Existence of steady symmetric vortex patch in a disk

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Abstract

In this paper, we construct a family of symmetric vortex patches for the 2D steady incompressible Euler equations in a disk. The result is obtained by studying a variational problem in which the kinetic energy of the fluid is maximized subject to some appropriate constraints for the vorticity. Moreover, we show that these vortex patches "shrink" to a given minimum point of the corresponding Kirchhoff-Routh function as the vorticity strength parameter goes to infinity.

Keywords: Euler equation, ideal fluid, vortex patch, Kirchhoff-Routh function, variational problem

1. Introduction

In this paper, we prove a result of existence of steady symmetric vortex patch for a planar ideal fluid moving in a disk. More specifically, by maximizing the kinetic energy subject to some appropriate constraints for the vorticity, we construct a steady flow in which the vorticity has the form

$$\omega^\lambda = \lambda I_{\{\psi^\lambda>\mu^\lambda\}} - \lambda I_{\{\psi^\lambda<-\mu^\lambda\}} \tag{1.1}$$

for some $\mu^\lambda \in \mathbb{R}$. Here I_A denotes the characteristic function of some measurable set A, i.e., $I_A(x) \equiv 1$ for $x \in A$ and $I_A \equiv 0$ elsewhere, λ is the vorticity strength parameter that is given, and ψ^λ is the stream function satisfying

$$-\Delta\psi^\lambda = \omega^\lambda. \tag{1.2}$$

In addition, ω^λ and ψ^λ are both even in x_1 and odd in x_2.

In history, the construction for dynamically possible steady vortex flows has been extensively studied. Roughly speaking, there are mainly two methods dealing with this problem. The first

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one is called the stream-function method. It consists in finding a solution to the following semilinear elliptic equation satisfied by the stream function:

$$-\Delta\psi = f(\psi), \tag{1.3}$$

where the nonlinearity f is given. To obtain a solution of (1.3), one can use the mountain pass lemma (see [1, 18, 20]), the constrained variational method (see [4, 5, 17, 21, 22]), or a finite dimensional reduction (see [11, 12]). The second one is called the vorticity method. It was put forward by Arnold (see [2, 3]) and further developed by many authors; see [6, 7, 8, 9, 13, 15, 16, 23, 24] for example. The basic idea of the vorticity method is to extremize the kinetic energy of the fluid on a suitable class for the vorticity. For this method, the distributional function for the vorticity is prescribed, and the stream function ψ still satisfies a semilinear elliptic equation (1.3), but the nonlinearity f is usually unknown. In this paper, we use the vorticity method to prove our main result.

In (1.1), the vorticity is a piecewise constant function, which is usually called a vortex patch. Such a special kind of solution has been studied by many authors. Here we recall some of the relevant and significant results. In [23], by using the vorticity method, Turkington constructed a family of vortex patch solutions in a planar bounded domain. Moreover, he showed that these solutions "shrink" to a global minimum of the Kirchhoff-Routh function with $k = 1$ (see Section 2 for the definition). Later in [16], based on a similar argument, Elcrat–Miller constructed steady multiple vortex solutions near a given strict local minimum point of the corresponding Kirchhoff-Routh function. In 2015, Cao–Peng–Yan [12] proved that for any given non-degenerate critical point of the Kirchhoff-Routh function for any k, there exists a family of steady vortex patches "shrinking" to this point. The method used in [12] is based on a finite dimensional reduction argument for the stream function.

Notice that both in [12] and [16], some non-degenerate condition is required for the concentration point. But for a very simple domain, an open disk, every critical point of the Kirchhoff-Routh function with $k \geq 2$ must be degenerate due to rotational invariance. A natural question arises: can we still prove existence of steady multiple vortex patches in a disk? Our main purpose in this paper is to give this question a positive answer. The new idea here is that we improve the vorticity method by adding extra symmetry constraints on the vorticity, such that the vortex patches obtained as the maximizers of the corresponding variational problem "shrink" to two given symmetric points P_1 and P_2, even if (P_1, P_2) is degenerate. Then by analyzing the limiting behavior we can show that these vortex patches have the form (1.1) if the vorticity strength is sufficiently large. At last by a result of Burton in [10] these vortex patches are steady solutions of the 2D Euler equations. Note that for this problem, one may also use the stream function method (as in [12]) to work in a suitable symmetric subspace to overcome the difficulty caused by the degeneracy of the critical points.

This paper is organized as follows. In Section 2, we give the mathematical formulation of the vortex patch problem and then state the main result. In Section 3, we solve a maximization problem for the vorticity and study the asymptotic behavior of the maximizers as the vorticity strength goes to infinity. In Section 4, we prove the main result. Finally in Section 5, we briefly discuss the existence of steady non-symmetric vortex patches.

2. Main Results

To begin with, we introduce some notation that will be used throughout this paper. Let D be the unit disk in the plane centered at the origin, that is,

$$D = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\}. \tag{2.4}$$

For $\mathbf{x} = (x_1, x_2) \in D$, we denote $\bar{\mathbf{x}} = (-x_1, x_2)$ and $\tilde{\mathbf{x}} = (x_1, -x_2)$.

Let G be the Green's function for $-\Delta$ in D with zero Dirichlet boundary condition, that is,

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}| - h(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in D, \tag{2.5}$$

where

$$h(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \ln |\mathbf{y}| - \frac{1}{2\pi} \ln \left| \mathbf{x} - \frac{\mathbf{y}}{|\mathbf{y}|^2} \right|, \quad \mathbf{x}, \mathbf{y} \in D. \tag{2.6}$$

