ON THE COMPOSITION OF ROUGH SINGULAR INTEGRAL OPERATORS

GUOEN HU, XUDONG LAI, AND QINGYING XUE

Abstract. In this paper, we investigate the behavior of the bounds of the composition for rough singular integral operators on the weighted space. More precisely, we obtain the quantitative weighted bounds of the composite operator for two singular integral operators with rough homogeneous kernels on $L^p(\mathbb{R}^d, w)$, $p \in (1, \infty)$, which is smaller than the product of the quantitative weighted bounds for these two rough singular integral operators. Moreover, at the endpoint $p = 1$, the $L \log L$ weighted weak type bound is also obtained, which has interests of its own in the theory of rough singular integral even in the unweighted case.

1. Introduction

This paper will be devoted to study the quantitative weighted bounds for the composition of rough singular integral operators. The theory of Calderón-Zygmund singular integral operator, which origins from the pioneering work of Calderón and Zygmund [4] in 1950s, has been developed extensively in the last sixty years (see for example the recently exposition [14],[15]).

The composition of singular integral operators arise typically in the algebra of singular integral (see [6],[2],[3]) and the non-coercive boundary-value problems for elliptic equations (see [31],[28]). In the past decades, considerable attention has been paid to the composition of singular integral operators. We refer the reader to see the work in [34, 31, 9, 30, 7, 28] and the references therein. This paper will be devoted to study the composition of the singular integral operator T_Ω with a rough convolution type kernel. Recall that T_Ω is defined by

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} f(y)dy, \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 42B20, Secondary 47B33.

Key words and phrases. Rough singular integral operator, composite operator, weighted bound, bilinear sparse operator.

The research of the first author was supported by NSFC (No. 11871108). The research of second author was supported by China Postdoctoral Science Foundation (Nos. 2017M621253, 2018T110279), National Natural Science Foundation of China (No. 11801118) and the Fundamental Research Funds for the Central Universities. The research of the third author was supported partly by NSFC (Nos. 11471041, 11671039, 11871101) and NSFC-DFG (No. 11761131002).

Xudong Lai is the corresponding author.

where Ω is homogeneous of degree zero, integrable and has mean value zero on the unit sphere S^{d-1}. This operator was introduced by Calderón and Zygmund [4], and then studied by many authors in the last sixty years (see e.g. [5], [11], [32], [10], [16], [13], [33]). The composite operator $T_{\Omega_1}T_{\Omega_2}$ has been first appeared in the work of Calderón and Zygmund [6] where the algebra of singular integrals was studied. However in this paper, we will study other properties of the composite operator $T_{\Omega_1}T_{\Omega_2}$. Our starting points of this paper are as follows:

(i). Calderón and Zygmund [5] proved that T_Ω is bounded on $L^p(\mathbb{R}^d)$ if $p \in (1, \infty)$ for rough kernel Ω. It is trivial to see that the composite operator $T_{\Omega_1}T_{\Omega_2}$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$. At the endpoint $p = 1$, it was quite later that Seeger [33] showed T_Ω is of weak type (1,1) by means of some deep idea of geometric microlocal decomposition and the Fourier transform. Nevertheless, no proper weak type estimate of $T_{\Omega_1}T_{\Omega_2}$ was known prior to this article when both Ω_1 and Ω_2 are rough kernels. In this paper, we will prove that $T_{\Omega_1}T_{\Omega_2}$ satisfies the $L\log L$ weak type estimate.

(ii). Recently there are numerous work related to seek the optimal quantitative weighted bound for singular integral operator (see e.g. [8, 25, 26, 27, 12, 21, 24, 22, 1, 18, 19]). Motivated by this, our interests are focused on the behavior of the quantitative weighted bound for $T_{\Omega_1}T_{\Omega_2}$ compared to that of single singular integral. We show that the quantitative weighted bound of $T_{\Omega_1}T_{\Omega_2}$ is smaller than the products of that of T_{Ω_1} and T_{Ω_2}, which has interests of its own.

We summary our main results as follows.

Theorem 1.1. *Let Ω_1, Ω_2 be homogeneous of degree zero, have mean value zero and $\Omega_1, \Omega_2 \in L^\infty(S^{d-1})$. Then for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$,*

$$\begin{aligned}\|T_{\Omega_1}T_{\Omega_2}f\|_{L^p(\mathbb{R}^d,\,w)} \quad &\lesssim \quad [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)\big([\sigma]_{A_\infty} + [w]_{A_\infty}\big)\\ &\qquad \times \min\big\{[\sigma]_{A_\infty},\, [w]_{A_\infty}\big\}\|f\|_{L^p(\mathbb{R}^d,\,w)},\end{aligned}$$

where $p' = p/(p-1)$, $\sigma = w^{-1/(p-1)}$, and the precise definitions of $A_p(\mathbb{R}^d)$ weight and A_p constants are listed in Section 2.

Remark 1.2. It is unknown whether the above quantitative weighted bound is optimal. However, from the recent result of Hytönen, Roncal, and Tapiola [24]: if $\Omega \in L^\infty(S^{d-1})$, then for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$,

$$\|T_\Omega f\|_{L^p(\mathbb{R}^d,\,w)} \lesssim [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)\big([\sigma]_{A_\infty} + [w]_{A_\infty}\big)\|f\|_{L^p(\mathbb{R}^d,\,w)},$$

in which the quantitative weighted bound was improved later by Li, Pérez, Rivera-Rios and Roncal [29] as follows,

$$[w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)\min\{[\sigma]_{A_\infty},\, [w]_{A_\infty}\}, \tag{1.2}$$

we can see that the quantitative weighted bound of $T_{\Omega_1}T_{\Omega_2}$ in Theorem 1.1 is smaller than the product of the quantitative weighted bounds of T_{Ω_1} and T_{Ω_2} in (1.2). In fact, for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$, by some elementary computation,

$$\max\{[w]_{A_\infty}, [\sigma]_{A_\infty}\} \le [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big),$$

which easily implies our desired estimate.

Theorem 1.3. *Let Ω_1, Ω_2 be homogeneous of degree zero, have mean value zero and $\Omega_1, \Omega_2 \in L^\infty(S^{d-1})$. Then for $w \in A_1(\mathbb{R}^d)$ and $\lambda > 0$,*

$$\begin{aligned} &w(\{x \in \mathbb{R}^d : |T_{\Omega_1}T_{\Omega_2}f(x)| > \lambda\}) \\ &\quad\lesssim [w]_{A_1}[w]_{A_\infty}^2 \log(\mathrm{e} + [w]_{A_\infty}) \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big) w(x)dx. \end{aligned}$$

Remark 1.4. To the best knowledge of the author, the $L\log L$ weak type estimate in Theorem 1.3 is new even in the unweighted case. We do not know whether this kind of $L\log L$ weak type estimate is optimal, but this estimate has no hope to be improved to the weak type (1,1) estimate even in the case $\Omega_1, \Omega_2 \in C^\infty(S^{d-1})$. In fact, it was shown by Phong and Stein [31] that in general the composite operator $T_{\Omega_1}T_{\Omega_2}$ is not of weak type (1,1). More ever, the authors of [31] gave a necessary and sufficient condition such that the composite operator is of weak type (1,1). If $\Omega_1, \Omega_2 \in C^\infty(S^{d-1})$, then by [14, Proposition 2.4.8], the symbols of T_{Ω_1} and T_{Ω_2} (thus is $\mathcal{F}[p.v.\Omega_1(\cdot)/|\cdot|^d]$ and $\mathcal{F}[p.v.\Omega_2(\cdot)/|\cdot|^d]$, where $\mathcal{F}[f]$ denote the Fourier transform of f) are $C^\infty(\mathbb{R}^d \setminus \{0\})$. By check the necessary and sufficient condition in [31, Theorem 1], we may show that $T_{\Omega_1}T_{\Omega_2}$ is not of weak type (1,1).

Previous results of quantitative weighted bounds for the composite operator is only known for the smooth singular integral operators, we refer to see [1],[18] and [19]. It should be pointed out that the argument for the smooth singular integral operators used in [1, 18, 19] essentially relies on the smooth condition of the kernel. Our strategy in this paper is to establish a decomposition of the composite operator by representing it as two operators which may have different kinds of bilinear sparse dominations: $(L(\log L)^\beta, L^r)$ and (L^1, L^r) type respectively (see Corollary 5.1). This decomposition is done basing on the weak type estimates of the grand maximal operator $\mathscr{M}_{T_\Omega,r}$ and T_Ω. In addition, we also show that the $(L(\log L)^\beta, L^r)$ type sparse domination could be applied to the operator that is of $(L(\log L)^\beta$ weak type to get quantitative weighted bounds. Our main arguments (see Sections 3 and 4) presented in this paper are stated in the abstract setting which have interest of its own. By applying them to the composite operator $T_{\Omega_1}T_{\Omega_2}$, we may get our main theorems.

This paper is organized as follows. In Section 2, we give some notation and lemmas. In Section 3, we will establish an quantitative weighted weak type estimate for the operator which enjoys a bilinear sparse domination.

Section 4 is devoted to give a decomposition of the composite operator. Finally as applications of the arguments in Sections 3 and 4, the proof of our main theorems are given in Section 5.

2. Preliminary

In this paper, we will work on $\mathbb{R}^d$, $d \geq 2$. C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leq CB$. Specially, we use $A \lesssim_{d,p} B$ to denote that there exists a positive constant C depending only on d, p such that $A \leq CB$. Constant with subscript such as c_1, does not change in different occurrences.

For any set $E \subset \mathbb{R}^d$, χ_E denotes its characteristic function. For a cube $Q \subset \mathbb{R}^d$ and $\lambda \in (0, \infty)$, we use $\ell(Q)$ (diam$Q$) to denote the side length (diameter) of Q, and λQ to denote the cube with the same center as Q and whose side length is λ times that of Q. For a fixed cube Q, denote by $\mathcal{D}(Q)$ the set of dyadic cubes with respect to Q, that is, the cubes from $\mathcal{D}(Q)$ are formed by repeating subdivision of Q and each of descendants into 2^d congruent subcubes.

