A note on steady vortex flows in two dimensions

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Abstract

In this note, we give a general criterion for steady vortex flows in a planar bounded domain. More specifically, we show that if the stream function satisfies "locally" a semilinear elliptic equation with monotone or Lipschitz nonlinearity, then the flow must be steady.

Keywords: Euler equations, steady vortex flow, semilinear elliptic equation, stream function

1. Introduction and Main Result

Let $D\subset\mathbb{R}^2$ be a simply connected and bounded domain with a smooth boundary ∂D. The motion of an incompressible nonviscous fluid of unit density in D is governed by the following Euler dynamical equations:

$$\begin{cases}\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla P, & (x,t)\in D\times\mathbb{R}^+,\\ \nabla\cdot\mathbf{v}=0,\\ \mathbf{v}(x,0)=\mathbf{v}_0(x), & x\in D,\\ \mathbf{v}\cdot\mathbf{n}=0, & (x,t)\in\partial D\times\mathbb{R}^+,\end{cases}\tag{1.1}$$

where $\mathbf{v}=(v_1,v_2)$ is the velocity field, P is the scalar pressure and $\mathbf{n}$ denotes the outward unit normal of ∂D. Here we impose the impermeability condition on the boundary.

Define the vorticity of the fluid as $\omega:=\partial_1v_2-\partial_2v_1$. Since $\mathbf{v}$ is divergence-free, there is a function ψ, called the stream function, such that $\mathbf{v}=J\nabla\psi:=(\partial_2\psi,-\partial_1\psi)$, where $J(a,b)=(b,-a)$ denotes clockwise rotation through $\frac{\pi}{2}$ for any planar vector (a,b). It is easy to see that ω and ψ satisfy the following Poisson's equation:

$$-\Delta\psi=\omega.$$

By the impermeability boundary condition, we deduce that ψ is a constant on ∂D. Without loss of generality, by adding a suitable constant we assume that ψ vanishes on the boundary.

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Thus ψ can be determined by ω in the following way

$$\psi(x,t)=G\omega(x,t):=\int_D G(x,y)\omega(y,t)dy,\ x\in D,$$

where G is the Green function for $-\Delta$ in D with zero Dirichlet boundary condition.

Taking the curl on both sides of the first equation in (1.1), we obtain

$$\partial_t\omega+J\nabla G\omega\cdot\nabla\omega=0, \tag{1.2}$$

which is a nonlinear transport equation for ω and is usually called the vorticity equation.

In this paper, we are concerned with the steady vorticity equation, that is,

$$J\nabla G\omega\cdot\nabla\omega=0. \tag{1.3}$$

To motivate the definition of weak solutions to the steady vorticity equation, we multiply any $\phi\in C_c^\infty(D)$ on both sides of (1.3) and integrate by parts formally to obtain

$$\int_D \omega J\nabla G\omega\cdot\nabla\phi dx=0. \tag{1.4}$$

It is not difficult to check that the integral in (1.4) makes sense if $\omega\in L^{4/3}(D)$. In fact, for $\omega\in L^{\frac{4}{3}}(D)$, by L^p estimate we have $G\omega\in W^{2,\frac{4}{3}}(D)$, thus $G\omega\in W^{1,4}(D)$ by Sobolev embedding, therefore the integral in (1.4) makes sense by Hölder's inequality.

Definition 1.1. We call $\omega\in L^{4/3}(D)$ a weak solution to the steady vorticity equation (1.3) if it satisfies

$$\int_D \omega J\nabla G\omega\cdot\nabla\phi dx=0,\ \forall\,\phi\in C_c^\infty(D). \tag{1.5}$$

In the past several decades, various methods have been proposed to construct steady vortex flows. The most commonly used method is to investigate the following semilinear elliptic problem

$$\begin{cases}-\Delta\psi=f(\psi), & x\in D,\\ \psi=0, & x\in\partial D,\end{cases} \tag{1.6}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function. It is obvious that

$$J\nabla\psi\cdot\nabla f(\psi)=J\nabla\psi\cdot(f'(\psi)\nabla\psi)=0 \text{ a.e. in } D,$$

which means that any solution to (1.6) corresponds to a steady vortex flow in classical sense with ψ as the stream function. See [6][8][15][14] and the references listed therein. Another efficient way to construct steady vortex flows is called the vorticity method, which was first established by Arnold [1][2]. See also [3][4][9][10][11][12][16]. Roughly speaking, the vorticity method states that any steady vortex flow is equivalent to a critical point of the kinetic energy subject to some appropriate constraints for the vorticity. For example, Turkington in [16] considered the maximization of the kinetic energy

$$E(\omega) := \frac{1}{2}\int_D\int_D G(x,y)\omega(x)\omega(x)dxdy$$

in the admissible class

$$K_\lambda(D) := \{\omega \in L^\infty(D) \mid 0 \leq \omega \leq \lambda \text{ a.e. in } D,\ \int_D \omega(x)dx = 1\}. \tag{1.7}$$