Let $k \geq 1$ be an integer and $\kappa_1, \kappa_2, ..., \kappa_k$ be k non-zero real numbers. Define the corresponding Kirchhoff-Routh function H_k as follows:

$$H_k(\mathbf{x_1}, \cdots, \mathbf{x_k}) := -\sum_{i \neq j} \kappa_i \kappa_j G(\mathbf{x_i}, \mathbf{x_j}) + \sum_{i=1}^{k} \kappa_i^2 h(\mathbf{x_i}, \mathbf{x_i}) \tag{2.7}$$

where $(\mathbf{x}_1, \cdots, \mathbf{x}_k) \in D^{(k)} := \underbrace{D \times D \times \cdots \times D}_{k}$ such that $\mathbf{x}_i \neq \mathbf{x}_j$ for $i \neq j$. In this paper we consider the case $k = 2$ and $\kappa_1 = -\kappa_2 = \kappa > 0$, then the Kirchhoff-Routh function can be written as

$$H_2(\mathbf{x}, \mathbf{y}) = 2\kappa^2 G(\mathbf{x}, \mathbf{y}) + \kappa^2 h(\mathbf{x}, \mathbf{x}) + \kappa^2 h(\mathbf{y}, \mathbf{y}), \tag{2.8}$$

where $(\mathbf{x}, \mathbf{y}) \in D^{(2)}$ and $\mathbf{x} \neq \mathbf{y}$. It is easy to see that

$$\lim_{|\mathbf{x}-\mathbf{y}| \to 0} H_2(\mathbf{x}, \mathbf{y}) = +\infty, \qquad \lim_{\mathbf{x} \to \partial D \text{ or } \mathbf{y} \to \partial D} H_2(\mathbf{x}, \mathbf{y}) = +\infty, \tag{2.9}$$

so H_2 attains its minimum in $\{(\mathbf{x}, \mathbf{y}) \in D^{(2)} \mid \mathbf{x} \neq \mathbf{y}\}$. Moreover, by Proposition A.1 in the Appendix, for any minimum point $(\mathbf{x}_0, \mathbf{y}_0)$ of H_2, there exists $\theta \in [0, 2\pi)$ such that

$$\mathbf{x}_0 = \sqrt{\sqrt{5} - 2}(\cos\theta, \sin\theta), \quad \mathbf{y}_0 = -\sqrt{\sqrt{5} - 2}(\cos\theta, \sin\theta).$$

Now we consider a steady ideal fluid with unit density in D with impermeability boundary condition, the motion of which is described by the following Euler equations:

$$\begin{cases} (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P & \text{in } D, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } D, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial D\ , \end{cases} \tag{2.10}$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity field, P is the scalar pressure, and $\mathbf{n}(\mathbf{x})$ is the outward unit normal at $\mathbf{x} \in \partial D$.

To simplify the Euler equations (2.10), we define the scalar vorticity $\omega := \partial_1 v_2 - \partial_2 v_1$. By using the identity $\frac{1}{2}\nabla|\mathbf{v}|^2 = (\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{v}^\perp\omega$, where $\mathbf{v}^\perp$ denotes the clockwise rotation of $\mathbf{v}$ through $\pi/2$, then the first equation of (1.1) can be written as

$$\nabla\left(\frac{1}{2}|\mathbf{v}|^2 + P\right) - \mathbf{v}^\perp\omega = 0. \tag{2.11}$$

Taking the curl in (2.11) we get

$$\nabla\cdot(\omega\mathbf{v}) = 0. \tag{2.12}$$

To recover the velocity field in terms of the vorticity, we define the stream function ψ by solving the following Poisson's equation with zero Dirichlet boundary condition

$$\begin{cases} -\Delta\psi = \omega & \text{in } D, \\ \psi = 0 & \text{on } \partial D. \end{cases} \tag{2.13}$$

Using Green's function we have

$$\psi(\mathbf{x}) = \mathcal{G}\omega(\mathbf{x}) := \int_D G(\mathbf{x},\mathbf{y})\omega(\mathbf{y})d\mathbf{y}. \tag{2.14}$$

Since D is simply-connected and $\mathbf{v}\cdot\mathbf{n} = 0$ on ∂D, it is easy to check that $\mathbf{v}$ is uniquely determined by ω in the following way (see also [19], Chapter 1, Theorem 2.2)

$$\mathbf{v} = \nabla^\perp\psi, \tag{2.15}$$

where $\nabla^\perp\psi := (\nabla\psi)^\perp = (\partial_2\psi, -\partial_1\psi)$.

From the above discussion we obtain the following *vorticity equation*

$$\nabla\cdot\left(\omega\nabla^\perp\mathcal{G}\omega\right) = 0. \tag{2.16}$$

In this paper, we interpret the vorticity equation (2.16) in the following weak sense.

Definition 2.1. We call $\omega \in L^\infty(D)$ a weak solution to (2.16) if

$$\int_D \omega\nabla^\perp\mathcal{G}\omega\cdot\xi d\mathbf{x} = 0 \tag{2.17}$$

for all $\xi \in C_0^\infty(D)$.

It should be noted that if $\omega \in L^\infty(D)$, then by the regularity theory for elliptic equations $\mathcal{G}\omega \in C^{1,\alpha}(\overline{D})$ for some $\alpha \in (0,1)$, therefore the integral in (2.17) makes sense.

From now on, we will focus on the existence of weak solutions to (2.16). The following lemma from [10] (see also [14]) gives a criterion for an L^∞ function to be a weak solution.

Lemma A. *Let $\omega \in L^\infty(D)$. Suppose that $\omega = f(\mathcal{G}\omega)$ a.e. in D for some monotone function $f : \mathbb{R} \to \mathbb{R}$, then ω is a weak solution to (2.16).*

Now we are ready to state our main result.

Theorem 2.2. *Let κ be a positive number. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, there exists $\omega^\lambda \in L^\infty(D)$ such that*

(1) ω^λ is a weak solution to (2.16);

(2) ω^λ is even in x_1 and odd in x_2, that is, $\omega^\lambda(\mathbf{x}) = \omega^\lambda(\bar{\mathbf{x}})$ and $\omega^\lambda(\mathbf{x}) = -\omega^\lambda(\tilde{\mathbf{x}})$;

(3) $\omega^\lambda = \omega_1^\lambda + \omega_2^\lambda$, where $\omega_1^\lambda = \lambda I_{\{\mathcal{G}\omega^\lambda > \mu^\lambda\}}$ and $\omega_2^\lambda = -\lambda I_{\{\mathcal{G}\omega^\lambda < -\mu^\lambda\}}$ for some $\mu^\lambda \in \mathbb{R}^+$ depending on λ, and

$$\int_D \omega_1^\lambda(\mathbf{x})d\mathbf{x} = \kappa, \quad \int_D \omega_2^\lambda(\mathbf{x})d\mathbf{x} = -\kappa;$$

(4) ω_1^λ "shrinks" to $P_1 := \left(0, \sqrt{\sqrt{5}-2}\right)$ and ω_2^λ "shrinks" to $P_2 := \left(0, -\sqrt{\sqrt{5}-2}\right)$ as λ goes to infinity. More precisely,

$$\operatorname{diam}\left(\operatorname{supp}\omega_1^\lambda\right) \le C\lambda^{-\frac{1}{2}}, \quad \operatorname{diam}\left(\operatorname{supp}\omega_2^\lambda\right) \le C\lambda^{-\frac{1}{2}},$$

$$\lim_{\lambda\to+\infty}\left|\frac{1}{\kappa}\int_D \mathbf{x}\omega_1^\lambda(\mathbf{x})d\mathbf{x} - P_1\right| = 0, \quad \lim_{\lambda\to+\infty}\left|-\frac{1}{\kappa}\int_D \mathbf{x}\omega_2^\lambda(\mathbf{x})d\mathbf{x} - P_2\right| = 0,$$

where C is a positive number not depending on λ.