For $\beta \in [0, \infty)$, cube $Q \subset \mathbb{R}^d$ and a suitable function g, $\|g\|_{L(\log L)^\beta, Q}$ is the norm defined by

$$\|g\|_{L(\log L)^\beta, Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q \frac{|g(y)|}{\lambda}\log^\beta\Big(\mathrm{e} + \frac{|g(y)|}{\lambda}\Big)dy \leq 1\Big\}.$$

$\langle |f|\rangle_Q$ denotes the mean value of $|f|$ on Q and $\langle |g|\rangle_{Q,r} = \big(\langle |g|^r\rangle_Q\big)^{\frac{1}{r}}$. We denote $\|g\|_{L(\log L)^0, Q}$ by $\langle |g|\rangle_Q$. Let M_β be the maximal operator defined by

$$M_\beta f(x) = \big[M(|f|^\beta)(x)\big]^{\frac{1}{\beta}},$$

where M is the Hardy-Littlewood maximal operator, and $M_{L(\log L)^\beta}$ be the maximal operator defined by

$$M_{L(\log L)^\beta} g(x) = \sup_{Q \ni x} \|g\|_{L(\log L)^\beta, Q}.$$

For simplicity, we denote $M_{L(\log L)^1}$ by $M_{L\log L}$. It is well known that $M_{L(\log L)^\beta}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$, and for any $\lambda > 0$,

$$\big|\{x \in \mathbb{R}^d : M_{L(\log L)^\beta} g(x) > \lambda\}\big| \lesssim \int_{\mathbb{R}^d} \frac{|g(x)|}{\lambda}\log^\beta\Big(\mathrm{e} + \frac{|g(x)|}{\lambda}\Big)dx. \tag{2.1}$$

Let w be a nonnegative, locally integrable function on $\mathbb{R}^d$. We say that $w \in A_p(\mathbb{R}^d)$ if the A_p constant $[w]_{A_p}$ is finite, where

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)dx\Big)^{p-1}, \ \ p \in (1, \infty),$$

the supremum is taken over all cubes in $\mathbb{R}^d$, and the A_1 constant is defined by

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^d}\frac{Mw(x)}{w(x)}.$$

A weight $u \in A_\infty(\mathbb{R}^d) = \cup_{p\geq 1}A_p(\mathbb{R}^d)$. We use the following definition of the A_∞ constant of u (see e.g. [35])

$$[u]_{A_\infty} = \sup_{Q\subset\mathbb{R}^d}\frac{1}{u(Q)}\int_Q M(u\chi_Q)(x)dx.$$

As usual, by a general dyadic grid $\mathscr{D}$, we mean a collection of cubes with the following properties: (i) for any cube $Q \in \mathscr{D}$, its side length $\ell(Q)$ is of the form 2^k for some $k \in \mathbb{Z}$; (ii) for any cubes $Q_1,\, Q_2 \in \mathscr{D}$, $Q_1 \cap Q_2 \in \{Q_1,\, Q_2,\, \emptyset\}$; (iii) for each $k \in \mathbb{Z}$, the cubes of side length 2^k form a partition of $\mathbb{R}^d$.

Let $\eta \in (0,\,1)$ and $\mathcal{S} = \{Q_j\}$ be a family of cubes. We say that $\mathcal{S}$ is η-sparse, if for each fixed $Q \in \mathcal{S}$, there exists a measurable subset $E_Q \subset Q$, such that $|E_Q| \geq \eta|Q|$ and E_Q's are pairwise disjoint. Associated with the sparse family $\mathcal{S}$ and constants $\beta \in [0,\,\infty)$ and $r \in [1,\,\infty)$, we define the bilinear sparse operator $\mathcal{A}_{\mathcal{S};\,L(\log L)^\beta,\,L^r}$ by

$$\mathcal{A}_{\mathcal{S};\,L(\log L)^\beta,L^r}(f,g) = \sum_{Q\in\mathcal{S}}|Q|\|f\|_{L(\log L)^\beta,\,Q}\langle|g|\rangle_{Q,r}.$$

Also, we define the operator $\mathcal{A}_{\mathcal{S},\,L^{r_1},\,L^{r_2}}$ by

$$\mathcal{A}_{\mathcal{S};\,L^{r_1},L^{r_2}}(f,g) = \sum_{Q\in\mathcal{S}}|Q|\langle|f|\rangle_{Q,r_1}\langle|g|\rangle_{Q,\,r_2}.$$

Let T be a sublinear operator acting on $\cup_{p\geq 1}L^p(\mathbb{R}^d)$, $\beta,\, q \in (0,\,\infty)$. We say that T enjoys a $(L(\log L)^\beta,\, L^q)$-bilinear sparse domination with bound A, if for each bounded function f with compact support, there exists a sparse family $\mathcal{S}$ of cubes, such that for all $g \in L^q_{\rm loc}(\mathbb{R}^d)$,

$$\Big|\int_{\mathbb{R}^d} g(x)Tf(x)dx\Big| \leq A\mathcal{A}_{\mathcal{S},\,L(\log L)^\beta,\,L^q}(f,\,g). \tag{2.2}$$

We will use the following lemmas in our proof.

Lemma 2.1 (see [23]). *Let $t \in (1,\,\infty)$. Then for $p \in (1,\,\infty)$ and weight w,*

$$\|Mf\|_{L^{p'}(\mathbb{R}^d,\,(M_tw)^{1-p'})} \leq c_d p t'^{\frac{1}{p'}}\|f\|_{L^{p'}(\mathbb{R}^d,\,w^{1-p'})}.$$

Lemma 2.2 (see [29] or [27]). *Let $p \in (1,\,\infty)$ and v be a weight. Let S be the operator defined by*

$$S(h) = v^{-\frac{1}{p}}M(hv^{\frac{1}{p}})$$

and R be the operator defined by

$$R(h) = \sum_{k=0}^{\infty}\frac{1}{2^k}\frac{S^k h}{\|S\|^k_{L^p(\mathbb{R}^d,\,v)\to L^p(\mathbb{R}^d,\,v)}}. \tag{2.3}$$

Then for any $h \in L^p(\mathbb{R}^d,\, v)$,

(i) $0 \le h \le R(h)$,

(ii) $\|R(h)\|_{L^p(\mathbb{R}^d,\, v)} \le 2\|h\|_{L^p(\mathbb{R}^d,\, v)}$,

(iii) $R(h)v^{\frac{1}{p}} \in A_1(\mathbb{R}^d)$ *with* $[R(h)v^{\frac{1}{p}}]_{A_1} \le c_d p'$. *Furthermore, when* $v = M_r w$ *for some* $r \in [1,\, \infty)$, *we also have that* $[Rh]_{A_\infty} \le c_d p'$.

Lemma 2.3 (see [22])**.** *Let* $w \in A_\infty(\mathbb{R}^d)$. *Then for any cube* Q *and* $\delta \in (1,\, 1+\frac{1}{2^{11+d}[w]_{A_\infty}}]$,

$$\Big(\frac{1}{|Q|}\int_Q w^\delta(x)dx\Big)^{\frac{1}{\delta}} \le \frac{2}{|Q|}\int_Q w(x)dx.$$

3. Endpoint estimates for sparse operators

The main purpose of this section is to establish a weighted weak type endpoint estimate for the operator which enjoys $(L(\log)^\beta, L^q)$-bilinear sparse domination. We begin with some lemmas.

Lemma 3.1. *Let* $\beta \in [0,\, \infty)$, $r \in [1,\, \infty)$ *and* w *be a weight. Then for any* $t \in (1,\, \infty)$ *and* $p \in (1,\, r')$ *such that* $t\frac{p'/r-1}{p'-1} > 1$,

$$\mathcal{A}_{\mathcal{S},L(\log L)^\beta,L^r}(f,\, g) \lesssim p'^{1+\beta}\big(\frac{p'}{r}\big)'\big(t\frac{p'/r-1}{p'-1}\big)'^{\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}^d,\, M_t w)}\|g\|_{L^{p'}(\mathbb{R}^d, w^{1-p'})}.$$

Proof. Let $p \in (1,\, r')$, $f \in C_0^\infty(\mathbb{R}^d)$ with $\|f\|_{L^p(\mathbb{R}^d,\, M_t w)} = 1$ and Rf be the function defined by (2.3). Recall that

$$\|f\|_{L(\log L)^\beta,\, Q} \lesssim \Big(1+\big(\frac{\beta}{s-1}\big)^\beta\Big)\Big(\frac{1}{|Q|}\int_Q |f(y)|^s dy\Big)^{\frac{1}{s}}.$$

Applying Lemma 2.2 with $v = M_t w$ and Lemma 2.3, we then get that

$$\begin{aligned}\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^\beta,\, Q}\langle|g|\rangle_{Q,\, r}|Q| &\lesssim s'^\beta\sum_{Q\in\mathcal{S}}\langle|g|\rangle_{Q,\, r}\langle|f|\rangle_{Q,\, s}|Q|\\ &\lesssim s'^\beta\sum_{Q\in\mathcal{S}}\langle|g|\rangle_{Q,\, r}\langle|Rf|\rangle_{Q,\, s}|Q|\\ &\lesssim p'^\beta\sum_{Q\in\mathcal{S}}\langle|g|\rangle_{Q,\, r}\int_Q Rf(y)dy,\end{aligned}$$

if we choose $s = 1+\frac{1}{2^{11+d}[Rf]_{A_\infty}}$. As in the proof of Lemma 4.1 in [23], we see that

$$\begin{aligned}\sum_{Q\in\mathcal{S}}\langle|g|\rangle_{Q,\, r}\int_Q Rf(x)dx &\lesssim \sum_{Q\in\mathcal{S}}\inf_{y\in Q}M_r g(y)\int_Q Rf(x)dx\\ &\lesssim [Rh]_{A_\infty}\int_{\mathbb{R}^d}M_r g(x)Rf(x)dx\\ &\lesssim p'\int_{\mathbb{R}^d}M_r g(x)Rf(x)dx.\end{aligned}$$