Turkington proved the existence of a maximizer and showed that any maximizer ω^λ must be of the form

$$\omega^\lambda = \lambda I_{A^\lambda},\ A^\lambda = \{x \in D \mid G\omega^\lambda(x) > \mu^\lambda\}, \tag{1.8}$$

where I_{A^λ} denotes the characteristic function of A^λ, and μ^λ is a Lagrange multiplier depending on λ. To show that ω^λ is a steady solution to the vorticity equation (1.3), one can not use the form (1.8) anymore, since the nonlinearity here is a Heaviside function with discontinuity at μ^λ and the regularity of ∂A^λ is unknown(still an open question). To show that ω^λ satisfies (1.5), Turkington used the fact that ω^λ is an energy maximizer in $K_\lambda(D)$. See [16] for the detailed proof.

In [5], Burton proved that if ω belongs to $L^{4/3}(D)$ and satisfies $\omega = f(G\omega)$ a.e. in D, where $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is a monotone function, then ω is a weak solution to (1.3). By Burton's result, in order to obtain a steady vortex flow from (1.8), we do not need additional information about the energy of ω^λ.

Another example of steady vortex flows with vorticity concentrated in multiple separated regions is given in [7]. Therein the authors studied the following elliptic problem

$$\begin{cases} -\Delta\psi = \lambda\sum_{i=1}^k I_{A_i},\ x \in D, \\ A_i = B_\delta(x_{0,i}) \cap \{x \in D \mid \psi(x) > \kappa_i\}, \end{cases} \tag{1.9}$$

where I_{A_i} denotes the characteristic function of A_i, λ is a given positive number, κ_i is a real number depending on λ, $x_{0,i}$ is a given point in D, δ is a very small positive number such that $B_\delta(x_{0,i}) \subset\subset D$ and $\overline{B_\delta(x_{0,i})} \cap \overline{B_\delta(x_{0,j})} = \varnothing$ if $i \neq j$. The authors constructed a solution to (1.9) for sufficiently large λ, such that each A_i is a simply connected domain bounded by a C^1 closed curve and is strictly contained in $B_\delta(x_{0,i})$(or equivalently, $dist(A_i, \partial B_\delta(x_{0,i})) > 0$). To show that ψ satisfies (1.5), one can integrate by parts directly since each ∂A_i is C^1 and ψ is continuous across ∂A_i.

Notice that in (1.9) the vorticity $\omega = -\Delta\psi$ is no longer a function of the stream function ψ, since the k Lagrange multipliers $\kappa_i, \cdots, \kappa_k$ may be different numbers. However, the vorticity is a function of the stream function "locally".

Our aim in this note is to give a general criterion for solutions of the steady vorticity equation (1.3), that is, if the stream function satisfies "locally" a semilinear elliptic equation with monotone or Lipschitz nonlinearity, then the corresponding flow must be steady.

Before stating the theorem, we give some notations for clarity. We will use $supp(f)$ to denote the support of some function f, and the distance between two planar sets A and B is defined by

$$dist(A,B) := \inf_{x\in A, y\in B} |x-y|.$$

Let $\delta > 0$ be a positive number, the notation A_δ denotes the δ-neighbourhood in D of some planar set A, or equivalently,

$$A_\delta := \{x \in D \mid dist(x, A) < \delta\}.$$

Theorem 1.2. *Let k be a positive integer. Suppose that $\omega \in L^{4/3}(D)$ satisfies*

$$\omega = \sum_{i=1}^{k} \omega_i, \ \min_{1\le i<j\le k}\{dist(supp(\omega_i), supp(\omega_j))\} > 0, \ \omega_i = f^i(G\omega), a.e. \ in \ supp(\omega_i)_\delta \tag{1.10}$$

for some $\delta > 0$, where each f^i is either monotone from $\mathbb{R}$ to $\mathbb{R} \cup \{\pm\infty\}$ or Lipschitz continuous from $\mathbb{R}$ to $\mathbb{R}$, then ω is a weak solution to the steady vorticity equation (1.3)*.*

Remark 1.3. In the above theorem, if f_i is Lipschitz continuous from $\mathbb{R}$ to $\mathbb{R}$, then ω_i must be bounded since $G\omega \in L^\infty(D)$ by L^p estimate and Sobolev embedding.

