MULTIPLE GROUND STATE SOLUTIONS WITH SIGN INFORMATION FOR DOUBLE PHASE ROBIN PROBLEMS

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ABSTRACT. We consider a nonlinear unbalanced double phase problem with a superlinear reaction and Robin boundary condition. Using suitable variants of the Nehari method, we show that the problem has three nontrivial solutions all with sign information (positive, negative and nodal).

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the following two phase Robin problem:

$$\begin{cases} -\operatorname{div}\left(a(z)|Du|^{p-2}Du + |Du|^{q-2}Du\right) + \xi(z)|u|^{p-2}u = f(z,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_\theta} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega, \quad 1 < q < p. \end{cases} \tag{1.1}$$

In this problem $a : \overline{\Omega} \to \mathbb{R}$ is a Lipschitz continuous map, $a(z) \geq 0$ for all $z \in \overline{\Omega}$. The potential function $\xi \in L^\infty(\Omega)$ and $\xi(z) \geq 0$ for a.a. $z \in \Omega$. The reaction (source) term $f(z,x)$ is a measurable function which is C^1 in the $x \in \mathbb{R}$ variable and $(p-1)$-superlinear near $\pm\infty$. In the boundary condition, $\frac{\partial u}{\partial n_\theta}$ is the conormal derivative of u corresponding to the modular function $\theta(z,x) = a(z)x^p + x^q$ for all $z \in \Omega$ and all $x \geq 0$. The conormal derivative is interpreted via the nonlinear Green's identity (see Papageorgiou-Radulescu-Repovs [26], p. 34) and

$$\frac{\partial u}{\partial n_\theta} = [a(z)|Du|^{p-2} + |Du|^{q-2}]\frac{\partial u}{\partial n} \quad \text{for all } u \in C^1(\overline{\Omega}),$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

The differential operator of (1.1) is related to the double phase functional

$$\rho_{pq}(u) = \int_\Omega [a(z)|Du|^p + |Du|^q]\, dz.$$

In this functional the integrant $G(z,y) = a(z)|y|^p + |y|^q$ for all $z \in \Omega$ and all $y \in \mathbb{R}^N$, exhibits unbalanced growth, namely it satisfies

$$|y|^q \leq G(z,y) \leq c_0[1 + |y|^p] \quad \text{for all } z \in \overline{\Omega}, \text{ all } y \in \mathbb{R}^N.$$

Marcellini [21] and Zhikov [30], [31] were the first to study such functionals, which provide models for describing a feature of strongly anisotropic materials and new examples of Lavrentiev phenomenon. The modulating coefficient $a(z) \geq 0$ dictates the geometry of the composite made by two different materials. More precisely, considering two different materials with power hardening exponents p and

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q, respectively, the variable coefficient $a(\cdot)$ dictates the geometry of a composite of the materials. In the region where $a(z)$ is positive, the p-material is present, otherwise the q-material is the only one making the composite.

More, recently Mingione and coworkers [2], [8], [9], produced some remarkable local regularity results for minimizers of two phase functionals. We also mention the recent works of Cencelj-Radulescu-Repovs [5] (on double phase Dirichlet problems with variable growth), Colasuonno-Squassina [7] (on eigenvalue problems for Dirichlet double phase operators) and Liu-Dai [17], [18] (on nodal solutions for double phase Dirichlet problems). A nice survey of some of the recent works on two phase equations, can be found in Mingione-Radulescu [22] and Radulescu [28].

On the other hand, as mentioned in [18], Eq. (1.1) is related to the non-Newtonian fluid. More precisely, consider a fluid that lies between two large plates of equal size, separated by a fixed distance h at each point, and the system was originally static. It is assumed that the above plate will move in the X direction at a constant speed of v. A thin layer of fluid that is tightly attached to the moving plate also moves at the same speed. The fluid near the moving plate has a higher velocity than the one far away from the plate, and the velocity drops to zero at the fixed plate. Let u denote the speed of this fluid at some layer l_1. Choose a layer of fluid such that it has distance Δy to l_1 and the speed is $u + \Delta u$. Letting $\Delta y \to 0$, according to Newton's law of viscosity, we know that

$$F = \mu A \frac{\partial u}{\partial y},$$

where F is internal friction with the opposite direction of u, μ is the viscosity coefficient, and A is the contact area between the plate and fluid. More generally, we have

$$\vec{F} = \mu A D u,$$

where Du is the shear rate. If the shear stress $\tau := \vec{F}/A$ and the shear rate is linear, passing through the origin, the fluid is called a Newtonian fluid. Otherwise, it is called a non-Newtonian fluid. Taking $\mu = |Du|^{q-2} + a(z)|Du|^{p-2}$, we have the following equation:

$$-\operatorname{div}(a(z)|Du|^{p-2}Du + |Du|^{q-2}Du) = \lambda f,$$

where $\lambda = 1/A$, and $f = -\operatorname{div} \vec{F}$.