Hölder's inequality, along with Lemma 2.1, tells us that

$$\begin{aligned}\int_{\mathbb{R}^d} M_r g(x) Rf(x)dx &\lesssim \Big[\int_{\mathbb{R}^d}\big[M_r g(x)\big]^{p'}(M_t w(x))^{1-p'}dx\Big]^{\frac{1}{p'}}\|Rf\|_{L^p(\mathbb{R}^d, M_t w)}\\ &= \Big[\int_{\mathbb{R}^d}\big[M(|g|^r)(x)\big]^{\frac{p'}{r}}\Big(M_{t\frac{p'/r-1}{p'-1}}(w^{\frac{p'-1}{p'/r-1}})(x)\Big)^{1-p'/r}dx\Big]^{\frac{1}{p'}}\\ &\lesssim \Big[(\frac{p'}{r})'(t\frac{p'/r-1}{p'-1})'^{\frac{r}{p'}}\Big]^{\frac{1}{r}}\|g\|_{L^{p'}(\mathbb{R}^d,\, w^{1-p'})}.\end{aligned}$$

Combining the estimates above leads to that

$$\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^\beta}\langle|g|\rangle_{Q,\,r}|Q| \lesssim p'^{1+\beta}(\frac{p'}{r})'(t\frac{p'/r-1}{p'-1})'^{\frac{1}{p'}}\|g\|_{L^{p'}(\mathbb{R}^d,\, w^{1-p'})}.$$

This, via homogeneity, implies our required estimate and completes the proof of Lemma 3.1. □

Let U be an operator on $\cup_{p\geq 1}L^p(\mathbb{R}^d)$. We say that U is sublinear, if for all functions $f_1,\ f_2$ and $x\in\mathbb{R}^d$,

$$|U(f_1+f_2)(x)|\leq |U(f_1)(x)|+|U(f_2)(x)|,$$

and for all $\lambda\in\mathbb{R}$ and function f,

$$|\lambda Uf(x)| = |U(\lambda f)(x)|.$$

Theorem 3.2. *Let $\alpha,\ \beta\in\mathbb{N}\cup\{0\}$, $t,\ r\in[1,\infty)$, $p_1\in(1,r')$ such that $t\frac{p_1'/r-1}{p_1'-1}>1$. Let U be a sublinear operator which enjoys a $(L(\log L)^\beta,\ L^r)$-sparse domination with bound D. Then for any weight u and bounded function f with compact support,*

$$\begin{aligned}&u(\{x\in\mathbb{R}^d:\ |Uf(x)|>1\})\\ &\quad\lesssim\Big(1+\Big\{Dp_1'^{1+\beta}(\frac{p_1'}{r})'(t\frac{p_1'/r-1}{p_1'-1})'^{\frac{1}{p_1'}}\Big\}^{p_1}\Big)\\ &\qquad\times\int_{\mathbb{R}^d}|f(y)|\log^\beta(\mathrm{e}+|f(y)|)M_t u(y)dy.\end{aligned}\tag{3.1}$$

Proof. Let f be a bounded function with compact support, and $\mathcal{S}$ be the sparse family such that for $g\in L^r_{\rm loc}(\mathbb{R}^d)$,

$$\Big|\int_{\mathbb{R}^d}Uf(x)g(x)dx\Big|\leq D\mathcal{A}_{\mathcal{S};\,L(\log L)^\beta,\,L^r}(f,\,g).$$

By the one-third trick (see [21, Lemma 2.5]), there exist dyadic grids $\mathscr{D}_1,\ \dots,\ \mathscr{D}_{3^d}$ and sparse families $\mathcal{S}_1,\ \dots,\ \mathcal{S}_{3^d}$, such that for $j=1,\ \dots,\ 3^d$, $\mathcal{S}_j\subset\mathscr{D}_j$, and

$$\mathcal{A}_{\mathcal{S};\,L(\log L)^\beta,\,L^r}(f,\,g)\lesssim\sum_{j=1}^{3^d}\mathcal{A}_{\mathcal{S}_j;\,L(\log L)^\beta,\,L^r}(f,\,g).$$

Now let $M_{\mathscr{D}_j,L(\log L)^\beta}$ be the maximal operator defined by

$$M_{\mathscr{D}_j,L(\log L)^\beta}h(x)=\sup_{Q\ni x,\,Q\in\mathscr{D}_j}\|h\|_{L(\log L)^\beta,\,Q}. \tag{3.2}$$

For each $j=1,\,\dots,\,3^d$, decompose the set $\{x\in\mathbb{R}^d:\,M_{\mathscr{D}_j,L(\log L)^\beta}f(x)>1\}$ as

$$\{x\in\mathbb{R}^d:\,M_{\mathscr{D}_j,L(\log L)^\beta}f(x)>1\}=\cup_kQ_{jk},$$

with Q_{jk} the maximal cubes in $\mathscr{D}_j$ such that $\|f\|_{L(\log L)^\beta,\,Q_{jk}}>1$. We have that

$$1<\|f\|_{L(\log L)^\beta,\,Q_{jk}}\lesssim 2^d.$$

Let

$$f_1^j(y)=f(y)\chi_{\mathbb{R}^d\backslash\cup_kQ_{jk}}(y),\;\;f_2^j(y)=\sum_kf(y)\chi_{Q_{jk}}(y),$$

and

$$f_3^j(y)=\sum_k\|f\|_{L(\log L)^\beta,\,Q_{jk}}\chi_{Q_{jk}}(y).$$

It is obvious that $\|f_1^j\|_{L^1(\mathbb{R}^d)}\lesssim\|f\|_{L^1(\mathbb{R}^d)}$, $\|f_1^j\|_{L^\infty(\mathbb{R}^d)}\lesssim 1$ and $\|f_3^j\|_{L^\infty(\mathbb{R}^d)}\lesssim 1$.

Let u be a weight and $p_1\in(1,\,\infty)$. It then follows from Lemma 3.1 that

$$\begin{aligned}&\mathcal{A}_{\mathcal{S};\,L(\log L)^\beta,\,L^r}(f_1^j,\,g)\\&\lesssim p_1'^{1+\beta}\big(\frac{p_1'}{r}\big)'\big(t\frac{p_1'/r-1}{p_1'-1}\big)'^{\frac{1}{p_1'}}\|f_1^j\|_{L^{p_1}(\mathbb{R}^d,\,M_tu)}\|g\|_{L^{p_1'}(\mathbb{R}^d,\,u^{1-p'})}.\end{aligned}\tag{3.3}$$

Let $E=\cup_{j=1}^{3^d}\cup_k4dQ_{jk}$ and $\widetilde{u}(y)=u(y)\chi_{\mathbb{R}^d\backslash E}(y)$. It is obvious that

$$u(E)\lesssim\sum_{j,k}\inf_{z\in Q_{jk}}Mu(z)|Q_{jk}|\lesssim\int_{\mathbb{R}^d}|f(y)|\log^\beta(\mathrm{e}+|f(y)|)Mu(y)dy. \tag{3.4}$$

Moreover, by the fact that

$$\inf_{y\in Q_{jk}}M_t\widetilde{u}(y)\approx\sup_{z\in Q_{jk}}M_t\widetilde{u}(z),$$

we obtain that for $\gamma\in[0,\,\infty)$,

$$\begin{aligned}\|f_3^j\|_{L^1(\mathbb{R}^d,\,M_t\widetilde{u})}&\lesssim\sum_k\inf_{z\in Q_{jk}}M_t\widetilde{u}(z)|Q_{jk}|\|f\|_{L(\log L)^\beta,\,Q_{jk}}\\&\lesssim\int_{\mathbb{R}^d}|f(y)|\log^\beta(\mathrm{e}+|f(y)|)M_tu(y)\mathrm{d}y.\end{aligned}\tag{3.5}$$

Let

$$\mathcal{S}_j^*=\{I\in\mathcal{S}_j:\,I\cap(\mathbb{R}^d\backslash E)\neq\emptyset\}.$$

Note that if $\operatorname{supp}g\subset\mathbb{R}^d\backslash E$, then

$$\mathcal{A}_{\mathcal{S}_j,L(\log L)^{\beta_1},\,L^r}(f_2^j,\,g)=\mathcal{A}_{\mathcal{S}_j^*,L(\log L)^{\beta_1},\,L^r}(f_2^j,\,g).$$

As in the argument in [17, pp. 160-161], we can verify that for each fixed $I \in \mathcal{S}_j^*$,

$$\|f_2^j\|_{L(\log L)^\beta,\,I} \lesssim \|f_3^j\|_{L(\log L)^\beta,\,I}.$$

Again by Lemma 3.1, we have that for $g \in L^1(\mathbb{R}^d)$ with $\operatorname{supp} g \subset \mathbb{R}^d \backslash E$,

$$\begin{aligned}(3.6)\quad \mathcal{A}_{\mathcal{S}_j, L(\log L)^\beta,\,L^r}(f_2^j,\, g) &\lesssim \mathcal{A}_{\mathcal{S}_j, L(\log L)^\beta,\,L^r}(f_3^j,\, g)\\ &\lesssim p_1'^{1+\beta}\big(\frac{p_1'}{r}\big)'\big(t\frac{p_1'/r-1}{p_1'-1}\big)'^{\frac{1}{p_1'}}\|f_3^j\|_{L^1(\mathbb{R}^d,\,M_t u)}^{\frac{1}{p_1}}\|g\|_{L^{p_1'}(\mathbb{R}^d\backslash E,\,u^{1-p'})}.\end{aligned}$$