3. Variational Problem

In this section, we study a maximization problem for the vorticity and give some estimates for the maximizers as the vorticity strength goes to infinity.

First we choose $\delta > 0$ sufficiently small such that $B_\delta(P_i) \subset\subset D$ for $i = 1, 2$ and $\overline{B_\delta(P_1)} \cap \overline{B_\delta(P_2)} = \varnothing$, where P_1 and P_2 are defined in Theorem 2.2. For example, we can choose $\delta = \frac{\sqrt{\sqrt{5}-2}}{2}$. In the rest of this paper, we use B_i to denote $B_\delta(P_i)$ for $i = 1, 2$ for simplicity.

For $\lambda > 0$ sufficiently large, we define the vorticity class K^λ as follows:

$$\begin{aligned} K^\lambda := \Big\{\omega \in L^\infty(D) \mid \omega = \omega_1 + \omega_2, \operatorname{supp}\omega_i \subset B_i \text{ for } i = 1, 2, \int_D \omega_1(\mathbf{x})d\mathbf{x} = \kappa, \\ 0 \le \omega_1 \le \lambda, \omega_1(\mathbf{x}) = \omega_1(\bar{\mathbf{x}}) \text{ and } \omega_2(\mathbf{x}) = \omega_1(\tilde{\mathbf{x}}) \text{ for } \mathbf{x} \in D\Big\}. \end{aligned} \tag{3.18}$$

It is easy to check that for any $\omega \in K^\lambda$, ω is even in x_1 and odd in x_2. It is also clear that K^λ is not empty if $\lambda > 0$ is large enough.

The kinetic energy of the fluid with vorticity ω is

$$E(\omega) = \frac{1}{2}\int_D\int_D G(\mathbf{x}, \mathbf{y})\omega(\mathbf{x})\omega(\mathbf{y})d\mathbf{x}d\mathbf{y}. \tag{3.19}$$

Integrating by parts we have

$$E(\omega) = \frac{1}{2}\int_D \mathcal{G}\omega(\mathbf{x})\omega(\mathbf{x})d\mathbf{x} = \frac{1}{2}\int_D |\nabla \mathcal{G}\omega(\mathbf{x})|^2 \, d\mathbf{x}. \tag{3.20}$$

In the rest of this section we will consider the maximization of E on K^λ and study the properties of the maximizer.

3.1. Existence of a maximizer

First we show the existence of a maximizer of E on K^λ.

Lemma 3.1. *There exists* $\omega^\lambda \in K^\lambda$ *such that* $E(\omega^\lambda) = \sup_{\omega\in K^\lambda} E(\omega)$.

Proof. To make it clear, we divide the proof into three steps.

Step 1: E is bounded from above on K^λ. In fact, since $\|\omega\|_{L^\infty(D)} \leq \lambda$ for any $\omega \in K^\lambda$, we have

$$E(\omega) = \frac{1}{2}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega(\mathbf{x})\omega(\mathbf{y})d\mathbf{x}d\mathbf{y} \leq \frac{1}{2}\lambda^2\int_D\int_D |G(\mathbf{x},\mathbf{y})|d\mathbf{x}d\mathbf{y} \leq C\lambda^2$$

for some generic constant C. Here we use the fact that $G \in L^1(D\times D)$. This gives

$$\sup_{\omega\in K^\lambda} E(\omega) < +\infty.$$

Step 2: K^λ is closed in the weak* topology of $L^\infty(D)$, or equivalently, for any sequence $\{\omega_n\} \subset K^\lambda$ and $\omega \in L^\infty(D)$ satisfying

$$\lim_{n\to+\infty}\int_D \omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \int_D \omega(\mathbf{x})\phi(\mathbf{x})d\mathbf{x},\ \forall\, \phi \in L^1(D), \tag{3.21}$$

we have $\omega \in K^\lambda$. To prove this, we first show that

$$\operatorname{supp}\omega \subset \cup_{i=1}^2 B_i. \tag{3.22}$$

In fact, for any $\phi \in C_0^\infty\left(D \setminus \cup_{i=1}^2 \overline{B_i}\right)$, by (3.21) we have

$$\int_D \omega(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \lim_{n\to+\infty}\int_D \omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = 0,$$

which means that $\omega = 0$ a.e. in $D \setminus \cup_{i=1}^2 \overline{B_i}$.

Now we define $\omega_i = \omega I_{B_i}$ for $i = 1, 2$. It is obvious that $\omega = \omega_1 + \omega_2$. By choosing $\phi = I_{B_1}$ in (3.21), we have

$$\int_D \omega_1(\mathbf{x})d\mathbf{x} = \int_D \omega(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \lim_{n\to+\infty}\int_D \omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \lim_{n\to+\infty}\int_{B_1} \omega_n(\mathbf{x})d\mathbf{x} = \kappa. \tag{3.23}$$

It is also easy to show that $0 \le \omega_1 \le \lambda$ in D. In fact, suppose that $|\{\omega_1 > \lambda\}| > 0$, then we can choose $\varepsilon_0, \varepsilon_1 > 0$ such that $|\{\omega_1 > \lambda + \varepsilon_0\}| > \varepsilon_1$. Denote $S = \{\omega_1 > \lambda + \varepsilon_0\} \subset B_1$, then by choosing $\phi = I_S$ in (3.21) we have

$$\int_S (\omega_1 - \omega_n)(\mathbf{x}) d\mathbf{x} \ge \varepsilon_0 |S| \ge \varepsilon_0 \varepsilon_1.$$

On the other hand, by (3.21)

$$\lim_{n\to+\infty} \int_S (\omega_1 - \omega_n)(\mathbf{x}) d\mathbf{x} = \lim_{n\to+\infty} \int_D (\omega - \omega_n)(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x} = 0,$$

which is a contradiction. So we have $\omega_1 \le \lambda$. Similarly we can prove $\omega_1 \ge 0$.