Remark 1.4. Examples of steady vortex flows satisfying (1.10) can also be found in [12], where each $f^i : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is an unknown nondecreasing function.

2. Proof of the Main Result

In this section, we give the proof of Theorem 1.2. The basic idea is to approximate each f^i by a sequence of bounded Lipschitz functions.

For a function $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$, we shall use the following notation in the rest of this paper for convenience:

$$C_f := \{s \in \mathbb{R} \mid f \text{ is continuous at } s\},$$

$$D_f := \{s \in \mathbb{R} \mid f \text{ is not continuous at } s\}.$$

Lemma 2.1. *Suppose that $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is a monotone function, then there exists a sequence of bounded and smooth functions $\{f_n\}$ such that*

$$|f_n(s)| \le |f(s)|, \ \forall s \in \mathbb{R},$$

$$\lim_{n\to+\infty} f_n(s) = f(s), \ \forall s \in C_f.$$

Proof. Without loss of generality we assume that f is nondecreasing and bounded(we can use truncation if f is unbounded).

First we consider the case f is nonnegative. Let ρ be the standard mollifier of one dimension, that is,

$$\rho(s)=\begin{cases} c_0 e^{-\frac{1}{1-s^2}}, & |s|<1,\\ 0, & |s|\geq 1,\end{cases}$$

where c_0 is a positive number such that $\int_{\mathbb{R}}\rho(s)ds=1$. Note that $\rho\in C_c^\infty(\mathbb{R})$. Define

$$f^\varepsilon(s)=\int_{-\infty}^{+\infty}\rho_\varepsilon(s-\varepsilon-r)f(r)dr=\int_{s-2\varepsilon}^{s}\rho_\varepsilon(s-\varepsilon-r)f(r)dr,$$

where $\rho_\varepsilon(s):=\varepsilon^{-1}\rho(s\varepsilon^{-1})$ with $\varepsilon>0$ as a parameter. It is easy to check that $f^\varepsilon\in C^\infty(\mathbb{R})$.

Since f is nonnegative and nondecreasing, we have

$$|f^\varepsilon(s)|=\left|\int_{s-2\varepsilon}^{s}\rho_\varepsilon(s-\varepsilon-r)f(r)dr\right|\leq\left|\int_{s-2\varepsilon}^{s}\rho_\varepsilon(s-\varepsilon-r)f(s)dr\right|=|f(s)|,\ \forall\, s\in\mathbb{R}.$$

Moreover, for any $s\in C_f$,

$$\begin{aligned}|f^\varepsilon(s)-f(s)|&=\left|\int_{s-2\varepsilon}^{s}\rho_\varepsilon(s-\varepsilon-r)f(r)dr-f(s)\right|\\&=\left|\int_{s-2\varepsilon}^{s}\rho_\varepsilon(s-\varepsilon-r)(f(r)-f(s))dr\right|\\&\leq\sup_{r\in[s-2\varepsilon,s]}|f(r)-f(s)|\end{aligned}$$

which goes to 0 as $\varepsilon\to 0^+$. Thus we have proved the lemma for nonnegative f.

For the case f is non-positive, we can define

$$f^\varepsilon(s)=\int_{-\infty}^{+\infty}\rho_\varepsilon(s+\varepsilon-r)f(r)dr=\int_{s}^{s+2\varepsilon}\rho_\varepsilon(s+\varepsilon-r)f(r)dr,$$

then by repeating the above argument we can prove the lemma for non-positive f.

When f is a general nondecreasing function, we write $f=f^+-f^-$, where $f^+(s):=\max\{f(s),0\}$ and $f^-(s):=-\min\{f(s),0\}$. According to the above discussion, we can choose two sequences of smooth functions $\{f_n^+\}$ and $\{f_n^-\}$ such that

$$|f_n^+(s)|\leq|f^+(s)|,\ |f_n^-(s)|\leq|f^-(s)|,\ \forall\, s\in\mathbb{R},$$

$$\lim_{n\to+\infty}f_n^+(s)=f^+(s),\ \lim_{n\to+\infty}f_n^-(s)=f^-(s),\ \forall\, s\in C_f.$$

Here we used the fact that f^+ and f^- are both continuous on C_f. The lemma is proved by choosing $f_n=f_n^+-f_n^-$.