Inequalities (3.3) and (3.6) tell us that

$$\begin{aligned}&\sup_{\|g\|_{L^{p_1'}(\mathbb{R}^d\backslash E, \widetilde{u}^{1-p_1'})}\le 1}\Big|\int_{\mathbb{R}^d} Uf(x)g(x)dx\Big|\\ &\lesssim D\sup_{\|g\|_{L^{p_1'}(\mathbb{R}^d\backslash E, \widetilde{u}^{1-p_1'})}\le 1}\sum_{j=1}^{3^d}\Big(\mathcal{A}_{\mathcal{S}_j,\,L(\log L)^\beta, L^r}(f_1^j, g)+\mathcal{A}_{\mathcal{S}_j,\,L(\log L)^\beta, L^r}(f_2^j, g)\Big)\\ &\lesssim Dp_1'^{1+\beta}\big(\frac{p_1'}{r}\big)'\big(t\frac{p_1'/r-1}{p_1'-1}\big)'^{\frac{1}{p_1'}}\Big(\|f_1^j\|_{L^1(\mathbb{R}^d,\,M_t u)}^{\frac{1}{p_1}}+\|f_3^j\|_{L^1(\mathbb{R}^d,\,M_t u)}^{\frac{1}{p_1}}\Big).\end{aligned}$$

Thus together with inequalities (3.4) and (3.5), we know that

$$\begin{aligned}&u(\{x \in \mathbb{R}^d : |Uf(x)| > 1\}) \le u(E) + \|Uf\|_{L^{p_1}(\mathbb{R}^d\backslash E,\,\widetilde{u})}^{p_1}\\ &\lesssim \Big(1+\Big\{Dp_1'^{1+\beta}\big(\frac{p_1'}{r}\big)'\big(t\frac{p_1'/r-1}{p_1'-1}\big)'^{\frac{1}{p_1'}}\Big\}^{p_1}\Big)\int_{\mathbb{R}^d}|f(y)|\log^\beta(\mathrm{e}+|f(y)|)M_t u(y)dy.\end{aligned}$$

This completes the proof of Theorem 3.2. □

Corollary 3.3. *Let $\alpha,\,\beta \in \mathbb{N}\cup\{0\}$ and U be a sublinear operator. Suppose that for any $r \in (1,\,3/2]$, U satisfies bilinear $(L(\log L)^\beta,\,L^r)$-sparse domination with bound r'^α. Then for any $w \in A_1(\mathbb{R}^d)$ and bounded function f with compact support,*

$$\begin{aligned}&w(\{x \in \mathbb{R}^d : |Uf(x)| > \lambda\})\\ &\quad\lesssim [w]_{A_\infty}^\alpha \log^{1+\beta}(\mathrm{e}+[w]_{A_\infty})[w]_{A_1}\int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log^\beta\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)w(x)dx.\end{aligned}$$

Proof. Let $w \in A_1(\mathbb{R}^d)$. Choose $t = 1+\frac{1}{2^{11+d}[w]_{A_\infty}}$, $r = (1+t)/2$ and $p_1 = 1+\frac{1}{\log(\mathrm{e}+[w]_{A_\infty})}$. We apply Theorem 3.2 and deduce that $t\frac{p_1'/r-1}{p_1'-1} > 1$ and

$$\big(t\frac{p_1'/r-1}{p_1'-1}\big)' = \frac{t(p_1'/r-1)}{t(p_1'/r-1)-p_1'+1} = \frac{t}{t-1}\frac{\frac{p_1'}{r}-1}{\frac{p_1'}{1+t}-1} = t'\frac{\frac{2p_1'}{1+t}-1}{\frac{p_1'}{1+t}-1} \le 5t'.$$

Note that $r' = \frac{t+1}{t-1} \le 2^{12+d}[w]_{A_\infty}$ and $p_1' \lesssim \log(\mathrm{e}+[w]_{A_\infty})$. Therefore,

$$\begin{aligned}\Big\{r'^{\alpha}p_1'^{1+\beta}\big(\frac{p_1'}{r}\big)'\big(t\frac{p_1'/r-1}{p_1'-1}\big)'^{\frac{1}{p_1'}}\Big\}^{p_1} &\lesssim (r'^{\alpha}p_1'^{1+\beta})^{p_1}t'^{p_1-1}\\ &\lesssim [w]_{A_\infty}^{\alpha}\log^{1+\beta}(\mathrm{e}+[w]_{A_\infty}).\end{aligned}$$

On the other hand, we know from Lemma 2.3 that $M_t w(y) \lesssim [w]_{A_1}w(y)$. This, via inequality (3.1) (with $u = w$) yields

$$\begin{aligned}&w(\{x\in\mathbb{R}^d:\, |Uf(x)|>1\})\\ &\quad\lesssim [w]_{A_1}[w]_{A_\infty}^{\alpha}\log^{1+\beta}(\mathrm{e}+[w]_{A_\infty})\int_{\mathbb{R}^d}|f(y)|\log^{\beta}(\mathrm{e}+|f(y)|)w(y)dy,\end{aligned}$$

which completes the proof of Corollary 3.3. □

4. DECOMPOSITION OF THE COMPOSITE OPERATOR

We begin with an endpoint estimate for composition of sublinear and linear operators.

Theorem 4.1. *Let U_1 be a sublinear operator and U_2 be a linear operator on $\cup_{p\ge1}L^p(\mathbb{R}^d)$. Suppose that the following conditions hold*

(i) *U_1 is bounded on $L^2(\mathbb{R}^d)$ with bound 1;*

(ii) *there exists a positive constant β_1, such that for any $\lambda>0$,*

$$|\{x\in\mathbb{R}^d:\, |U_1f(x)|>\lambda\}| \le \int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log^{\beta_1}\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)dx; \tag{4.1}$$

(iii) *for some $q\in(1,\frac{3}{2}]$, U_2 enjoys a bilinear $(L(\log L)^{\beta_2},\, L^q)$-sparse domination with bound 1.*

Then we get that: for any $\lambda>0$,

$$|\{x\in\mathbb{R}^d:\, |U_1U_2f(x)|>\lambda\}| \lesssim \int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log^{1+\beta_1+\beta_2}\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)dx. \tag{4.2}$$

Proof. Let f be a bounded function and $\mathcal{S}$ be a sparse family of cubes such that for any function g,

$$\Big|\int_{\mathbb{R}^n}g(x)U_2f(x)dx\Big| \le \mathcal{A}_{\mathcal{S},\,L(\log L)^{\beta_2},\,L^q}(f,\,g).$$

By the sparseness of $\mathcal{S}$, we get that

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}g(x)U_2f(x)dx\Big| &\lesssim \int_{\mathbb{R}^n}M_{L(\log L)^{\beta_2}}f(y)M_qf(y)dy\\ &\lesssim \|f\|_{L^2(\mathbb{R}^n)}\|M_qf\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Therefore, U_2 is bounded on $L^2(\mathbb{R}^n)$ with bound C_q. On the other hand, we know from Theorem 3.2 that for any $\lambda>0$,

$$|\{x\in\mathbb{R}^d:\, |U_2f(x)|>\lambda\}| \le \int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log^{\beta_2}\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)dx. \tag{4.3}$$

Since U_1 is a sublinear operator and U_2 is a linear operator, $\lambda^{-1}|U_1U_2f| = |U_2U_2(\lambda^{-1}f)|$. Therefore it suffices to consider inequality (4.2) for $\lambda = 1$. Let f be a bounded function with compact support, and $\mathcal{S}$ be the sparse family such that (2.2) holds true. Then there exist dyadic grids $\mathscr{D}_1, \dots, \mathscr{D}_{3^d}$ and sparse families $\mathcal{S}_1, \dots, \mathcal{S}_{3^d}$, such that for $j = 1, \dots, 3^d$, $\mathcal{S}_j \subset \mathscr{D}_j$, and

$$\mathcal{A}_{\mathcal{S};\,L(\log L)^{\beta_2},\,L^q}(f,\,g) \lesssim \sum_{j=1}^{3^d} \mathcal{A}_{\mathcal{S}_j;\,L(\log L)^{\beta_2},\,L^q}(f,\,g).$$

Now let $M_{\mathscr{D}_j,L(\log L)^{\beta_2}}$ be the maximal operator defined by (3.2). For each $j = 1, \dots, 3^d$, decompose the set $\{x \in \mathbb{R}^d : M_{\mathscr{D}_j,L(\log L)^{\beta_2}}f(x) > 1\}$ as

$$\{x \in \mathbb{R}^d : M_{\mathscr{D}_j,L(\log L)^{\beta_2}}f(x) > 1\} = \cup_k Q_{jk},$$

with Q_{jk} the maximal cubes in $\mathscr{D}_j$ such that $\|f\|_{L(\log L)^{\beta_2},\,Q_{jk}} > 1$. We have that

$$1 < \|f\|_{L(\log L)^{\beta_2},\,Q_{jk}} \lesssim 2^d.$$

Let $E = \cup_{j=1}^{3^d} \cup_k 16dQ_{jk}$, f_1^j, f_2^j, f_3^j and $\mathcal{S}_j^*$ ($j = 1, \dots, 3^d$) be the same as we have done in the proof of Theorem 3.2. Write

$$|U_1U_2f(x)| \le |U_1\big(\chi_E U_2 f\big)(x)| + |U_1\big(\chi_{\mathbb{R}^d\setminus E}U_2f\big)(x)| =: \mathrm{I}_1f(x) + \mathrm{I}_2f(x).$$

Recall that U_1 satisfies the estimate (4.1), and by the fact (2.1)

$$|E| \lesssim \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda}\log^{\beta_2}\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)dx.$$

It then follows that

$$\begin{aligned}
&|\{x \in \mathbb{R}^d : |\mathrm{I}_1f(x)| > 1/2\}| \le \int_E |U_2f(x)|\log^{\beta_1}\big(\mathrm{e} + |U_2f(x)|\big)dx\\
=&\int_0^\infty \big|\{x \in E :\, |U_2f(x)| > s\}\big|d(s\log^{\beta_1}(\mathrm{e}+s))\\
\lesssim&\ |E| + \int_{\mathrm{e}^{2\beta_1}}^\infty |\{x \in E :\, |U_2f(x)| > 2s\}|d(s\log^{\beta_1}(\mathrm{e}+s))\\
\lesssim&\ |E| + \int_{\mathrm{e}^{2\beta_1}}^\infty |\{x \in E : |U_2(f\chi_{\{|f|>s\}})(x)| > s\}|d(s\log^{\beta_1}(\mathrm{e}+s))\\
&\quad + \int_{\mathrm{e}^{2\beta_1}}^\infty |\{x \in E : |U_2(f\chi_{\{|f|\le s\}})(x)| > s\}|d(s\log^{\beta_1}(\mathrm{e}+s)).
\end{aligned}$$