To finish Step 2, it suffices to show that ω is even in x_1 and odd in x_2. For fixed $\mathbf{x} \in D \cap \{x_1 > 0\}$, define $\phi = \frac{1}{\pi s^2} I_{B_s(\mathbf{x})} - \frac{1}{\pi s^2} I_{B_s(\bar{\mathbf{x}})}$, where $s > 0$ is sufficiently small. Since ω_n is even in x_1 for each n and ϕ is odd in x_1, by (3.21) we have

$$\int_D \omega(\mathbf{y}) \phi(\mathbf{y}) d\mathbf{y} = \lim_{n\to+\infty} \int_D \omega_n(\mathbf{y}) \phi(\mathbf{y}) d\mathbf{y} = 0, \tag{3.24}$$

which means that

$$\frac{1}{|B_s(\mathbf{x})|} \int_{B_s(\mathbf{x})} \omega(\mathbf{y}) d\mathbf{y} = \frac{1}{|B_s(\bar{\mathbf{x}})|} \int_{B_s(\bar{\mathbf{x}})} \omega(\mathbf{y}) d\mathbf{y}. \tag{3.25}$$

By Lebesgue differential theorem, for a.e. $\mathbf{x} \in D \cap \{x_1 > 0\}$, we have

$$\lim_{s\to 0^+} \frac{1}{|B_s(\mathbf{x})|} \int_{B_s(\mathbf{x})} \omega(\mathbf{y}) d\mathbf{y} = \omega(\mathbf{x}), \tag{3.26}$$

and

$$\lim_{s\to 0^+} \frac{1}{|B_s(\bar{\mathbf{x}})|} \int_{B_s(\bar{\mathbf{x}})} \omega(\mathbf{y}) d\mathbf{y} = \omega(\bar{\mathbf{x}}). \tag{3.27}$$

Combining (3.25), (3.26) and (3.27) we get

$$\omega(\mathbf{x}) = \omega(\bar{\mathbf{x}}) \text{ a.e. } x \in D \cap \{x_1 > 0\}, \tag{3.28}$$

which means that ω is even in x_1. Similarly we can prove that ω is odd in x_2.

From all the above arguments we know that $\omega \in K^\lambda$.

Step 3: E is sequentially continuous on K^λ in the weak* topology of $L^\infty(D)$, that is, for any sequence $\{\omega_n\} \subset K^\lambda$ such that $\omega_n \to \omega$ weakly* in $L^\infty(D)$ as $n \to +\infty$, we have

$$\lim_{n\to+\infty} E(\omega_n) = E(\omega).$$

In fact, by (3.21) as $n \to +\infty$ we know that $\omega_n \to \omega$ weakly in $L^2(D)$, then $\psi_n \to \psi$ weakly in $W^{2,2}(D)$ thus strongly in $L^2(D)$, where $\psi_n = \mathcal{G}\omega_n$ and $\psi = \mathcal{G}\omega$. So we get as $n \to +\infty$

$$E(\omega_n)=\frac{1}{2}\int_D\psi_n(\mathbf{x})\omega_n(\mathbf{x})d\mathbf{x}\to\frac{1}{2}\int_D\psi(\mathbf{x})\omega(\mathbf{x})d\mathbf{x}=E(\omega).$$

Now we finish the proof of Lemma 3.1 by using the standard maximization technique. By Step 1, we can choose a maximizing sequence $\{\omega_n\}\subset K^\lambda$ such that

$$\lim_{n\to+\infty}E(\omega_n)=\sup_{\omega\in K^\lambda}E(\omega)$$

Since K^λ is bounded thus weakly* sequentially compact in $L^\infty(D)$, we can choose a subsequence $\{\omega_{n_k}\}$ such that as $k\to+\infty$ $\omega_{n_k}\to\omega^\lambda$ weakly$*$ in $L^\infty(D)$ for some $\omega^\lambda\in L^\infty(D)$. By Step 2, we have $\omega^\lambda\in K^\lambda$. Finally by Step 3 we have

$$E(\omega^\lambda)=\lim_{k\to+\infty}E(\omega_{n_k})=\sup_{\omega\in K^\lambda}E(\omega),$$

which is the desired result.

□

3.2. Profile of ω^λ

Since $\omega^\lambda\in K^\lambda$, we know that ω^λ has the form $\omega^\lambda=\omega_1^\lambda+\omega_2^\lambda$ with ω_1^λ and ω_2^λ satisfying

(1) $\operatorname{supp}\omega_i^\lambda\subset B_i$ for $i=1,2$,

(2) $\int_D\omega_1^\lambda(\mathbf{x})d\mathbf{x}=-\int_D\omega_2^\lambda(\mathbf{x})d\mathbf{x}=\kappa$,

(3) $\omega_1^\lambda(\mathbf{x})=\omega_1^\lambda(\bar{\mathbf{x}})$, $\omega_1^\lambda(\mathbf{x})=-\omega_2^\lambda(\tilde{\mathbf{x}})$ for any $\mathbf{x}\in D$.

In fact, we can prove that ω^λ has a special form.

Lemma 3.2. *There exists $\mu^\lambda\in\mathbb{R}$ depending on λ such that*

$$\omega_1^\lambda=\lambda I_{\{\psi^\lambda>\mu^\lambda\}\cap B_1},\quad \omega_2^\lambda=-\lambda I_{\{\psi^\lambda<-\mu^\lambda\}\cap B_2},$$

where $\psi^\lambda:=\mathcal{G}\omega^\lambda$.