□

Proof of Theorem 1.2: For $i = 1, \cdots, k$, if f^i is a monotone function, by Lemma 2.1 we can choose a sequence of bounded and smooth functions $\{f_n^i\}$ such that

$$|f_n^i(s)| \leq |f^i(s)|, \ \forall\, s \in \mathbb{R}, \tag{2.1}$$

$$\lim_{n\to+\infty} f_n^i(s) = f^i(s), \ \forall\, s \in C_f. \tag{2.2}$$

If f^i is Lipschitz continuous, we can also choose a sequence of bounded Lipschitz functions $\{f_n^i\}$ satisfying (2.1)(2.2) by using truncation.

Since $\omega \in L^{4/3}(D)$, by L^p estimate we have $G\omega \in W^{2,4/3}(D)$, then by Sobolev embedding we obtain $G\omega \in W^{1,4}(D)$. By the chain rule for Sobolev functions(see [13], 4.22), it is easy to verify that

$$J\nabla G\omega \cdot \nabla(f_n^i(G\omega)) = (f_n^i)'(G\omega)J\nabla G\omega \cdot \nabla G\omega = 0 \text{ for a.e. } x \in supp(\omega_i)_\delta, \tag{2.3}$$

where we used the fact $J\nabla G\omega \cdot \nabla G\omega \equiv 0$. Since $|f_n^i(s)| \leq |f^i(s)|$ for each $s \in \mathbb{R}$ and $n = 1, 2, \cdots$, we deduce that

$$supp(f_n^i(G\omega)) \cap supp(\omega_i)_\delta \subset supp(\omega_i).$$

Define $\omega_n = \sum_{i=1}^k f_n^i(G\omega)I_{supp(\omega_i)_\delta}$. Taking into account (1.10) and (2.3), we can easily check that ω_n belongs to $W^{1,4}(D)$ and satisfies

$$|\omega_n| \leq \sum_{i=1}^k |f_n^i(G\omega)I_{supp(\omega_i)_\delta}| \leq \sum_{i=1}^k |f^i(G\omega)I_{supp(\omega_i)_\delta}| = |\omega| \ \text{ a.e. } x \in D, \tag{2.4}$$

$$J\nabla G\omega \cdot \nabla\omega_n = 0 \text{ a.e. in } D. \tag{2.5}$$

Therefore we obtain

$$\int_D \omega_n J\nabla G\omega \cdot \nabla\phi dx = 0, \ \forall\, \phi \in C_c^\infty(D). \tag{2.6}$$

Now we claim that

$$\lim_{n\to+\infty} \omega_n = \omega \text{ a.e. in } D. \tag{2.7}$$

In fact, it suffices to show that for each i

$$\lim_{n\to+\infty} f_n^i(G\omega(x)) = f^i(G\omega(x)) \text{ for a.e. } x \in supp(\omega_i)_\delta.$$

For $x \in (G\omega)^{-1}(C_{f^i})$, by (2.2) we have $f_n^i(G\omega(x)) \to f^i(G\omega(x))$. So we need just consider the case $x \in (G\omega)^{-1}(D_{f^i})$. Since each f^i is either monotone or Lipschitz continuous, the set D_{f^i} is countable, thus it suffices to show that for each $s \in D_{f^i}$ there holds

$$\lim_{n\to+\infty} f_n^i(G\omega(x)) = f^i(G\omega(x)) \text{ for a.e. } x \in (G\omega)^{-1}(s).$$

To show this, first we use the property of Sobolev functions(see [13], 4.22) to obtain

$$\omega = -\Delta G\omega = 0 \text{ a.e. on } (G\omega)^{-1}(s),$$

then by (2.4) we have

$$\omega_n = 0 \text{ a.e. on } (G\omega)^{-1}(s),$$

therefore

$$\lim_{n\to+\infty} \omega_n = \omega \text{ a.e. on } (G\omega)^{-1}(s).$$

Combining (2.4),(2.6) and (2.7), we are able to apply the dominated convergence theorem to obtain

$$\int_D \omega J\nabla G\omega \cdot \nabla\phi dx = 0,$$

which is the desired result.

□

Remark 2.2. According to the proof of Theorem 1.2, we need only impose the following two abstract conditions on f^i:

(1) D_{f^i} is a countable set;

(2) there exist a sequence of bounded Lipschitz functions $\{f_n^i\}$ and a constant $C > 0$ such that

$$|f_n^i(s)| \le C|f^i(s)|, \ \forall\, s \in \mathbb{R},$$
$$\lim_{n\to+\infty} f_n^i(s) = f^i(s), \ \forall\, s \in C_f.$$

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