We deduce from (4.3) for U_2 that

$$\begin{aligned}&\int_{\mathrm{e}^{2\beta_1}}^{\infty}|\{x\in E:|U_2(f\chi_{\{|f|>s\}})(x)|>s\}|d(s\log^{\beta_1}(\mathrm{e}+s))\\&\lesssim\int_{\mathrm{e}^{2\beta_1}}^{\infty}\int_{|f(x)|>s}\frac{|f(x)|}{s}\log^{\beta_2}\big(\mathrm{e}+\frac{|f(x)|}{s}\big)dxd(s\log^{\beta_1}(\mathrm{e}+s))\\&\lesssim\int_{\mathbb{R}^d}|f(x)|\log^{\beta_2}(\mathrm{e}+|f(x)|)\int_{\mathrm{e}}^{|f(x)|}\frac{1}{s}d(s\log^{\beta_1}(\mathrm{e}+s))dx\\&\lesssim\int_{\mathbb{R}^d}|f(x)|\log^{\beta_1+\beta_2+1}(\mathrm{e}+|f(x)|)dx,\end{aligned}$$

Trivial computations leads to that

$$d(s\log^{\beta_1}(\mathrm{e}+s))\lesssim\frac{1}{s^2}\log^{\beta_1}(\mathrm{e}+s)ds,$$

and when $s\in[\mathrm{e}^{2\beta_1},\,\infty)$,

$$\begin{aligned}-d(\frac{1}{s}\log^{\beta_1}(\mathrm{e}+s))&=\Big[-\frac{1}{s^2}\log^{\beta_1}(\mathrm{e}+s)-\frac{\beta_1}{s(\mathrm{e}+s)}\log^{\beta_1-1}(\mathrm{e}+s)\Big]ds\\&\geq\frac{1}{2s^2}\log^{\beta_1}(\mathrm{e}+s)ds\end{aligned}$$

It follows from the $L^2(\mathbb{R}^d)$ boundedness of U_2 that

$$\begin{aligned}&\int_{\mathrm{e}^{2\beta_1}}^{\infty}|\{x\in E:\,|U_2(f\chi_{\{|f|\leq s\}})(x)|>s\}|d(s\log^{\beta_1}(\mathrm{e}+s))\\&\lesssim\int_{\mathrm{e}^{2\beta_1}}^{\infty}\frac{1}{s^2}\int_{|f(x)|\leq s}|f(x)|^2dxd(s\log^{\beta_1}(\mathrm{e}+s))\\&=\int_{\mathbb{R}^n}|f(x)|^2\int_{\max\{|f(x)|,\,\mathrm{e}^{2\beta_1}\}}\frac{1}{s^2}\log^{\beta_1}(\mathrm{e}+s)ds\\&\lesssim\int_{\mathbb{R}^d}|f(x)|\log^{\beta_1}(\mathrm{e}+|f(x)|)dx.\end{aligned}$$

Therefore, we conclude the estimate of I_1 as follows

$$|\{x\in\mathbb{R}^d:\,|\mathrm{I}_1f(x)|>1/2\}|\lesssim\int_{\mathbb{R}^d}|f(x)|\log^{\beta_1+\beta_2+1}(\mathrm{e}+|f(x)|)dx.$$

We turn our attention to term I_2f. By the $L^2(\mathbb{R}^d)$ boundedness of U_1, we know that

$$\begin{aligned}|\{x\in\mathbb{R}^d:\,|\mathrm{I}_2f(x)|>1/2\}|&\lesssim\int_{\mathbb{R}^d\backslash E}|U_2f(x)|^2dx\\&\lesssim\Big(\sup_{\|g\|_{L^2(\mathbb{R}^d\backslash E)\leq1}}\Big|\int_{\mathbb{R}^d\backslash E}g(x)U_2f(x)dx\Big|\Big)^2.\end{aligned}$$

For $g\in L^2(\mathbb{R}^d\backslash E)$, we can write

$$\begin{aligned}\Big|\int_{\mathbb{R}^d\backslash E}g(x)U_2f(x)dx\Big| &\le \sum_{j=1}^{3^d}\mathcal{A}_{\mathcal{S}_j,\,L(\log L)^{\beta_2},\,L^q}(f_1^j,\,g)\\ &\quad+\sum_{j=1}^{3^d}\mathcal{A}_{\mathcal{S}_j,\,L(\log L)^{\beta_2},\,L^q}(f_2^j,\,g).\end{aligned}$$

Recall that if $\operatorname{supp} g\subset\mathbb{R}^d\backslash E$, then

$$\mathcal{A}_{\mathcal{S}_j,L(\log L)^{\beta_2},\,L^q}(f_2^j,\,g)=\mathcal{A}_{\mathcal{S}_j^*,L(\log L)^{\beta_2},\,L^q}(f_2^j,\,g)\lesssim\mathcal{A}_{\mathcal{S}_j,L(\log L)^{\beta_2},\,L^q}(f_3^j,\,g).$$

On the other hand, the sparseness of $\mathcal{S}_j$ states that

$$\begin{aligned}\mathcal{A}_{\mathcal{S}_j,L(\log L)^{\beta_2},\,L^q}(f_3^j,\,g) &\lesssim \int_{\mathbb{R}^d}M_{L(\log L)^{\beta_2}}f_3^j(x)M_qg(x)dx\\ &\lesssim \|M_{L(\log L)^{\beta_2}}f_3^j\|_{L^2(\mathbb{R}^d)}\|M_qg\|_{L^2(\mathbb{R}^d)}\\ &\lesssim \|f_3^j\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Since $\|f_1^j\|_{L^1(\mathbb{R}^d)}\le\|f\|_{L^1(\mathbb{R}^d)}$, and

$$\|f_3^j\|_{L^1(\mathbb{R}^d)}\lesssim\int_{\mathbb{R}^d}|f(y)|\log^{\beta_2}(\mathrm{e}+|f(y)|)dy,$$

we finally obtain that

$$\begin{aligned}|\{x\in\mathbb{R}^d:\,|\mathrm{I}_2f(x)|>1/2\}| &\lesssim \Big(\sum_{j=1}^{3^d}(\|f_1^j\|_{L^2(\mathbb{R}^d)}+\|f_3^j\|_{L^2(\mathbb{R}^d)})\Big)^2\\ &\lesssim \Big(\sum_{j=1}^{3^d}(\|f_1^j\|^{\frac12}_{L^1(\mathbb{R}^d)}+\|f_3^j\|^{\frac12}_{L^2(\mathbb{R}^d)})\Big)^2\\ &\lesssim \int_{\mathbb{R}^d}|f(y)|\log^{\beta_2}(\mathrm{e}+|f(y)|)\mathrm{d}y.\end{aligned}$$

Combining estimates for I and II completes the proof of Theorem 4.1. □

For a linear operator T, we define the corresponding grand maximal operator $\mathscr{M}_{T,r}$ by

$$\mathscr{M}_{T,\,r}f(x)=\sup_{Q\ni x}|Q|^{-\frac1r}\|T(f\chi_{\mathbb{R}^d\backslash 3Q})\chi_Q\|_{L^r(\mathbb{R}^d)},$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^d$ containing x. $\mathscr{M}_{T,\,r}$ was introduced by Lerner [25] and is useful in establishing bilinear sparse domination of rough operator T_Ω. Let T_1, T_2 be two linear operators. We define the grand maximal operator $\mathscr{M}^*_{T_1T_2,r}$ by

$$\mathscr{M}^*_{T_1T_2,\,r}f(x)=\sup_{Q\ni x}\Big(\frac{1}{|Q|}\int_Q|T_1\big(\chi_{\mathbb{R}^d\backslash 3Q}T_2(f\chi_{\mathbb{R}^d\backslash 9Q})\big)(\xi)|^rd\xi\Big)^{\frac1r}.$$

Lemma 4.2. *Let $s \in [0, \infty)$ and $A \in (1, \infty)$, S be a sublinear operator which satisfies that for any $\lambda > 0$,*

$$\big|\{x \in \mathbb{R}^d : |Sf(x)| > \lambda\}\big| \lesssim \int_{\mathbb{R}^d} \frac{A|f(x)|}{\lambda} \log^s \Big(\mathrm{e} + \frac{A|f(x)|}{\lambda}\Big) dx.$$

Then for any $\varrho \in (0, 1)$ and cube $Q \subset \mathbb{R}^d$,

$$\Big(\frac{1}{|Q|} \int_Q |S(f\chi_Q)(x)|^{\varrho} dx\Big)^{\frac{1}{\varrho}} \lesssim A\|f\|_{L(\log L)^s,\, Q}.$$

Proof. Lemma 4.2 was proved essentially in [20, p. 643]. We present the proof here mainly to make clear the bound. By homogeneity, we may assume that $\|f\|_{L(\log L)^s,\, Q} = 1$, which means that

$$\int_Q |f(x)| \log^s(\mathrm{e} + |f(x)|) dx \le |Q|.$$

A trivial computation leads to that

$$\begin{aligned} \int_Q |S(f\chi_Q)(x)|^{\varrho} dx &= \int_0^A |\{x \in Q : |S(f\chi_Q)(x)| > t\}| t^{\varrho - 1} dt \\ &\quad + \int_A^{\infty} |\{x \in \mathbb{R}^d : |S(f\chi_Q)(x)| > t\}| t^{\varrho - 1} dt \\ &\lesssim |Q|A^{\varrho} + \int_A^{\infty} \int_Q \frac{A|f(x)|}{t} \log^s \Big(\mathrm{e} + \frac{A|f(x)|}{t}\Big) dx t^{\varrho - 1} dt \\ &\lesssim |Q|A^{\varrho}. \end{aligned}$$