Our work here extends the one by Papageorgiou-Radulescu [25] where the authors deal with a Robin problem driven by a nonhomogeneous differential operator, which includes as a special case the (p, q)-Laplacian (that is, the sum of a p-Laplacian and of a q-Laplacian) and with a reaction term which is superlinear. In their work the operator is z-independent (autonomous) and has balanced growth. For such operators the nonlinear regularity theory of Lieberman [16] and the nonlinear maximum principle of Pucci-Serrin [27] are available. However, in the present setting these tools are no longer valid and so our approach here is different and uses the Nehari method. To obtain meaningful results on double phase problems, it is crucial that a bound on the exponents (p, q) should to be imposed (see assumption H(a) in Section 2). Eventually we show the existence of three ground state solutions, all with sign information (positive, negative and nodal (sign changing)). Moreover, for the nodal solution we show that it changes sign only once (that is, it has two nodal domains).

Our main result is stated as follows.

Theorem 1.1. *If hypotheses* $\mathrm{H(a)}, \mathrm{H}_0, \mathrm{H}_1$ *(see Section 2) hold, then problem* (1.1) *has at least three nontrivial solutions*

$$\begin{aligned} &\hat{u} \in W^{1,\theta}(\Omega) \quad \text{with } \hat{u}(z) > 0 \quad \text{for a.a. } z \in \Omega, \\ &\hat{v} \in W^{1,\theta}(\Omega) \quad \text{with } \hat{v}(z) < 0 \quad \text{for a.a. } z \in \Omega, \\ &y_0 \in W^{1,\theta}(\Omega) \quad \text{nodal with two nodal domains.} \end{aligned}$$

2. Mathematical background and hypotheses

To deal with unbalanced two phase differential operators, we have to use Musielak-Orlicz-Sobolev spaces (see [24]). So, let $\theta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$\theta(z, x) = a(z)x^p + x^q \quad \text{for all } z \in \Omega, \text{ all } x \geq 0.$$

This is a generalized N-function (see Musielak [24]) and

$$\theta(z, 2x) \leq 2^p \theta(z, x) \quad \text{for all } z \in \Omega, \text{ all } x \geq 0.$$

This property is known as the (Δ_2)-property (see Musielak [24], p. 52). Then the Musielak-Orlicz space $L^\theta(\Omega)$ is defined by

$$L^\theta(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega \theta(z, |u|)\, dz < \infty \right\}.$$

This space is equipped with the Luxemburg norm

$$\|u\|_\theta = \inf \left\{ \lambda > 0 : \int_\Omega \theta\Big(z, \frac{|u|}{\lambda}\Big)\, dz \leq 1 \right\}.$$

Having $L^\theta(\Omega)$ we can define the corresponding Sobolev-type space $W^{1,\theta}(\Omega)$ by setting

$$W^{1,\theta}(\Omega) = \left\{ u \in L^\theta(\Omega) : |Du| \in L^\theta(\Omega) \right\}.$$

This is a "Musielak-Orlicz-Sobolev" space and it is furnished with the norm $\|\cdot\|$ defined by

$$\|u\| = \|u\|_\theta + \|Du\|_\theta.$$

Here by $\|Du\|_\theta$ we mean $\||Du|\|_\theta$. The spaces $L^\theta(\Omega)$ and $W^{1,\theta}(\Omega)$ are separable and uniformly convex (hence reflexive). If $\hat{\theta}(z, x)$ is another modular function, we say that "$\hat{\theta}$ is weaker than θ" and write $\hat{\theta} \preceq \theta$ if there exist constants $c_1, c_2 > 0$ and $h \in L^1(\Omega)$ such that

$$\hat{\theta}(z, x) \leq c_1 \theta(z, c_2 x) + h(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

Then we have

$$L^\theta(\Omega) \hookrightarrow L^{\hat{\theta}}(\Omega) \quad \text{and} \quad W^{1,\theta}(\Omega) \hookrightarrow W^{1,\hat{\theta}}(\Omega).$$

Using the above inclusions and the classical Sobolev embedding theorem we infer the following embeddings of spaces.

