbb{R}^N\setminus B^i}\frac{v_i^{p-1}(y,t)}{|x-y|^{N+sp}}dy\\
&\le C2^{(N+sp)i}\operatorname*{ess\,sup}_{t\in\Gamma_i}\int_{\mathbb{R}^N\setminus B^i}\frac{v_i^{p-1}(y,t)}{|y-\bar{x}_0|^{N+sp}}dy\\
&\le C2^{(N+sp)i}\left[\operatorname*{ess\,sup}_{t\in\Gamma_i}\int_{B_j\setminus B_{j+1}}\frac{v_i^{p-1}(y,t)}{|y-\bar{x}_0|^{N+sp}}dy+\operatorname*{ess\,sup}_{t\in\Gamma_i}\int_{\mathbb{R}^N\setminus B_j}\frac{v_i^{p-1}(y,t)}{|y-\bar{x}_0|^{N+sp}}dy\right]. \qquad (5.34)
\end{aligned}$$

From the estimate $v_i\le2\varepsilon\omega\left(r_j\right)$ in $B_j\times(\bar{t}_0-2t_{j+1},\bar{t}_0+2t_{j+1})$, it follows that

$$\operatorname*{ess\,sup}_{t\in\Gamma_i}\int_{B_j\setminus B_{j+1}}\frac{v_i^{p-1}(y,t)}{|y-\bar{x}_0|^{N+sp}}dy\le Cr_{j+1}^{-sp}\left[\varepsilon\omega\left(r_j\right)\right]^{p-1}. \qquad (5.35)$$

Since $v_i \le |u_j| + 2\varepsilon\omega(r_j)$ in $\mathbb{R}^N \times (\bar{t}_0 - 2t_{j+1}, \bar{t}_0 + 2t_{j+1})$, there holds that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in\Gamma_i} \int_{\mathbb{R}^N\setminus B_j} \frac{v_i^{p-1}(y,t)}{|y-\bar{x}_0|^{N+sp}} dy \\
&\le C r_{j+1}^{-sp} \varepsilon^{p-1} [\omega(r_j)]^{p-1} + \operatorname*{ess\,sup}_{t\in\Gamma_i} \int_{\mathbb{R}^N\setminus B_j} \frac{|u_j(y,t)|^{p-1}}{|y-\bar{x}_0|^{N+sp}} dy \\
&\le C r_{j+1}^{-sp} \varepsilon^{p-1} [\omega(r_j)]^{p-1} + C r_j^{-sp} \left[\operatorname{Tail}_\infty(u_j; \bar{x}_0, r_j, \bar{t}_0 - 2t_{j+1}, \bar{t}_0 + 2t_{j+1}) \right]^{p-1} \\
&\le C r_{j+1}^{-sp} \left(1 + \frac{\sigma^{sp-\alpha(p-1)}}{\varepsilon^{p-1}} \right) [\varepsilon\omega(r_j)]^{p-1} \\
&\le C r_{j+1}^{-sp} [\varepsilon\omega(r_j)]^{p-1}, \qquad (5.36)
\end{aligned}$$

where we used that estimate given in (5.11), specifically,

$$\begin{aligned}
\left[\operatorname{Tail}_\infty(u_j; \bar{x}_0, r_j, \bar{t}_0 - 2t_{j+1}, \bar{t}_0 + 2t_{j+1}) \right]^{p-1} &\le \left[\operatorname{Tail}_\infty(u_j; \bar{x}_0, r_j, \bar{t}_0 - t_j, \bar{t}_0 + t_j) \right]^{p-1} \\
&\le C\sigma^{-\alpha(p-1)} [\omega(r_j)]^{p-1}.
\end{aligned}$$

A combination of (5.33)-(5.36) gives that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\substack{t\in\Gamma_i \\ x\in \operatorname{supp}\psi_i}} \int_{\mathbb{R}^N\setminus B^i} \frac{v_i^{p-1}(y,t)}{|x-y|^{N+sp}} dy \int_{\Gamma_i} \fint_{B^i} v_i(x,t)\psi_i^p(x)\eta_i^2(t) dx dt \\
&\le C 2^{(sp+N)i} r_{j+1}^{-sp} [\varepsilon\omega(r_j)]^p \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t) dx dt. \qquad (5.37)
\end{aligned}$$