Proof. First we show that ω_1^λ has the form $\omega_1^\lambda=\lambda I_{\{\psi^\lambda>\mu^\lambda\}\cap B_1}$ for some $\mu^\lambda\in\mathbb{R}$. To this end, choose $\alpha,\beta\in L^\infty(D)$ satisfying

$$\begin{cases}\alpha,\beta\geq 0,\ \int_D\alpha(\mathbf{x})d\mathbf{x}=\int_D\beta(\mathbf{x})d\mathbf{x},\\ \operatorname{supp}\alpha,\operatorname{supp}\beta\subset B_1\cap\{x_1>0\},\\ \alpha=0\quad\text{in } D\setminus\{\omega_1^\lambda\leq\lambda-a\},\\ \beta=0\quad\text{in } D\setminus\{\omega_1^\lambda\geq a\},\end{cases}\tag{3.29}$$

where a is a small positive number, and define a family of test functions $\omega_s=\omega^\lambda+s(g_1-g_2)$, where $s>0$ sufficiently small and

$$g_1(x)=\alpha(\mathbf{x})+\alpha(\bar{\mathbf{x}})-\alpha(\tilde{\mathbf{x}})-\alpha(\tilde{\bar{\mathbf{x}}})$$

and

$$g_2(x)=\beta(\mathbf{x})+\beta(\bar{\mathbf{x}})-\beta(\tilde{\mathbf{x}})-\beta(\tilde{\bar{\mathbf{x}}}).$$

Note that g_1, g_2 are both even in x_1 and odd in x_2, and $\operatorname{supp} g_1, \operatorname{supp} g_2 \subset B_1 \cup B_2$. It is easy to check that $\omega_s \in K^\lambda$ if s is sufficiently small (depending on α, β and a), so we have

$$0 \geq \left.\frac{dE(\omega_s)}{ds}\right|_{s=0^+} = \int_D g_1(\mathbf{x})\psi^\lambda(\mathbf{x})d\mathbf{x} - \int_D g_2(\mathbf{x})\psi^\lambda(\mathbf{x})d\mathbf{x}. \tag{3.30}$$

On the other hand ,

$$\begin{aligned}&\int_D g_1(\mathbf{x})\psi^\lambda(\mathbf{x})d\mathbf{x} - \int_D g_2(\mathbf{x})\psi^\lambda(\mathbf{x})d\mathbf{x} = \int_{D_1} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x}\\ &+\int_{D_2} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x}+\int_{D_3} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x}+\int_{D_4} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x},\end{aligned} \tag{3.31}$$

where

$$D_1 = B_1 \cap \{x_1>0\}, \quad D_2 = B_1 \cap \{x_1<0\},$$
$$D_3 = B_2 \cap \{x_1<0\}, \quad D_4 = B_2 \cap \{x_1>0\}.$$

Since ψ^λ is also even in x_1 and odd in x_2 (this can be proved directly using the symmetry of the Green's function), we have

$$\begin{aligned}&\int_{D_1} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x} = \int_{D_2} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x}\\ =&\int_{D_3} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x} = \int_{D_4} \psi^\lambda(\mathbf{x})(g_1-g_2)(\mathbf{x})d\mathbf{x}.\end{aligned} \tag{3.32}$$

Combining (3.30), (3.31) and (3.32) we conclude that

$$\int_{D_1} \psi^\lambda(\mathbf{x})g_1(\mathbf{x})d\mathbf{x} \leq \int_{D_1} \psi^\lambda(\mathbf{x})g_2(\mathbf{x})d\mathbf{x}, \tag{3.33}$$

or equivalently

$$\int_{D_1} \psi^\lambda(\mathbf{x})\alpha(\mathbf{x})d\mathbf{x} \leq \int_{D_1} \psi^\lambda(\mathbf{x})\beta(\mathbf{x})d\mathbf{x}. \tag{3.34}$$

By the choice of α and β, inequality (3.33) holds if and only if

$$\sup_{\{\omega^\lambda<\lambda\}\cap D_1} \psi^\lambda \leq \inf_{\{\omega^\lambda>0\}\cap D_1} \psi^\lambda. \tag{3.35}$$

Combining the continuity of ψ^λ in $\{\omega^\lambda<\lambda\}\cap D_1$, we obtain

$$\sup_{\{\omega^\lambda<\lambda\}\cap D_1} \psi^\lambda = \inf_{\{\omega^\lambda>0\}\cap D_1} \psi^\lambda. \tag{3.36}$$

Now we define μ^λ as follows

$$\mu^\lambda := \sup_{\{\omega^\lambda<\lambda\}\cap D_1} \psi^\lambda = \inf_{\{\omega^\lambda>0\}\cap D_1} \psi^\lambda. \tag{3.37}$$

It is easy to see that

$$\begin{cases} \omega^\lambda = 0 & \text{a.e. in } \{\psi^\lambda < \mu^\lambda\} \cap D_1, \\ \omega^\lambda = \lambda & \text{a.e. in } \{\psi^\lambda > \mu^\lambda\} \cap D_1. \end{cases} \tag{3.38}$$

On $\{\psi_1^\lambda = \mu^\lambda\} \cap D_1$, ψ^λ is a constant, so we have $\nabla\psi^\lambda = 0$ a.e., therefore $\omega^\lambda = -\Delta\psi^\lambda = 0$ a.e.. Hence we conclude that

$$\begin{cases} \omega^\lambda = 0 & \text{a.e. in } \{\psi^\lambda \le \mu^\lambda\} \cap D_1, \\ \omega^\lambda = \lambda & \text{a.e. in } \{\psi^\lambda > \mu^\lambda\} \cap D_1. \end{cases} \tag{3.39}$$

Finally by the symmetry of ω^λ and ψ^λ we get the desired result.

□

3.3. Asymptotic behavior as $\lambda \to +\infty$

Now we give some asymptotic estimates on ω^λ as $\lambda \to +\infty$. We will use C to denote various positive numbers not depending on λ.