(a) If $q \neq n$, then $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ for all $1 \leq r \leq q^* = \begin{cases} \frac{Nq}{N-q} & \text{if } q < N \\ +\infty & \text{if } N \leq q \end{cases}$.

(b) If $q = N$, then $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ for all $1 \leq r < \infty$.

(c) If $q \leq N$, then $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ compactly for all $1 \leq r < q^*$.

(d) If $q > N$, $W^{1,\theta}(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly.

(e) $W^{1,\theta}(\Omega) \hookrightarrow W^{1,q}(\Omega)$.

In addition, we will use the weighted Lebesgue space $L^p_a(\Omega)$ defined by

$$L^p_a(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable}, \|u\|_{p,a} = \left[\int_\Omega a(z)|u|^p \, dz \right]^{1/p} < \infty \right\}.$$

We have

$$L^p(\Omega) \hookrightarrow L^\theta(\Omega) \hookrightarrow L^p_a(\Omega) \cap L^q(\Omega).$$

Another quantity that will be useful in our arguments, is the modular integral functional $\rho_\theta(y)$ defined by

$$\rho_\theta(y) = \int_\Omega \theta(z, |y|) \, dz = \int_\Omega [a(z)|y|^p + |y|^q] \, dz.$$

Remark 2.1. *The "Luxemburg norm" is basically due to Luxemburg* [20]*, which is used to introduce a norm on such "Orlicz-type spaces", where the modular function is not positively homogeneous (see also* [1, 13]*). In fact, it is more generally known as the Minkowski functional of the set* $\{u \in W^{1,\theta}(\Omega) : \rho_\theta(u) \le 1\}$. *Minkowski functionals are well-known from the theory of locally convex spaces.*

The norm $\|\cdot\|$ and $\rho_\theta(\cdot)$ are closely related.

Proposition 2.2. (a) *If* $y \ne 0$, *then* $\|y\|_\theta = \lambda$ *if and only if* $\rho_\theta\left(\frac{y}{\lambda}\right) = 1$.

(b) $\|y\|_\theta < 1$ (resp. $> 1, = 1$) *if and only if* $\rho_\theta(y) < 1$ (resp. $> 1, = 1$).

(c) $\|y\|_\theta < 1 \Rightarrow \|y\|^p_\theta \le \rho_\theta(y) \le \|y\|^q_\theta$.

(d) $\|y\|_\theta > 1 \Rightarrow \|y\|^q_\theta \le \rho_\theta(y) \le \|y\|^p_\theta$.

(e) $\|y\|_\theta \to 0$ *if and only if* $\rho_\theta(y) \to 0$.

(f) $\|y\|_\theta \to +\infty$ *if and only if* $\rho_\theta(y) \to +\infty$.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$ and using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^s(\partial\Omega)$ $(1 \le s \le \infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,q}(\Omega) \to L^q(\partial\Omega)$, known as the "trace map" such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,q}(\Omega) \cap C(\overline{\Omega})$. We know that

$$\operatorname{im} \gamma_0 = W^{\frac{1}{q'},q}(\partial\Omega) \left(\frac{1}{q} + \frac{1}{q'} = 1 \right) \quad \text{and} \quad \ker \gamma_0 = W^{1,q}_0(\Omega).$$

Moreover, the trace map γ_0 is compact into $L^s(\partial\Omega)$ for all $s \in [1, \frac{(N-1)q}{N-q})$ if $q < N$ and for all $s \in [1, +\infty)$ if $q \ge N$. In the sequel for notational economy we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

In the study of problem (1.1), we will also use some basic tools of Nonsmooth Analysis. More precisely, we will use the subdifferential theory for locally Lipschitz functions due to Clarke [6]. So, let us briefly recall some basic definitions and facts from that theory. Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for pair (X^*, X). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, we can find an open neighborhood $U(x)$ of x and a constant $k_x > 0$ such that

$$|\varphi(y) - \varphi(u)| \le k_x \|y - u\|_X \quad \text{for all } y, u \in U(x).$$

Here by $\|\cdot\|_X$ we denote the norm of X. If $\varphi : X \to \mathbb{R}$ is Lipschitz continuous on every bounded set in X, then φ is locally Lipschitz and if X is finite dimensional, then the converse is also true. A continuous convex function $\varphi : X \to \mathbb{R}$ and a function $\varphi \in C^1(X, \mathbb{R})$ are both locally Lipschitz.