This gives the desired conclusion and completes the proof of Lemma 4.2. □

Lemma 4.3. *Let T_1, T_2 be two linear operators. Suppose that for some $\alpha, \beta \in [0, \infty)$ and $r \in (1, 2]$,*

$$|\{x \in \mathbb{R}^d : \mathscr{M}_{T_1, r} T_2 f(x) > t\}| \lesssim \int_{\mathbb{R}^d} \frac{r^{\alpha}|f(x)|}{t} \log^{\beta} \Big(\mathrm{e} + \frac{r^{\alpha}|f(x)|}{t}\Big) dx. \tag{4.4}$$

Then

$$|\{x \in \mathbb{R}^d : \mathscr{M}^*_{T_1 T_2, r} f(x) > t\}| \lesssim \int_{\mathbb{R}^d} \frac{r^{\alpha}|f(x)|}{t} \log^{\beta} \Big(\mathrm{e} + \frac{r^{\alpha}|f(x)|}{t}\Big) dx.$$

Proof. Let $\tau \in (0, 1)$, $x \in \mathbb{R}^d$ and $Q \subset \mathbb{R}^d$ be a cube containing x. We know by (4.4) and Lemma 4.2 that

$$\Big(\frac{1}{|Q|} \int_Q \big[\mathscr{M}_{T_1, r} T_2(f\chi_{9Q})(\xi)\big]^{\tau} d\xi\Big)^{\frac{1}{\tau}} \lesssim r^{\alpha} M_{L(\log L)^{\beta}} f(x).$$

A straightforward computation leads to that

$$\begin{aligned}\Big[\frac{1}{|Q|}\int_Q|T_1\big(\chi_{\mathbb{R}^d\backslash 3Q}T_2(f\chi_{\mathbb{R}^d\backslash 9Q})\big)(\xi)|^rd\xi\Big]^{\frac{1}{r}}&\le\inf_{\xi\in Q}\mathscr{M}_{T_1,\,r}\big(T_2(f\chi_{\mathbb{R}^d\backslash 9Q})\big)(\xi)\\&\lesssim\Big(\frac{1}{|Q|}\int_Q\big[\mathscr{M}_{T_1,\,r}T_2f(\xi)\big]^{\tau}d\xi\Big)^{\frac{1}{\tau}}+\Big(\frac{1}{|Q|}\int_Q\big[\mathscr{M}_{T_1,\,r}\big(T_2(f\chi_{9Q})\big)(\xi)\big]^{\tau}d\xi\Big)^{\frac{1}{\tau}}\\&\lesssim M_{\tau}\mathscr{M}_{T_1,\,r}T_2f(x)+r^{\alpha}M_{L(\log L)^{\beta}}f(x).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\big|\{x\in\mathbb{R}^d:\,M_{\tau}\mathscr{M}_{T_1,\,r}T_2f(x)>\lambda\}\big|\\&\quad\lesssim\lambda^{-1}\sup_{t\ge 2^{-1/\tau}\lambda}t|\{x\in\mathbb{R}^d:\,\mathscr{M}_{T_1,\,r}T_2f(x)>t\}|\\&\quad\lesssim\int_{\mathbb{R}^d}\frac{r^{\alpha}|f(x)|}{\lambda}\log^{\beta}\Big(\mathrm{e}+\frac{r^{\alpha}|f(x)|}{\lambda}\Big)dx,\end{aligned}$$

where the first inequality follows from inequality (11) in [20]. This, along with (2.1) gives us the desired conclusion. □

Theorem 4.4. *Let T_1, T_2 be two linear operators, $r\in(1,\,3/2]$, $\beta_1,\,\beta_2,\,\gamma\in[0,\,\infty)$. Suppose that the following conditions hold*

(i) *T_1, is bounded on $L^{r'}(\mathbb{R}^d)$ with bound A;*

(ii) *for each $\lambda>0$,*

$$|\{x\in\mathbb{R}^d:\,|T_1T_2f(x)|>\lambda\}|\lesssim\int_{\mathbb{R}^d}\frac{A_0|f(x)|}{\lambda}\log^{\beta_1}\Big(\mathrm{e}+\frac{A_0|f(x)|}{\lambda}\Big)dx;$$

(iii) *for each $\lambda>0$,*

$$|\{x\in\mathbb{R}^d:\,\mathscr{M}_{T_1,\,r'}T_2f(x)>\lambda\}|\lesssim\int_{\mathbb{R}^d}\frac{A_1|f(x)|}{\lambda}\log^{\beta_1}\Big(\mathrm{e}+\frac{A_1|f(x)|}{\lambda}\Big)dx,$$

and

$$|\{x\in\mathbb{R}^d:\,\mathscr{M}_{T_2,\,r'}f(x)>\lambda\}|\lesssim\int_{\mathbb{R}^d}\frac{A_2|f(x)|}{\lambda}\log^{\beta_2}\Big(\mathrm{e}+\frac{A_2|f(x)|}{\lambda}\Big)dx.$$

Then for a bounded function f with compact support, there exists a $\frac{1}{2}\frac{1}{9^d}$-sparse family of cubes $\mathcal{S}=\{Q\}$, and functions H_1 and H_2, such that for each function g,

$$\Big|\int_{\mathbb{R}^n}H_1(x)g(x)dx\Big|\lesssim(A_0+A_1)\mathcal{A}_{\mathcal{S};\,L(\log L)^{\beta_1},\,L^r}(f,\,g),$$

$$\Big|\int_{\mathbb{R}^n}H_2(x)g(x)dx\Big|\lesssim AA_2\mathcal{A}_{\mathcal{S};\,L(\log L)^{\beta_2},\,L^r}(f,\,g),$$

and for a. e. $x\in\mathbb{R}^n$,

$$T_1T_2f(x)=H_1(x)+H_2(x).$$

Proof. We will employ the argument in [25], together with some ideas in [19]. For a fixed cube Q_0, define the local analogy of $\mathscr{M}_{T_2,r'}$ and $\mathscr{M}^*_{T_1T_2,r'}$ by

$$\mathscr{M}_{T_2;\,r';\,Q_0}f(x)=\sup_{Q\ni x,\,Q\subset Q_0}|Q|^{-\frac{1}{r'}}\|\chi_Q T_2(f\chi_{3Q_0\backslash 3Q})\|_{L^{r'}(\mathbb{R}^d)},$$

and

$$\mathscr{M}^*_{T_1T_2,\,r';\,Q_0}f(x)=\sup_{Q\ni x,Q\subset Q_0}\Big(\frac{1}{|Q|}\int_Q|T_1\big(\chi_{\mathbb{R}^d\backslash 3Q}T_2(f\chi_{9Q_0\backslash 9Q})\big)(\xi)|^{r'}d\xi\Big)^{\frac{1}{r'}}$$

respectively. Let $E=\cup_{j=1}^3E_j$ with

$$E_1=\big\{x\in Q_0:\,|T_1T_2(f\chi_{9Q_0})(x)|>DA_0\|f\|_{L(\log L)^{\beta_1},\,9Q_0}\big\},$$

$$E_2=\{x\in Q_0:\,\mathscr{M}_{T_2,\,r';\,Q_0}f(x)>DA_2\|f\|_{L(\log L)^{\beta_2},\,9Q_0}\},$$

$$E_3=\big\{x\in Q_0:\,\mathscr{M}^*_{T_1T_2,\,r';\,Q_0}f(x)>DA_1\|f\|_{L(\log L)^{\beta_1},9Q_0}\big\},$$

with D a positive constant. Our hypothesis, vie Lemma 4.3 tells us that

$$|E|\le\frac{1}{2^{d+2}}|Q_0|,$$

if we choose D large enough. Now on the cube Q_0, we apply the Calderón-Zygmund decomposition to χ_E at level $\frac{1}{2^{d+1}}$, and obtain pairwise disjoint cubes $\{P_j\}\subset\mathcal{D}(Q_0)$, such that

$$\frac{1}{2^{d+1}}|P_j|\le|P_j\cap E|\le\frac{1}{2}|P_j|$$

and $|E\backslash\cup_jP_j|=0$. Observe that $\sum_j|P_j|\le\frac12|Q_0|$. Let

$$\begin{aligned}G_1(x)&=T_1T_2(f\chi_{9Q_0})(x)\chi_{Q_0\backslash\cup_lP_l}(x)\\&\quad+\sum_l T_1\Big(\chi_{\mathbb{R}^n\backslash 3P_l}T_2(f\chi_{9Q_0\backslash 9P_l})\Big)(x)\chi_{P_l}(x).\end{aligned}$$

The facts that $P_l\cap E^c\neq\emptyset$ and $|E\backslash\cup_jP_j|=0$ imply that for any $g\in L^r(\mathbb{R}^d)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}G_1(x)g(x)dx\Big|&\lesssim\int_{\mathbb{R}^d}|T_1T_2(f\chi_{9Q_0})(x)g(x)|\chi_{Q_0\backslash\cup_lP_l}(x)dx\\&\quad+\sum_l\inf_{\xi\in P_l}|P_l|^{\frac{1}{r'}}\mathscr{M}^*_{T_1T_2;Q_0,r'}f(\xi)\Big(\int_{P_l}|g(x)|^rdx\Big)^{\frac1r}\\&\lesssim(A_0+A_1)\|f\|_{L(\log L)^{\beta_1},\,9Q_0}\langle|g|\rangle_{r,\,Q_0}|Q_0|.\end{aligned}\tag{4.5}$$

Also, we define function G_2 by

$$G_2(x)=\sum_l T_1\big(\chi_{3P_l}T_2\big(f\chi_{9Q_0\backslash 9P_l}\big)\big)(x)\chi_{P_l}(x).$$