Lemma 3.3. *Let $\varepsilon = \sqrt{\frac{\kappa}{\lambda\pi}}$. Suppose that ω^λ is the one obtained in Lemma 3.1. Then*

(1) $E(\omega^\lambda) \ge -\frac{1}{2\pi}\kappa^2 \ln\varepsilon - C$;

(2) $\mu^\lambda \ge -\frac{1}{2\pi}\kappa \ln\varepsilon - C$;

(3) there exists a positive number $R > 1$, not depending on λ, such that $\operatorname{diam}(\operatorname{supp}\omega_i^\lambda) < R\varepsilon$ *for $i = 1, 2$;*

(4) $\frac{1}{\kappa}\int_D \mathbf{x}\omega_1^\lambda(\mathbf{x})d\mathbf{x} \to \mathbf{x}_1$ *and* $-\frac{1}{\kappa}\int_D \mathbf{x}\omega_2^\lambda(\mathbf{x})d\mathbf{x} \to \mathbf{x}_2$ *as $\lambda \to +\infty$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy $\mathbf{x}_1 \in \overline{B_1}, \mathbf{x}_2 \in \overline{B_2}$ and* $H_2(\mathbf{x}_1, \mathbf{x}_2) = \min_{(\mathbf{x},\mathbf{y})\in D^{(2)}} H_2(\mathbf{x}, \mathbf{y})$.

Proof. The proofs are identical to the ones in Section 2.4, [13], therefore we omit them. □

4. Proof for Theorem 2.2

In this section we finish the proof of Theorem 2.2. The key point is to show that the maximizer ω^λ obtained in Lemma 3.1 satisfies the condition in Lemma A.

Proof of Theorem 2.2. First, we show that the support of ω_i^λ shrinks to P_i for $i = 1, 2$. By (3) and (4) of Lemma 3.3, $\operatorname{supp}\omega_i^\lambda$ shrinks to $\mathbf{x}_i$, where $(\mathbf{x}_1, \mathbf{x}_2)$ is a minimum point of H_2 in $D^{(2)}$. Then by Proposition A.1 in the Appendix, there exists some $\theta \in [0, 2\pi)$ such that

$$\mathbf{x}_1 = \sqrt{\sqrt{5}-2}(\cos\theta, \sin\theta), \mathbf{x}_2 = -\sqrt{\sqrt{5}-2}(\cos\theta, \sin\theta).$$

But by the symmetry of ω^λ (see the definition of K^λ), we have $\cos\theta=0$, $\sin\theta>0$, so $\theta=\pi/2$. Consequently

$$\mathbf{x}_1=\left(0,\sqrt{\sqrt{5}-2}\right),\mathbf{x}_2=\left(0,-\sqrt{\sqrt{5}-2}\right). \tag{4.40}$$

Second, we show that ω^λ is a weak solution to (2.16). To begin with, we show that

$$|\psi^\lambda|\le C \text{ on } \partial B_1\cup\partial B_2, \tag{4.41}$$

where C is a positive number not depending on λ. In fact, by (3) and (4) of Lemma 3.3 and (4.40), we have for large λ

$$\operatorname{dist}(\operatorname{supp}\omega_i^\lambda,\partial B_i)>\delta_0,\ i=1,2, \tag{4.42}$$

where $\delta_0\in(0,1)$ does not depend on λ. Then for $\mathbf{x}\in\partial B_1$,

$$\begin{aligned}|\psi^\lambda(\mathbf{x})|&=\left|\int_D G(\mathbf{x},\mathbf{y})\omega^\lambda(\mathbf{y})d\mathbf{y}\right|\\&=\left|\int_D G(\mathbf{x},\mathbf{y})\omega_1^\lambda(\mathbf{y})d\mathbf{y}+\int_D G(\mathbf{x},\mathbf{y})\omega_2^\lambda(\mathbf{y})d\mathbf{y}\right|\\&=\left|\int_D -\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{y}|\omega_1^\lambda(\mathbf{y})d\mathbf{y}-\int_D h(\mathbf{x},\mathbf{y})\omega_1^\lambda(\mathbf{y})d\mathbf{y}+\int_D G(\mathbf{x},\mathbf{y})\omega_2^\lambda(\mathbf{y})d\mathbf{y}\right|\\&\le\left|-\frac{1}{2\pi}\ln\delta_0\int_D\omega_1^\lambda(\mathbf{y})d\mathbf{y}\right|+\int_{\operatorname{supp}\omega_1^\lambda}|h(\mathbf{x},\mathbf{y})|\omega_1^\lambda(\mathbf{y})d\mathbf{y}+\int_{\operatorname{supp}\omega_2^\lambda}\left|G(\mathbf{x},\mathbf{y})\omega_2^\lambda(\mathbf{y})\right|d\mathbf{y}\\&\le-\frac{\kappa}{2\pi}\ln\delta_0+C.\end{aligned} \tag{4.43}$$

Here we used the fact that $|h|\le C$ on $\partial B_1\times\operatorname{supp}\omega_1^\lambda$ and $|G|\le C$ on $\partial B_1\times\operatorname{supp}\omega_2^\lambda$. Similarly $|\psi^\lambda|\le C$ on ∂B_2. So (4.41) is proved.

By Lemma 3.2, ω^λ has the form

$$\omega^\lambda=\lambda I_{\{\psi^\lambda>\mu^\lambda\}\cap B_1}-\lambda I_{\{\psi^\lambda<-\mu^\lambda\}\cap B_2}. \tag{4.44}$$

Since $|\psi^\lambda|\le C$ on $\partial B_1\cup\partial B_2$, by the maximum principle we know that

$$\psi^\lambda\le C \text{ in } D\setminus B_1, \psi^\lambda\ge -C \text{ in } D\setminus B_2. \tag{4.45}$$

On the other hand, by (2) of Lemma 3.3 we have $\lim_{\lambda\to+\infty}\mu^\lambda=+\infty$, so combining (4.45) we get

$$\{\psi^\lambda>\mu^\lambda\}\cap B_1=\{\psi^\lambda>\mu^\lambda\},\ \{\psi^\lambda<-\mu^\lambda\}\cap B_2=\{\psi^\lambda<-\mu^\lambda\} \tag{4.46}$$

provided that λ is sufficiently large. So in fact ω^λ has the form

$$\omega^\lambda=\lambda I_{\{\psi^\lambda>\mu^\lambda\}}-\lambda I_{\{\psi^\lambda<-\mu^\lambda\}}, \tag{4.47}$$

or equivalently,

$$\omega^\lambda = f(\psi^\lambda), \tag{4.48}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function defined by

$$f(t) = \begin{cases} \lambda, & t > \mu^\lambda, \\ 0, & t \in [-\mu^\lambda, \mu^\lambda], \\ -\lambda, & t < -\mu^\lambda. \end{cases} \tag{4.49}$$

By Lemma A, ω^λ is a weak solution to (2.16).