For a locally Lipschitz function $\varphi(\cdot)$, we define

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $\varphi^0(x; \cdot)$ is sublinear (that is, subadditive and positively homogeneous) and Lipschitz continuous. So, we can define the set

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x; h) \quad \text{for all } h \in X\}.$$

The Hahn-Banach theorem implies that for all $x \in X, \partial\varphi(x) \neq \emptyset$ and it is convex and w^*-compact. The multifunction $x \to \partial\varphi(x)$ is known as the "subdifferential" of φ at $x \in X$. If $\varphi(\cdot)$ is continuous, convex, then it coincides with the subdifferential in the sense of Convex Analysis (see Gasinski-Papageorgiou [14], p. 523). Also, if $\varphi \in C^1(X, \mathbb{R})$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

If $x \in \mathbb{R}$, then $x^{\pm} = \max\{\pm x, 0\}$. Given $u \in W^{1,\theta}(\Omega)$ we define $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. Then $u^{\pm} \in W^{1,\theta}(\Omega)$ and $u = u^+ - u^-, |u| = u^+ + u^-$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and if $\varphi \in C^1(W^{1,\theta}(\Omega))$, then

$$K_\varphi = \left\{u \in W^{1,\theta}(\Omega) : \varphi'(u) = 0\right\} \quad \text{(the critical set of } \varphi).$$

Now we can introduce our hypotheses on the data of (1.1).

<u>H(a)</u>: $a : \overline{\Omega} \to \mathbb{R}_+$ is Lipschitz continuous and $q > \frac{Np}{N+p-1}$ (i.e., $\frac{p}{q} < 1 + \frac{p-1}{N}$).

<u>H_0</u>: $\xi \in L^\infty(\Omega), \xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and either $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

<u>H_1</u>: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$ $f(z, 0) = 0, f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \leq c_3[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ and with $c_3 > 0$, $p < r < q^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)\, ds$, then $\lim_{x \to \pm\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) $(p - 1)f(z, x)x \leq f'_x(z, x)x^2$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$;

(iv) $f(z, x)x \geq \eta(z)|x|^p$ for a.a. $z \in \Omega$ with $\eta \in L^\infty(\Omega)$ such that

$$\int_\Omega \xi(z)\, dz + \int_{\partial\Omega} \beta(z)\, d\sigma < \int_\Omega \eta(z)\, dz$$

and

$$\lim_{x \to 0} \frac{f(z, x)}{|x|^{q-2}x} = 0 \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 2.3. *The condition on the exponents p, q implies that $W^{1,\theta}(\Omega) \hookrightarrow L^p(\partial\Omega)$ compactly. Also we have $p < q^*$.*

Remark 2.4. *This hypothesis incorporates in our setting Neumann problems when $\beta \equiv 0$.*

Remark 2.5. *Let* $e(z,x) = f(z,x)x - pF(z,x)$. *On account of hypothesis* H_1(iii), *for a.a.* $z \in \Omega$ $e(z,\cdot)$ *is nondecreasing on* $\mathbb{R}_+$ *and nonincreasing on* $\mathbb{R}_-$. *These facts and hypothesis* H_1(ii) *imply that*

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \quad \textit{uniformly for a.a. } z \in \Omega.$$

Hence the reaction of problem (1.1) *is* $(p-1)$*-superlinear near* $\pm\infty$. *Another consequence of hypothesis* H_1(iv) *is that for a.a.* $z \in \Omega$ $x \to \frac{f(z,x)x}{|x|^p}$ *is nondecreasing in* $|x|$.

Let $\varphi : W^{1,\theta}(\Omega) \to \mathbb{R}$ be the energy functional of problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in W^{1,\theta}(\Omega),$$

where

$$\gamma_p(u) = \int_\Omega a(z)|Du|^p\,dz + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma.$$

Since we will be looking for constant sign solutions of (1.1), we will also need the following truncations of $\varphi(\cdot)$

$$\varphi_\pm(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,\pm u^\pm)\,dz.$$

Note that $\varphi \in C^2(W^{1,\theta}(\Omega))$ and $\varphi_\pm \in C^1(W^{1,\theta}(\Omega))$.