For each function g, we have by Hölder's inequality that

$$\begin{aligned}(4.6)\qquad &\Big|\int_{\mathbb{R}^d} G_2(x)g(x)dx\Big|\\ &\le A\sum_l\Big(\int_{3P_l}\big|T_2\big(f\chi_{9Q_0\backslash 9P_l}\big)(x)\big|^{r'}dx\Big)^{\frac{1}{r'}}\Big(\int_{P_l}|g(y)|dy\Big)^{\frac{1}{r}}\\ &\lesssim A\sum_l|P_l|^{\frac{1}{r'}}\inf_{\xi\in P_l}\mathscr{M}_{T_2,\,r';\,Q_0}f(\xi)\Big(\int_{P_l}|g(y)|dy\Big)^{\frac{1}{r}}\\ &\lesssim AA_2\|f\|_{L(\log L)^{\beta_2};\,9Q_0}\langle|g|\rangle_{r,\,Q_0}|Q_0|.\end{aligned}$$

It is obvious that

$$T_1T_2(f\chi_{9Q_0})(x)\chi_{Q_0}(x)=G_1(x)+G_2(x)+\sum_l T_1T_2(\chi_{9P_l})(x)\chi_{P_l}(x).$$

As in [19], we now repeat the argument above with $T_1T_2(f\chi_{9Q_0})(x)\chi_{Q_0}$ replaced by each $T_1T_2(\chi_{9P_l})(x)\chi_{P_l}(x)$, and so on. Let $\{Q_0^{j_1}\}=\{P_j\}$, and for fixed $j_1,\,\dots,\,j_{m-1}$, $\{Q_0^{j_1\dots j_{m-1}j_m}\}_{j_m}$ be the cubes obtained at the m-th stage of the decomposition process to the cube $Q_0^{j_1\dots j_{m-1}}$. For each fixed $j_1\dots,j_m$, define the functions $H^{j_1\dots j_m}_{Q_0,1}f$ and $H^{j_1\dots j_m}_{Q_0,2}f$ by

$$H^{j_1\dots j_m}_{Q_0,1}f(x)=T_1\big(\chi_{\mathbb{R}^n\backslash 3Q_0^{j_1\dots j_m}}T_2(f\chi_{9Q_0^{j_1\dots j_{m-1}}\backslash 9Q_0^{j_1\dots j_m}})\big)(x)\chi_{Q_0^{j_1\dots j_m}}(x),$$

and

$$H^{j_1\dots j_m}_{Q_0,2}f(x)=T_1\Big(\chi_{3Q_0^{j_1\dots j_m}}T_2(f\chi_{9Q_0^{j_1\dots j_{m-1}}\backslash 9Q_0^{j_1\dots j_m}})\Big)(x)\chi_{Q_0^{j_1\dots j_m}}(x),$$

respectively. Set $\mathcal{F}=\{Q_0\}\cup_{m=1}^{\infty}\cup_{j_1,\dots,j_m}\{Q_0^{j_1\dots j_m}\}$. Then $\mathcal{F}\subset\mathcal{D}(Q_0)$ is a $\frac{1}{2}$-sparse family. Let

$$\begin{aligned}H_{Q_0,\,1}(x)&=T_1T_2(f\chi_{9Q_0})\chi_{Q_0\backslash\cup_{j_1}Q_0^{j_1}}(x)\\ &\quad+\sum_{m=1}^{\infty}\sum_{j_1,\dots,j_m}T_1T_2(f\chi_{9Q_0^{j_1\dots j_m}})\chi_{Q_0^{j_1\dots j_m}\backslash\cup_{j_{m+1}}Q_0^{j_1\dots j_{m+1}}}(x)\\ &\quad+\sum_{m=1}^{\infty}\sum_{j_1,\dots,j_m}H^{j_1\dots j_m}_{Q_0,1}f(x)\chi_{Q_0^{j_1\dots j_m}}(x).\end{aligned}$$

Also, we define the function $H_{Q_0,\,2}$ by

$$H_{Q_0,\,2}(x)=\sum_{m=1}^{\infty}\sum_{j_1\dots j_m}H^{j_1\dots j_m}_{Q_0,2}f(x)\chi_{Q_0^{j_1\dots j_m}}(x).$$

Then for a. e. $x\in Q_0$,

$$T_1T_2(f\chi_{9Q_0})(x)=H_{Q_0,\,1}(x)+H_{Q_0,\,2}(x).$$

Moreover, as in the inequalities (4.5) and (4.6), the process of producing $\{Q_0^{j_1\ldots j_m}\}$ leads to that

(4.7)
$$\Big|\int_{\mathbb{R}^d} H_{Q_0,\,1}f(x)\chi_{Q_0}(x)dx\Big| \lesssim (A_0+A_1)\sum_{Q\in\mathcal{F}}\|f\|_{L(\log L)^{\beta_1},\,9Q}\langle|g|\rangle_{r,\,Q}|Q|,$$

and for the function g,

$$\Big|\int_{\mathbb{R}^d} g(x)H_{Q_0,\,2}(x)dx\Big| \lesssim AA_2\sum_{Q\in\mathcal{F}}|Q|\|f\|_{L(\log L)^{\beta_2},\,9Q}\langle|g|\rangle_{r,\,Q}. \tag{4.8}$$

We can now conclude the proof of Theorem 4.4. In fact, as in [25], we decompose $\mathbb{R}^d$ by cubes $\{R_l\}$, such that supp$f\subset 3R_l$ for each l, and R_l's have disjoint interiors. Then for a. e. $x\in\mathbb{R}^d$,

$$T_1T_2f(x)=\sum_l H_{R_l,\,1}f(x)+\sum_l H_{R_l,\,2}f(x):=H_1f(x)+H_2f(x).$$

Our desired conclusion follows from inequalities (4.7) and (4.8) directly. □

5. Proof of Theorems

Applying Theorem 4.1 to the rough singular integral operators T_{Ω_1} and T_{Ω_2}, we get the following result.

Corollary 5.1. *Let Ω_1, Ω_2 be homogeneous of degree zero, have mean value zero and $\Omega_1,\,\Omega_2\in L^\infty(S^{d-1})$. Let $r\in(1,\,3/2]$. Then for each bounded function f with compact support, there exists a $\frac{1}{2}\frac{1}{9^d}$-sparse family of cubes $\mathcal{S}=\{Q\}$, and functions J_1 and J_2, such that for each function g,*

$$\Big|\int_{\mathbb{R}^d}J_1(x)g(x)dx\Big|\lesssim r'\mathcal{A}_{\mathcal{S};\,L\log L,\,L^r}(f,\,g),$$

$$\Big|\int_{\mathbb{R}^d}J_2(x)g(x)dx\Big|\lesssim r'^2\mathcal{A}_{\mathcal{S};\,L^1,\,L^r}(f,\,g),$$

and for a. e. $x\in\mathbb{R}^d$,

$$T_{\Omega_1}T_{\Omega_2}f(x)=J_1(x)+J_2(x).$$

Proof. Let $r\in(1,\,3/2]$. Lerner [25] proved that if $\Omega\in L^\infty(S^{d-1})$, then

$$\|\mathscr{M}_{T_\Omega,\,r'}f\|_{L^{1,\,\infty}(\mathbb{R}^d)}\lesssim r'\|\Omega\|_{L^\infty(S^{d-1})}\|f\|_{L^1(\mathbb{R}^d)}. \tag{5.1}$$

On the other hand, since T_Ω is bounded on $L^{r'}(\mathbb{R}^d)$ with bound $\max\{r,\,r'\}$, we deduce that

$$\mathscr{M}_{T_\Omega,\,r'}f(x)\le M_{r'}T_\Omega f(x)+\max\{r,\,r'\}M_{r'}f(x).$$

Therefore, $\mathscr{M}_{T_\Omega,\,r'}$ is bounded on $L^{2r}(\mathbb{R}^d)$ with bound Cr'. This, via estimate (5.1), leads to that

$$\|\mathscr{M}_{T_\Omega,\,r'}f\|_{L^2(\mathbb{R}^d)}\lesssim r'\|f\|_{L^2(\mathbb{R}^d)}. \tag{5.2}$$

We now conclude the proof of Corollary 5.1. Let I be the identity operator. It is obvious that $\mathscr{M}_{I,\,r'}T_{\Omega_2}f(x)=0$. Applying (5.1) and Theorem 4.4 with $T_1=I$, $T_2=T_{\Omega_2}$, we know that T_{Ω_2} satisfies a $(L^1,\,L^r)$-bilinear sparse domination with bound r'. Thus by Theorem 4.1 with the fact T_{Ω_1} is of weak type (1,1) (see e.g. [33]), we have that for any $\lambda>0$,

$$|\{x\in\mathbb{R}^d:\,|T_{\Omega_1}T_{\Omega_2}f(x)|>\lambda\}|\lesssim\int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)dx.$$

Furthermore, it follows from Theorem 4.1, (5.1) and (5.2) that for any $\lambda>0$,

$$|\{x\in\mathbb{R}^d:\,\mathscr{M}_{T_{\Omega_1},\,r'}T_{\Omega_2}f(x)>\lambda\}|\lesssim\int_{\mathbb{R}^d}\frac{r'|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{r'|f(x)|}{\lambda}\Big)dx.$$

Recall that T_{Ω_1} is bounded on $L^r(\mathbb{R}^d)$ with bound Cr'. Another application of Theorem 4.4 yields desired conclusion. □

Proof of Theorem 1.1. For $p\in(1,\,\infty)$ and $w\in A_p(\mathbb{R}^d)$, let $\tau_w=2^{11+d}[w]_{A_\infty}$ and $\tau_\sigma=2^{11+d}[\sigma]_{A_\infty}$, $\varepsilon_1=\frac{p-1}{2p\tau_\sigma+1}$, and $\varepsilon_2=\frac{p'-1}{2p'\tau_w+1}$. It was proved in [18] that

$$\mathcal{A}_{\mathcal{S};L^{1+\varepsilon_1},\,L^{1+\varepsilon_2}}(f,\,g)\lesssim[w]_{A_p}^{1/p}([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'})\|f\|_{L^p(\mathbb{R}^d,\,w)}\|g\|_{L^{p'}(\mathbb{R}^d,\sigma)}.\tag{5.3}$$

Note that

$$\mathcal{A}_{\mathcal{S};L\log L,\,L^{1+\varepsilon_2}}(f,\,g)\lesssim\frac{1}{\varepsilon_1}\mathcal{A}_{\mathcal{S};L^{1+\varepsilon_1},\,L^{1+\varepsilon_2}}(f,\,g).$$