By simple calculations, we can tell that

$$\omega(r_j) = \sigma^{j\alpha}\omega(r_0) \ge \frac{1}{\varepsilon} \left(\frac{r_{j+1}}{r_0} \right)^{\frac{sp}{p-1}} \omega(r_0) \quad \text{with any } j \ge 0.$$

This combined with $\omega(r_0) \ge 1$ and $r < 1$ ensures that

$$r_{j+1}^{-sp} [\varepsilon\omega(r_j)]^{p-1} \ge 1.$$

Thus, there holds

$$\begin{aligned}
\int_{\Gamma_i} \fint_{B^i} h(x,t) v_i(x,t)\psi_i^p(x)\eta^2(t) dx dt &\le C\varepsilon\omega(r_j) \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t) dx dt \\
&\le C[\varepsilon\omega(r_j)]^p r_{j+1}^{-sp} \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t) dx dt. \qquad (5.38)
\end{aligned}$$

As a consequence of (5.4), we obtain

$$\begin{aligned}
&\int_{\Gamma_i} \fint_{B^i} v_i^2(x,t)\psi_i^p(x)\eta_i(t)|\partial_t \eta_i(t)| dx dt \\
&\le C 2^{spi} k_i^2 t_{j+1}^{-1} \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t) dx dt \\
&\le C 2^{spi} [\varepsilon\omega(r_j)]^2 d_{j+1}^{-1} r_{j+1}^{-sp} \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t) dx dt \\
&\le C 2^{spi} [\varepsilon\omega(r_j)]^p r_{j+1}^{-sp} \int_{\Gamma_i} \fint_{B^i} \chi_{\{u_j \le k_i\}}(x,t) dx dt. \qquad (5.39)
\end{aligned}$$

Utilizing the fact $\psi_i \equiv 1$ in B^{i+1}, we can see

$$\begin{aligned}&\int_{\Gamma_{i+1}}\int_{B^{i+1}}⨍_{B^{i+1}}\frac{|v_i(x,t)-v_i(y,t)|^p}{|x-y|^{N+sp}}dxdydt+\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}⨍_{B^{i+1}}v_i^2(x,t)dx\\&\leq C2^{(sp+N)i}[\varepsilon\omega\left(r_j\right)]^p r_{j+1}^{-sp}\int_{\Gamma_i}⨍_{B^i}\chi_{\{u_j\leq k_i\}}(x,t)dxdt.\end{aligned}\tag{5.40}$$

Still by (5.4), it can be deduced that

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}⨍_{B^{i+1}}v_i^p(x,t)dx&\leq k_i^{p-2}\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}⨍_{B^{i+1}}v_i^2(x,t)dx\\&\leq Cd_{j+1}^{-1}\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}⨍_{B^{i+1}}v_i^2(x,t)dx\end{aligned}\tag{5.41}$$

and

$$\begin{aligned}\int_{\Gamma_i}⨍_{B^{i+1}}v_i^p(x,t)dxdt&\leq Ck_i^p\int_{\Gamma_i}⨍_{B^i}\chi_{\{u_j\leq k_i\}}(x,t)dxdt\\&\leq C\left[\varepsilon\omega\left(r_j\right)\right]^p\int_{\Gamma_i}⨍_{B^i}\chi_{\{u_j\leq k_i\}}(x,t)dxdt.\end{aligned}\tag{5.42}$$