As we already mentioned in the Introduction, we will follow the Nehari method. For this reason we introduce the Nehari manifold $N \subseteq W^{1,\theta}(\Omega)$ defined by

$$N = \left\{u \in W^{1,\theta}(\Omega) : \langle \varphi'(u), u\rangle = 0, u \neq 0\right\}.$$

On account of hypothesis H_1(iv), N does not contain constant functions. Indeed, if $c \in \mathbb{R}\backslash\{0\}$, then

$$\begin{aligned}\langle \varphi'(c), c\rangle &= |c|^p \left[\int_\Omega \xi(z)\,dz + \int_{\partial\Omega} \beta(z)\,d\sigma\right] - \int_\Omega f(z,c)c\,dz \\ &\leq |c|^p \left[\int_\Omega \xi(z)\,dz + \int_{\partial\Omega} \beta(z)\,d\sigma - \int_\Omega \eta(z)\,dz\right] < 0.\end{aligned}$$

To produce constant sign ground state solutions for problem (1.1), we introduce also the following Nehari type manifolds

$$\begin{aligned}N_+ &= \left\{u \in W^{1,\theta}(\Omega) : u \geq 0, u \neq 0, \langle \varphi_+'(u), u\rangle = 0\right\}, \\ N_- &= \left\{v \in W^{1,\theta}(\Omega) : v \leq 0, v \neq 0, \langle \varphi_-'(v), v\rangle = 0\right\}.\end{aligned}$$

Clearly $N_\pm$ do not contain constant functions (see hypothesis H_1(iv)).

Since our aim is also to produce nodal solutions, following Liu-Wang-Wang [19], we introduce also the set

$$N_0 = \left\{u \in W^{1,\theta}(\Omega) : u^+ \in N, u^- \in -N\right\}.$$

Finally we mention that by a "solution" (weak solution) of (1.1) we mean a function $u \in W^{1,\theta}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega a(z)|Du|^{p-2}(Du,Dh)_{\mathbb{R}^N}\,dz + \int_\Omega |Du|^{q-2}(Du,Dh)_{\mathbb{R}^N}\,dz + \int_\Omega \xi(z)|u|^{p-2}uh\,dz \\ &\quad + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh\,d\sigma = \int_\Omega f(z,u)h\,dz \quad \text{for all } h \in W^{1,\theta}(\Omega).\end{aligned}$$

3. Constant sign solutions

In this section we produce two constant sign solutions (positive and negative) which minimize the energy functional $\varphi(\cdot)$ on the Nehari manifold N (ground state solutions).

Proposition 3.1. *If hypotheses* $\mathrm{H(a)}, \mathrm{H}_0, \mathrm{H}_1$ *hold and* $u \in W^{1,\theta}(\Omega)$*,* u *is not constant (that is,* $u \notin \mathbb{R}$*), then there exists a unique* $t_0 = t_0(u) > 0$ *such that* $t_0 u \in N$*.*

Proof. Consider the function

$$k(t) = t^{p-1}\gamma_p(u) + t^{q-1}\|Du\|_q^q - \int_\Omega f(z,tu)u\,dz, \quad t > 0.$$

Recall that on account of hypothesis H_1(iii) for a.a. $z \in \Omega$ we have that

$$x \to \frac{f(z,x)x}{|x|^p}$$

is nondecreasing in $|x| > 0$. So, if $t \in (0,1)$, then

$$\begin{aligned} &\frac{f(z,tu)(tu)}{t^p|u|^p} \le \frac{f(z,u)u}{|u|^p}, \\ &\Rightarrow f(z,tu)u \le t^{p-1} f(z,u)u. \end{aligned} \tag{3.1}$$

We have

$$\begin{aligned} &k(t) \ge t^{q-1}\|Du\|_q^q - t^{p-1}\int_\Omega f(z,u)u\,dz \\ &\qquad \text{(see (3.1) and hypotheses } \mathrm{H(a)}, \mathrm{H}_0) \\ &\Rightarrow k(t) > 0 \quad \text{for all } t \in (0,1) \text{ small} \quad \text{(recall } q < p). \end{aligned} \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} &\frac{k(t)}{t^{p-1}} = \gamma_p(u) + \frac{1}{t^{p-q}}\|Du\|_q^q - \int_\Omega \frac{f(z,tu)}{t^{p-2}} u\,dz, \\ &\Rightarrow \lim_{t\to+\infty} \frac{k(t)}{t^{p-1}} = -\infty \quad \text{(recall that for a.a. } z \in \Omega, f(z,\cdot) \text{ is } (p-1)\text{-superlinear)} \\ &\Rightarrow k(t) < 0 \quad \text{for } t > 0 \text{ big.} \end{aligned} \tag{3.3}$$