Invoking Corollary 5.1 and inequality (5.3), we deduce that for bounded functions f and g,

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}g(x)T_{\Omega_1}T_{\Omega_2}f(x)dx\Big|&\lesssim[w]_{A_p}^{1/p}([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'})[w]_{A_\infty}\\&\quad\times\big([w]_{A_\infty}+[\sigma]_{A_\infty}\big)\|f\|_{L^p(\mathbb{R}^d,\,w)}\|g\|_{L^{p'}(\mathbb{R}^d,\sigma)}.\end{aligned}$$

Recall that $w\in A_p(\mathbb{R}^n)$ implies that $\sigma\in A_{p'}(\mathbb{R}^n)$ and $[\sigma]_{A_{p'}}^{1/p'}=[w]_{A_p}^{1/p}$. Applying Corollary 5.1 to $T_{\Omega_2}T_{\Omega_1}$, we obtain that

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}f(x)T_{\Omega_2}T_{\Omega_1}g(x)dx\Big|&\lesssim[\sigma]_{A_{p'}}^{1/p'}([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'})[\sigma]_{A_\infty}\\&\quad\times\big([w]_{A_\infty}+[\sigma]_{A_\infty}\big)\|f\|_{L^p(\mathbb{R}^d,\,w)}\|g\|_{L^{p'}(\mathbb{R}^d,\sigma)}\\&\lesssim[w]_{A_p}^{1/p}([\sigma]_{A_\infty}^{1/p}+[w]_{A_\infty}^{1/p'})[\sigma]_{A_\infty}\\&\quad\times\big([w]_{A_\infty}+[\sigma]_{A_\infty}\big)\|f\|_{L^p(\mathbb{R}^d,\,w)}\|g\|_{L^{p'}(\mathbb{R}^d,\sigma)}.\end{aligned}$$

Combining the last two inequalities yields desired conclusion. □

Proof of Theorem 1.3. Let $w \in A_1(\mathbb{R}^d)$. We obtain from Corollary 5.1 and Corollary 3.3 that

$$\begin{aligned} & w\big(\{x \in \mathbb{R}^d : \, |T_{\Omega_1} T_{\Omega_2} f(x)| > \lambda\}\big) \\ & \quad \le w\big(\{x \in \mathbb{R}^d : \, |J_1(x)| > \lambda/2\}\big) + u\big(\{x \in \mathbb{R}^d : \, |J_2(x)| > \lambda/2\}\big) \\ & \quad \lesssim [w]_{A_\infty} \log^2(\mathrm{e} + [w]_{A_\infty})[w]_{A_1} \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big) u(x)dx \\ & \qquad + [w]^2_{A_\infty} \log(\mathrm{e} + [w]_{A_\infty})[w]_{A_1} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} w(x)dx \\ & \quad \lesssim [w]^2_{A_\infty} \log(\mathrm{e} + [w]_{A_\infty})[w]_{A_1} \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big) w(x)dx, \end{aligned}$$

with J_1 and J_2 the functions defined in Corollary 5.1. This completes the proof of Theorem 1.3. □

Added in Proof. After this paper was prepared, we learned that Li et al. [29] established the weighted bounds for linear operators satisfying the assumptions in Corollary 3.3 with $\beta = 0$, which coincides the conclusion in Corollary 3.3 for $\beta = 0$. The argument in [29] is different from the argument in the proof of Corollary 3.3 and is of independent interest.

The authors would like to thank Dr. Kangwei Li for his helpful comments and suggestions.

References

1. C. Benea and F. Bernicot, Conservation de certaines propriétés á travers un contrôle épars d'un opérateur et applications au projecteur de Leray-Hopf, arXiv: 1703:00228. 2, 3
2. A.P. Calderón, Algebras of singular integral operators. Singular integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), pp. 18-55. Amer. Math. Soc., Providence, R.I., 1967. 1
3. A.P. Calderón, Algebras of singular integral operators. 1968 Proc. Internat. Congr. Math. (Moscow, 1966) pp. 393-395 Izdat. "Mir", Moscow. 1
4. A. P. Calderón and A. Zygmund, On the existence of certain singular integrals, Acta Math. **88** (1952), 85-139. 1, 2
5. A. P. Calderón and A. Zygmund, On singular integrals, Amer. J. Math. **78** (1956), 289-309. 2
6. A.P. Calderón and A. Zygmund, Algebras of certain singular operators. Amer. J. Math. 78 (1956), 310-320. 1, 2
7. N. Carozza and A. Passarelli di Napoli, Composition of maximal operators, Publ. Mat. **40** (1996), 397-409. 1
8. J. M. Conde-Alonso, A. Culiuc, F. Di Plinio and Y. Ou, A sparse domination principle for rough singular integrals, Anal. PDE. **10** (2017), 1255-1284. 2
9. M. Christ, Inversion in some algebras of singular integral operators. Rev. Mat. Iberoamericana **4** (1988), no. 2, 219-225. 1
10. M. Christ and J.-L. Rubio de Francia, Weak type $(1,1)$ bounds for rough operators, II, Invent. Math. **93** (1988), 225-237. 2
11. W. C. Connett, Singular integrals near L^1, Proc. Sympos. Pure Math. of Amer. Math. Soc., (S. Wainger and G. Weiss eds), Vol 35 I(1979), 163-165. 2

12. F. Di Plinio, T. Hytönen and K. Li, Sparse bounds for maximal rough singular integrals via the Fourier transform, arXiv:1706.09064. 2
13. D. Fan and Y. Pan, Singular integral operators with rough kernels supported by subvarieties, Amer. J. Math. **119** (1997), 799-839. 2
14. L. Grafakos, *Classic Fourier Analysis*, Graduate Texts in Mathematics, Vol. **249** (Third edition), Springer, New York, 2014. 1, 3
15. L. Grafakos, *Modern Fourier Analysis*, Graduate Texts in Mathematics, Vol. **250** (Third edition), Springer, New York, 2014.. 1
16. L. Grafakos and A. Stefanov, L^p bounds for singular integrals and maximal singular integrals with rough kernels, Indiana Univ. Math. J. **47**(1998), 455-469. 2
17. G. Hu, Weighted vector-valued estimates for a non-standard Calderón-Zygmund operator, Nonlinear Anal. **165** (2017), 143-162. 9
18. G. Hu, Quantitative weighted bounds for the composition of Calderón-Zygmund operators, Banach J. Math. Anal. **13** (2019), 133-150. 2, 3, 19
19. G. Hu, Weighted weak type endpoint estimates for the composition of Calderón-Zygmund operators, J. Aust. Math. Soc, https://doi.org/10.1017/S1446788719000107. 2, 3, 16, 17
20. G. Hu and D. Li, A Cotlar type inequality for the multilinear singular integral operators and its applications, J. Math. Anal. Appl. **290** (2004), 639-653. 14, 15
21. T. Hytönen, M. T. Lacey and C. Pérez, Sharp weighted bounds for the q-variation of singular integrals, Bull. Lond. Math. Soc. **45** (2013), 529-540. 2, 7
22. T. Hytönen and C. Pérez, Sharp weighted bounds involving A_1, Anal. PDE. **6** (2013), 777-818. 2, 6
23. T. Hytönen and C. Pérez, The $L(\log L)^\epsilon$ endpoint estimate for maximal singular integral operators, J. Math. Anal. Appl. **428** (2015), 605-626. 5, 6
24. T. Hytönen, L. Roncal, and O. Tapiola, Quantitative weighted estimates for rough homogeneous singular integrals, Israel J. Math. **218** (2017), 133-164. 2
25. A. K. Lerner, A weak type estimates for rough singular integrals, Rev. Mat. Iberoam., to appear, available at arXiv: 1705:07397. 2, 13, 16, 18
26. A. K. Lerner, A note on weighted bounds for rough singular integrals, Comptes Rend. Math. **356** (2018), 77-80. 2
27. A. K. Lerner, S. Ombrosi, and C. Pérez,A_1 bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden, Math. Res. Lett. **16** (2009), 149-156. 2, 5
28. A. Nagel, F. Ricci, E. M. Stein, S. Wainger, Algebras of singular integral operators with kernels controlled by multiple norms. Mem. Amer. Math. Soc. 256 (2018), no. 1230, vii+141 pp. 1
29. K. Li, C. Pérez, Isreal P. Rivera-Rios and L. Roncal, Weighted norm inequalities for rough singular integral operators, J. Geom. Anal. https://doi.org/10.1007/s12220-018-0085-4. 2, 5, 20
30. R. Oberlin, Estimates for compositions of maximal operators with singular integrals, Canad. Math. Bull. **56** (2013), 801-813. 1
31. D. H. Phone and E. M. Stein, Some further classes of pseudo-differential and singular integral operators arising in boundary-value problems I, composition of operators, Amer. J. Math. **104** (1982), 141-172. 1, 3
32. F. Ricci and G. Weiss, A characterization of $H^1(S^{n-1})$, Proc. Sympos. Pure Math. of Amer. Math. Soc., (S. Wainger and G. Weiss eds), Vol 35 I(1979), 289-294. 2
33. A. Seeger, Singular integral operators with rough convolution kernels, J. Amer. Math. Soc. **9** (1996), 95-105. 2, 19
34. R. S. Strichartz, Compositions of singular integral operators, J. Funct. Anal. **49** (1982), 91-127. 1
35. M. J. Wilson, Weighted inequalities for the dyadic square function without dyadic A_∞, Duke Math. J. **55** (1987), 19-50. 5

Guoen Hu: School of Applied Mathematics, Beijing Normal University, Zhuhai 519087, P. R. China
E-mail address: huguoen@yahoo.com

Xudong Lai: Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin, 150001, People's Republic of China
E-mail address: xudonglai@hit.edu.cn xudonglai@mail.bnu.edu.cn

Qingying Xue: School of Mathematics, Beijing Normal university, Beijing 100875, People's Republic of China
E-mail address: qyxue@bnu.edu.cn