Finally, combining the properties of ω^λ obtained in Section 2, we finish the proof of Theorem 2.2. □

5. Non-symmetric Case

In this section, we briefly discuss the existence of steady non-symmetric vortex patch.

Theorem 5.1. *Let κ_1, κ_2 be two real numbers such that $\kappa_1 > 0$ and $\kappa_2 < 0$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, there exists $\omega^\lambda \in L^\infty(D)$ such that*

(1) ω^λ is a weak solution to (2.16)*;*

(2) ω^λ is even in x_1;

(3) $\omega^\lambda = \omega_1^\lambda + \omega_2^\lambda$, where $\omega_1^\lambda = \lambda I_{\{\mathcal{G}\omega^\lambda > \mu_1^\lambda\}}$ and $\omega_2^\lambda = -\lambda I_{\{\mathcal{G}\omega^\lambda < -\mu_2^\lambda\}}$ for some $\mu_1^\lambda, \mu_2^\lambda \in \mathbb{R}^+$ depending on λ, and

$$\int_D \omega_1^\lambda(\mathbf{x})d\mathbf{x} = \kappa_1, \quad \int_D \omega_2^\lambda(\mathbf{x})d\mathbf{x} = \kappa_2;$$

(4) ω_1^λ "shrinks" to $P := (0, p)$ and ω_2^λ "shrinks" to $Q := (0, q)$ as λ goes to infinity, where $p > 0, q < 0$ depend only on κ_2/κ_1. More precisely,

$$\operatorname{diam}\left(\operatorname{supp}\omega_1^\lambda\right) \le C\lambda^{-\frac{1}{2}}, \quad \operatorname{diam}\left(\operatorname{supp}\omega_2^\lambda\right) \le C\lambda^{-\frac{1}{2}},$$

$$\lim_{\lambda\to+\infty}\left|\frac{1}{\kappa_1}\int_D \mathbf{x}\omega_1^\lambda(\mathbf{x})d\mathbf{x} - P\right| = 0, \quad \lim_{\lambda\to+\infty}\left|\frac{1}{\kappa_2}\int_D \mathbf{x}\omega_2^\lambda(\mathbf{x})d\mathbf{x} - Q\right| = 0,$$

where C is a positive number not depending on λ.

Proof. The construction of ω^λ here is similar to the symmetric case in Section 2. For simplicity we only sketch the proof.

First we choose (P, Q) as a minimum point of the corresponding Kirchhoff-Routh function

$$H_2(\mathbf{x}, \mathbf{y}) := -2\kappa_1\kappa_2 G(\mathbf{x}, \mathbf{y}) + {\kappa_1}^2 h(\mathbf{x}, \mathbf{x}) + {\kappa_2}^2 h(\mathbf{y}, \mathbf{y}),\ \mathbf{x}, \mathbf{y} \in D, \mathbf{x} \ne \mathbf{y}. \tag{5.50}$$

Without loss of generality, we assume that P and Q lie on the x_2-axis, that is, $P = (0, p)$ and $Q = (0, q)$ with $p > 0, q < 0$ determined by κ_2/κ_1 (see Proposition A.1 in the Appendix). Now we choose $\delta > 0$ sufficiently small such that $B_\delta(P), B_\delta(Q) \subset\subset D$ and $\overline{B_\delta(P)} \cap \overline{B_\delta(Q)} = \varnothing$.

Consider the maximization of E on the following class

$$\begin{aligned} M^\lambda := \big\{\omega \in L^\infty(D) \mid \ \omega = \omega_1 + \omega_2, \operatorname{supp}\omega_1 \subset B_\delta(P), \operatorname{supp}\omega_2 \subset B_\delta(Q), \\ \int_D \omega_i(\mathbf{x})d\mathbf{x} = \kappa_i, 0 \le sgn(\kappa_i)\omega_i \le \lambda, \text{ for } i = 1, 2, \ \omega(\mathbf{x}) = \omega(\bar{\mathbf{x}}) \text{ for } x \in D\big\}. \end{aligned} \tag{5.51}$$

Then by repeating the procedures in Section 2, we can prove that there exists a maximizer ω^λ and this maximizer satisfies (1)–(4) in Theorem 5.1 if λ is sufficiently large.

□

A. Minimum Points of H_2

In this appendix we calculate the minimum points of the function H_2.

Proposition A.1. *Let $D = \{\mathbf{x} \in \mathbb{R}^2 \mid |x| < 1\}$, $\kappa_1 > 0, \kappa_2 < 0$ be two real numbers, and $\gamma = -\frac{\kappa_2}{\kappa_1}$. Denote M the set of minimum points of H_2, where*

$$H_2(\mathbf{x}, \mathbf{y}) := -2\kappa_1\kappa_2 G(\mathbf{x}, \mathbf{y}) + {\kappa_1}^2 h(\mathbf{x}, \mathbf{x}) + {\kappa_2}^2 h(\mathbf{y}, \mathbf{y}), \ \mathbf{x}, \mathbf{y} \in D, \mathbf{x} \neq \mathbf{y}. \tag{A.1}$$

Then there exists $p \in (0, 1), q \in (-1, 0)$ depending only on γ, such that

$$M = \{(P, Q) \in D^{(2)} \mid P = p(\cos\theta, \sin\theta), Q = q(\cos\theta, \sin\theta), \theta \in [0, 2\pi)\}. \tag{A.2}$$

If $\gamma = 1$, then

$$p = -q = \sqrt{\sqrt{5} - 2}.$$

Proof. First, it is easy to see that

$$\lim_{|\mathbf{x}-\mathbf{y}|\to 0} H_2(\mathbf{x}, \mathbf{y}) = +\infty, \qquad \lim_{\mathbf{x}\to\partial D \text{ or } \mathbf{y}\to\partial D} H_2(\mathbf{x}, \mathbf{y}) = +\infty, \tag{A.3}$$

so M is not empty and $M \subset \{(\mathbf{x}, \mathbf{y}) \in D^{(2)} \mid \mathbf{x} \neq \mathbf{y}\}$.