From (3.2), (3.3) and Bolzano's theorem, we infer that there exists $t_0 > 0$ such that

$$\begin{aligned} &k(t_0) = 0, \\ &\Rightarrow \gamma_p(t_0 u) + \|D(t_0 u)\|_q^q = \int_\Omega f(z,t_0 u)(t_0 u)\,dz, \\ &\Rightarrow t_0 u \in N. \end{aligned}$$

Note that

$$k(t) = 0 \quad \text{if and only if} \quad \gamma_p(u) = \int_\Omega \frac{f(z,tu)u}{t^{p-1}}\,dz - \frac{1}{t^{p-q}}\|Du\|_q^q.$$

Hypothesis H_1(iii) and the fact that u is not constant imply that the right hand side of the above equivalence is strictly increasing. Therefore $t_0 = t_0(u)$ is unique. □

Proposition 3.2. *If hypotheses* $\mathrm{H(a)}, \mathrm{H}_0, \mathrm{H}_1$ *hold, then* $\varphi|_N$ *is coercive.*

Proof. We need to show that if

$$\{u_n\}_{n\geq 1}\subseteq N \quad \text{and} \quad \varphi(u_n)\leq M_1 \text{ for some } M_1>0,\ \text{all } n\in\mathbb{N},$$

then $\{u_n\}_{n\geq 1}\subseteq W^{1,\theta}(\Omega)$ is bounded.

We have

$$\gamma_p(u_n)+\frac{p}{q}\|Du_n\|_q^q-\int_\Omega pF(z,u_n)\,dz\leq pM_1 \quad \text{for all } n\in\mathbb{N}. \tag{3.4}$$

Also since $u_n\in N$, we have

$$\begin{aligned}&\langle\varphi'(u_n),u_n\rangle=0,\\ \Rightarrow &-\gamma_p(u_n)-\|Du_n\|_q^q+\int_\Omega f(z,u_n)u_n\,dz=0 \quad \text{for all } n\in\mathbb{N}.\end{aligned} \tag{3.5}$$

We add (3.4) and (3.5) and recalling that $q<p$, we obtain

$$\int_\Omega e(z,u_n)\,dz\leq pM_1 \quad \text{for all } n\in\mathbb{N}. \tag{3.6}$$

<u>Claim</u>: $\{u_n\}_{n\geq 1}\subseteq W^{1,\theta}(\Omega)$ is bounded.

Arguing by contradiction, suppose that by passing to a suitable subsequence if necessary, we may assume that

$$\|u_n\|\to\infty.$$

Let $y_n=\frac{u_n}{\|u_n\|}$ $n\in\mathbb{N}$. Then $\|y_n\|=1$ for all $n\in\mathbb{N}$ and so we may assume that

$$y_n\xrightarrow{w} y \text{ in } W^{1,\theta}(\Omega) \quad \text{and} \quad y_n\to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \tag{3.7}$$

(see hypothesis H(a)).

First assume that $y\neq 0$ and let $\hat{\Omega}=\{z\in\Omega : |y(z)|>0\}$. Then we have $|\hat{\Omega}|_N>0$ and from (3.7) we infer that

$$\begin{aligned}&|u_n(z)|\to+\infty \quad \text{for a.a. } z\in\hat{\Omega},\\ \Rightarrow\ &\frac{F(z,u_n(z))}{|u_n(z)|^p}\to+\infty \quad \text{for a.a. } z\in\hat{\Omega} \quad (\text{see hypothesis } H_1(\text{ii})).\end{aligned} \tag{3.8}$$

Then (3.8), hypothesis H_1(ii) and Fatou's lemma imply that

$$\int_{\hat{\Omega}}\frac{F(z,u_n)}{\|u_n\|^p}\,dz\to+\infty \quad \text{as } n\to\infty. \tag{3.9}$$

Hypotheses H_1(i), (ii) imply that we can find $M_2>0$ such that

$$F(z,x)\geq -M_2 \quad \text{for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R}. \tag{3.10}$$