We conclude from (5.40)-(5.42) that

$$\begin{aligned}&r_{j+1}^{sp}⨍_{\Gamma_{i+1}}\int_{B^{i+1}}⨍_{B^{i+1}}\frac{|v_i(x,t)-v_i(y,t)|^p}{|x-y|^{N+sp}}dxdydt\\&\quad+⨍_{\Gamma_{i+1}}⨍_{B^{i+1}}v_i^p(x,t)dxdt+\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}⨍_{B^{i+1}}v_i^p(x,t)dx\\&\leq d_{j+1}^{-1}\int_{\Gamma_{i+1}}\int_{B^{i+1}}⨍_{B^{i+1}}\frac{|v_i(x,t)-v_i(y,t)|^p}{|x-y|^{N+sp}}dxdydt\\&\quad+d_{j+1}^{-1}r_{j+1}^{-sp}\int_{\Gamma_{i+1}}⨍_{B^{i+1}}v_i^p(x,t)dxdt+d_{j+1}^{-1}\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}⨍_{B^{i+1}}v_i^2(x,t)dx\\&\leq C2^{(sp+N)i}[\varepsilon\omega\left(r_j\right)]^p r_{j+1}^{-sp}d_{j+1}^{-1}\int_{\Gamma_i}⨍_{B^i}\chi_{\{u_j\leq k_i\}}(x,t)dxdt\\&\leq C2^{(sp+N)i}[\varepsilon\omega\left(r_j\right)]^p A_i.\end{aligned}\tag{5.43}$$

According to Lemma 2.4, there holds that

$$\begin{aligned}⨍_{\Gamma_{i+1}}⨍_{B^{i+1}}v_i^{p(1+\frac{sp}{N})}(x,t)dxdt&\leq C\Bigg(r_{j+1}^{sp}⨍_{\Gamma_{i+1}}\int_{B^{i+1}}⨍_{B^{i+1}}\frac{|v_i(x,t)-v_i(y,t)|^p}{|x-y|^{N+sp}}dxdydt\\&\quad+⨍_{\Gamma_{i+1}}⨍_{B^{i+1}}v_i^p(x,t)dxdt\Bigg)\times\left(\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}⨍_{B^{i+1}}v_i^p(x,t)dx\right)^{\frac{sp}{N}}.\end{aligned}\tag{5.44}$$

By applying (5.43) and (5.44), we have

$$\begin{aligned}A_{i+1}(k_i-k_{i+1})^{p(1+\frac{sp}{N})}&\leq⨍_{\Gamma_{i+1}}⨍_{B^{i+1}\cap\{u_j\leq k_{i+1}\}}v_i^{p(1+\frac{sp}{N})}(x,t)dxdt\\&\leq C\Bigg(r_{j+1}^{sp}⨍_{\Gamma_{i+1}}\int_{B^{i+1}}⨍_{B^{i+1}}\frac{|v_i(x,t)-v_i(y,t)|^p}{|x-y|^{N+sp}}dxdydt\end{aligned}$$

$$+\fint_{\Gamma_{i+1}}\fint_{B^{i+1}} v_i^p(x,t)dxdt\Big)\times\left(\operatorname*{ess\,sup}_{t\in\Gamma_{i+1}}\fint_{B^{i+1}} v_i^p(x,t)dx\right)^{\frac{sp}{N}}$$
$$\leq C\Big(2^{(sp+N)i}[\varepsilon\omega(r_j)]^p A_i\Big)^{1+\frac{sp}{N}},$$

which leads to the recursive inequality that

$$A_{i+1}\leq \tilde{C}2^{(p+sp+N)(1+\frac{sp}{N})i}A_i^{1+\frac{sp}{N}},$$

where $\tilde{C}$ depends on s,p,Λ,N,h and the difference of $sp/(p-1)$ and α. Let

$$\nu^*:=\tilde{C}^{-N/(sp)}2^{-N(sp+N)(p+sp+N)/(s^2p^2)}.$$

Then we choose

$$\sigma=\min\left\{1/4,\sigma_*,\exp\left(-C^*/\nu^*\right)\right\},$$

and derive from (5.30) that

$$A_0=\frac{\left|2Q_{j+1}\cap\{u_j\leq 2\varepsilon\omega\left(r_j\right)\}\right|}{|2Q_{j+1}|}\leq\nu^*. \tag{5.45}$$