For $\mathbf{x}, \mathbf{y} \in D, \mathbf{x} \neq \mathbf{y}$,

$$\begin{aligned} H_2(\mathbf{x}, \mathbf{y}) &= -2\kappa_1\kappa_2 G(\mathbf{x}, \mathbf{y}) + {\kappa_1}^2 h(\mathbf{x}, \mathbf{x}) + {\kappa_2}^2 h(\mathbf{y}, \mathbf{y}) \\ &= -2\kappa_1\kappa_2 \left(-\frac{1}{2\pi} \ln \frac{|\mathbf{x} - \mathbf{y}|}{|\mathbf{y}||\mathbf{x} - \mathbf{y}^*|}\right) - \kappa_1^2 \left(\frac{1}{2\pi} \ln |\mathbf{x}||\mathbf{x} - \mathbf{x}^*|\right) - \kappa_2^2 \left(\frac{1}{2\pi} \ln |\mathbf{y}||\mathbf{y} - \mathbf{y}^*|\right) \\ &= \frac{\kappa_1^2}{\pi} \ln \left(\frac{|\mathbf{y}|^{2\gamma}|\mathbf{x} - \mathbf{y}^*|^{2\gamma}}{|\mathbf{x} - \mathbf{y}|^{2\gamma}|\mathbf{x}||\mathbf{y}|^{\gamma^2}|\mathbf{y} - \mathbf{y}^*|^{\gamma^2}}\right), \end{aligned} \tag{A.4}$$

where $\mathbf{x}^* = \mathbf{x}/|\mathbf{x}|^2$ and $\mathbf{y}^* = \mathbf{y}/|\mathbf{y}|^2$. So it suffices to consider the minimum points of the following function

$$T(\mathbf{x},\mathbf{y}) := \frac{|\mathbf{y}|^{2\gamma}|\mathbf{x}-\mathbf{y}^*|^{2\gamma}}{|\mathbf{x}-\mathbf{y}|^{2\gamma}|\mathbf{x}||\mathbf{y}|^{\gamma^2}|\mathbf{y}-\mathbf{y}^*|^{\gamma^2}}.$$

By using the polar coordinates,

$$T(\mathbf{x},\mathbf{y}) := \frac{(1+\rho_1^2\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2))^\gamma}{(\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2))^\gamma(1-\rho_1^2)(1-\rho_2^2)^{\gamma^2}},$$

where $\mathbf{x} = \rho_1(\cos\theta_1, \sin\theta_1)$ and $\mathbf{y} = \rho_2(\cos\theta_2, \sin\theta_2)$.

Now we show that if $(\mathbf{x},\mathbf{y})$ is a minimum point of H_2, then $\theta_1 - \theta_2 = \pi$. In fact, it is not hard to check that for fixed ρ_1 and ρ_2, T is strictly increasing in $\cos(\theta_1-\theta_2)$, so at any minimum point we must have

$$\cos(\theta_1-\theta_2) = -1.$$

To finish the proof, it suffices to calculate the minimum points of the following function:

$$R(\rho_1,\rho_2) := \frac{(1+\rho_1\rho_2)^{2\gamma}}{(\rho_1+\rho_2)^{2\gamma}(1-\rho_1^2)(1-\rho_2^2)^{\gamma^2}}$$

for $\rho_1, \rho_2 \in (0,1)$.

Case 1: $\gamma = 1$. In this simple case, $R(\rho_1,\rho_2)$ becomes

$$R(\rho_1,\rho_2) := \frac{(1+\rho_1\rho_2)^2}{(\rho_1+\rho_2)^2(1-\rho_1^2)(1-\rho_2^2)}, \quad \rho_1,\rho_2 \in (0,1).$$

By direct calculation, we obtain

$$\partial_{\rho_1} R = 0 \Leftrightarrow \rho_1\rho_2 + \rho_1^2 + \rho_1^3\rho_2 + 2\rho_2^2 = 1, \tag{A.5}$$

$$\partial_{\rho_2} R = 0 \Leftrightarrow \rho_2\rho_1 + \rho_2^2 + \rho_2^3\rho_1 + 2\rho_1^2 = 1. \tag{A.6}$$

Subtracting the two expressions in (A.5) and (A.6) we get

$$(1+\rho_1\rho_2)(\rho_1^2-\rho_2^2) = 0,$$

which gives $\rho_1 = \rho_2$. Now we can see that ρ_1 satisfies

$$\rho_1^4 + 4\rho_1^2 - 1 = 0,$$

so $\rho_1 = \rho_2 = \sqrt{\sqrt{5}-2}$.

Case 2: $\gamma > 0$ is arbitrary. In this case, we show that $R(\rho_1,\rho_2)$ has a unique minimum point for $\rho_1,\rho_2 \in (0,1)$. Existence is obvious since we have proved that M is not empty. Now we show the uniqueness. In fact, it suffices to prove that the critical point of R in $(0,1)\times(0,1)$ is unique.

Direct calculation gives

$$\partial_{\rho_1} R = 0 \Leftrightarrow \rho_1\rho_2 + (1+\gamma)\rho_1^2 + \gamma\rho_2^2 + \rho_1^3\rho_2 + (1-\gamma)\rho_1^2\rho_2^2 - \gamma = 0,$$

$$\partial_{\rho_2} R = 0 \Leftrightarrow \gamma\rho_1\rho_2 + \rho_1^2 + \rho_2^2 + (\gamma-1)\rho_1^2\rho_2^2 + \gamma^2\rho_2^2 + \gamma\rho_1\rho_2^3 - 1 = 0.$$

For simplicity we write

$$F_1(\rho_1,\rho_2) := \rho_1\rho_2 + (1+\gamma)\rho_1^2 + \gamma\rho_2^2 + \rho_1^3\rho_2 + (1-\gamma)\rho_1^2\rho_2^2 - \gamma$$

and

$$F_2(\rho_1,\rho_2) := \gamma\rho_1\rho_2 + \rho_1^2 + \rho_2^2 + (\gamma-1)\rho_1^2\rho_2^2 + \gamma^2\rho_2^2 + \gamma\rho_1\rho_2^3 - 1.$$

It is not hard to check that F_1, F_2 are both strictly monotone in ρ_1 for fixed ρ_2 and in ρ_2 for fixed ρ_1 if $\rho_1, \rho_2 \in (0,1)$, which means that the system $F_1(\rho_1,\rho_2) = 0, F_2(\rho_1,\rho_2) = 0$ has at most one solution. In other words, under the condition $\theta_1 - \theta_2 = \pi$, H_2 has a unique minimum point, which completes the proof. □

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