Then we have

$$\begin{aligned}\int_\Omega\frac{F(z,u_n)}{\|u_n\|^p}\,dz&=\int_{\hat{\Omega}}\frac{F(z,u_n)}{\|u_n\|^p}\,dz+\int_{\Omega\setminus\hat{\Omega}}\frac{F(z,u_n)}{\|u_n\|^p}\,dz\\ &\geq\int_{\hat{\Omega}}\frac{F(z,u_n)}{\|u_n\|^p}\,dz-\frac{M_2}{\|u_n\|^p}|\Omega|_N \quad (\text{see } (3.10)),\\ \Rightarrow\int_\Omega\frac{F(z,u_n)}{\|u_n\|^p}\,dz&\to+\infty \quad \text{as } n\to\infty \quad (\text{see } (3.9)),\\ \Rightarrow\int_\Omega\frac{f(z,u_n)u_n}{\|u_n\|^p}\,dz&\to+\infty \quad \text{as } n\to\infty \quad (\text{see hypothesis } H_1(\text{iii})).\end{aligned} \tag{3.11}$$

However, from (3.5) we have

$$\int_\Omega \frac{f(z,u_n)u_n}{\|u_n\|^p}\,dz = \gamma_p(y_n) + \frac{1}{\|u_n\|^{p-q}}\|Dy_n\|_q^q \le M_3 \tag{3.12}$$

for some M_3, all $n \in \mathbb{N}$.

Comparing (3.11) and (3.12), we have a contradiction.

Now assume that $y = 0$. Let $k \ge 1$ and define

$$v_n = (pk)^{1/p} y_n \quad \text{for all } n \in \mathbb{N}.$$

We have

$$v_n \xrightarrow{w} 0 \text{ in } W^{1,\theta}(\Omega) \quad \text{and} \quad v_n \to 0 \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \tag{3.13}$$

(see (3.7) and recall that $y = 0$).

From (3.13) ir follows that

$$\int_\Omega F(z,v_n)\,dz \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Let $\hat{\varphi}(u) = \frac{1}{p}[\gamma_p(u) + \|Du\|_q^q] - \int_\Omega F(z,u)\,dz$ for all $u \in W^{1,\theta}(\Omega)$. Note that $\hat{\varphi} \le \varphi$ (since $q < p$). Let $t_n \in [0,1]$ be such that

$$\hat{\varphi}(t_n u_n) = \max\,[\hat{\varphi}(tu_n) : 0 \le t \le 1]. \tag{3.15}$$

Since $\|u_n\| \to \infty$, we can find $n_0 \in \mathbb{N}$ such that

$$0 < \frac{(pk)^{1/p}}{\|u_n\|} \le 1 \quad \text{for all } n \ge n_0. \tag{3.16}$$

From (3.15) and (3.16) it follows that

$$\begin{aligned}
\hat{\varphi}(t_n u_n) &\ge \hat{\varphi}(v_n) \\
&= k\gamma_p(y_n) + \frac{k^{q/p}}{p^{\frac{p-q}{p}}}\|Dy_n\|_q^q - \int_\Omega F(z,v_n)\,dz \\
&\ge \frac{k^{q/p}}{p^{\frac{p-q}{p}}}\left[\rho_\theta(|Dy_n|) + \int_\Omega \xi(z)|y_n|^p\,dz + \int_{\partial\Omega}\beta(z)|y_n|^p\,d\sigma\right] \\
&\quad - \int_\Omega F(z,v_n)\,dz.
\end{aligned}$$

If $\rho_\theta(|Dy_n|) \ge 1$, then

$$\hat{\varphi}(t_n u_n) \ge \frac{k^{q/p}}{p^{\frac{p-q}{p}}} - \int_\Omega F(z,v_n)\,dz \quad \text{for all } n \ge n_0.$$

If $\rho_\theta(|Dy_n|) < 1$, then $\rho_\theta(|Dy_n|) \ge \|Dy_n\|_\theta^p$ and so

$$\begin{aligned}
\hat{\varphi}(t_n u_n) &\ge \frac{k^{q/p}}{p^{\frac{p-q}{p}}}\left[\|Dy_n\|_\theta^p + \int_\Omega \xi(z)|y_n|^p\,dz + \int_{\partial\Omega}\beta(z)|y_n|^p\,d\sigma\right] \\
&\quad - \int_\Omega F(z,v_n)\,dz \\
&\ge \frac{k^{q/p}}{p^{\frac{p-q}{p}}} c_4 \|y_n\|^p - \int_\Omega F(z,v_n)\,dz \quad \text{for some } c_4 > 0 \\
&\qquad \text{(see [23], Lemma 4.11 and [15], Proposition 2.4)}
\end{aligned}$$

$$= \frac{k^{q/p}}{p^{\frac{p-q}{p}}} c_4 - \int_\Omega F(z, v_n)\, dz \quad \text{for all } n \geq n_0.$$