This combined with Lemma 4.1 guarantees that $A_i\to 0$ as $i\to\infty$, which directly tells that

$$u_j(x,t)\geq\varepsilon\omega(r_j)\quad\text{in}\quad Q_{j+1}. \tag{5.46}$$

Recalling the definition of u_j, it can be deduced by (5.46) that

$$\operatorname*{ess\,osc}_{Q_{j+1}} u\leq(1-\varepsilon)\omega\left(r_j\right)=(1-\varepsilon)\sigma^{-\alpha}\omega\left(r_{j+1}\right). \tag{5.47}$$

Thus, we can choose $\sigma<\min\{\sigma_*,1/4\}$ and $\alpha<sp/(p-1)$ small enough such that (5.45) holds and

$$\sigma^\alpha\geq 1-\varepsilon=1-\sigma^{\frac{sp}{p-1}-\alpha},$$

which along with (5.47) ensures

$$\operatorname*{ess\,osc}_{Q_{j+1}} u\leq\omega\left(r_{j+1}\right). \tag{5.48}$$

Finally, the estimate (5.48) proves the induction step and finishes the proof. □

Proof of Theorem 3. Assume that u is a local weak solution to (1.1) with $p>2$ and f satisfies the assumptions in Theorem 3. Let $(x_0,t_0)\in Q_T$, $R\in(0,1)$ and $Q_R\equiv B_R(x_0)\times(t_0-R^{sp},t_0+R^{sp})$ with the property $\overline{Q}_R\subseteq Q_T$. By invoking Lemma 5.1 with $r=R$, we can find positive constants $\alpha<sp/(p-1),\sigma<1/4$ and $\bar{C}\geq 1$ only depending on s,p,Λ,N,h such that

$$\operatorname*{ess\,osc}_{Q_j} u\leq\bar{C}\Big(\frac{r_j}{R}\Big)^\alpha\omega\Big(\frac{R}{2}\Big)\quad\text{for all}\quad j\in\mathbb{N}, \tag{5.49}$$

where r_j, Q_j are given in (5.1), (5.3) and

$$\omega\Big(\frac{R}{2}\Big)=\mathrm{Tail}_\infty(u;x_0,R/2,t_0-R^{sp},t_0+R^{sp})+\Big(\fint_{Q_R}|u|^p dxdt\Big)^{\frac{1}{2}}\vee 1. \tag{5.50}$$

For any $\rho\in(0,R/2]$, there exists $j_0\in\mathbb{N}$ such that $\rho\in(r_{j_0+1},r_{j_0}]$. By taking $d=[\bar{C}\omega(R/2)]^{2-p}$, we can verify that $Q_{\rho,d\rho^{sp}}\subseteq Q_{j_0}$. Thus, it follows by (5.49) that

$$\operatorname*{ess\,osc}_{Q_{\rho,d\rho^{sp}}} u\leq\operatorname*{ess\,osc}_{Q_{j_0}} u\leq\bar{C}\sigma^{-\alpha}\Big(\frac{r_{j_0+1}}{R}\Big)^\alpha\omega\Big(\frac{R}{2}\Big)\leq\bar{C}\sigma^{-\alpha}\Big(\frac{\rho}{R}\Big)^\alpha\omega\Big(\frac{R}{2}\Big). \tag{5.51}$$

This together with (5.50) clearly leads to the claim. □

We give the proof of Proposition 1.1 as a direct application of Theorems 1 and 3.

Proof of Proposition 1.1. Assume that u is a local weak solution to (1.1) with $p > 2$ and f satisfies the assumptions in Proposition 1.1. According to Theorem 1, we clearly have $u \in L^\infty_{\mathrm{loc}}(Q_T)$. Now, we rewrite $\tilde{f}(x,t) = f\big(x,t,u(x,t)\big)$ in $\mathbb{R}^N \times (0,T)$, which combined with the structural condition on f and the boundedness of u implies that u can work as a local weak solution to the equation (1.1) with the nonhomogeneous term $\tilde{f} \in L^\infty_{\mathrm{loc}}(Q_T)$. Thus, the assumptions required in Theorem 3 are satisfied. Based on the oscillation estimate established in Theorem 3, we arrive at our claim. □

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