So, finally we can say that

$$\hat{\varphi}(t_n u_n) \geq \frac{k^{q/p}}{p^{\frac{p-q}{p}}} c_5 - \int_\Omega F(z, v_n)\, dz \quad \text{with } c_5 = \min\{1, c_4\}, \text{ all } n \geq n_0,$$

$$\Rightarrow \hat{\varphi}(t_n u_n) \geq \frac{k^{q/p}}{2p^{\frac{p-q}{p}}} c_5 \quad \text{for all } n \geq n_1 \geq n_0 \quad \text{(see (3.14))}.$$

Since $k \geq 1$ is arbitrary we infer that

$$\hat{\varphi}(t_n u_n) \to +\infty \quad \text{as } n \to \infty. \tag{3.17}$$

Note that

$$\hat{\varphi}(0) = 0 \quad \text{and} \quad \hat{\varphi}(u_n) \leq \varphi(u_n) \leq M_1 \quad \text{for all } n \in \mathbb{N} \quad \text{(see (3.4))}. \tag{3.18}$$

Then (3.17) and (3.18) imply that

$$t_n \in (0, 1) \quad \text{for all } n \geq n_2. \tag{3.19}$$

But this contradicts the fact that $u_n \in N$ for all $n \in \mathbb{N}$ (see Proposition 3.1). Therefore the Claim is true and so we conclude that $\varphi|_N$ is coercive. □

On account of hypothesis H_1(iii) we have $F \geq 0$. Therefore

$$\varphi \leq \varphi_\pm.$$

So from Proposition 3.2 we infer the following Corollary.

Corollary 3.3. *If hypotheses* H(a), H_0, H_1 *hold, then* $\varphi_\pm|_N$ *are both coercive.*

Proposition 3.4. *If hypotheses* H(a), H_0, H_1 *hold and* $u \in N$, *then* $\varphi(tu) \leq \varphi(u)$ *for all* $t > 0$.

Proof. Consider the fibering function

$$\tau_u(t) = \varphi(tu) \quad \text{for all } t > 0.$$

Since $u \in N$, with $k(\cdot)$ as in the proof of Proposition 3.1, we have

$$\tau_u'(1) = k(1) = 0 \tag{3.20}$$

and this is the unique critical point of $\tau_u(\cdot)$ (see Proposition 3.1 and its proof).

On account of hypotheses H_1(i), (ii), we see that given any $\mu > 0$, we can find $c_\mu > 0$ such that

$$F(z, x) \geq \frac{\mu}{p}|x|^p - c_\mu \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.21}$$

Therefore for $t > 0$ we have

$$\begin{aligned} \tau_u(t) &= \varphi(tu) \\ &\leq \frac{t^p}{p}[\gamma_p(u) - \mu\|u\|_p^p] + \frac{t^q}{q}\|Du\|_q^q + c_\mu|\Omega|_N \quad \text{(see (3.21))}. \end{aligned}$$

Choosing $\mu > 0$ big, we obtain

$$\tau_u(t) \leq c_5 t^q - c_6 t^p + c_\mu|\Omega|_N \quad \text{for some } c_5, c_6 > 0.$$

Since $q < p$, it folows that

$$\tau_u(t) < 0 \quad \text{for } t > 0 \text{ big}. \tag{3.22}$$

On the other hand hypotheses H_1(i), (iv) imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z,x) \leq \frac{\varepsilon}{q}|x|^q + c_\varepsilon |x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

So, we have

$$\tau_u(t) = \varphi(tu) \geq \frac{t^p}{p}\gamma_p(u) + \frac{t^q}{q}[\|Du\|_q^q - \varepsilon\|u\|_q^q] - c_\varepsilon t^r \|u\|_r^r.$$

Recall that $u \in N$. So, as we established earlier (see the last part of Section 2), u is not constant. Hence choosing $\varepsilon > 0$ small, we have

$$\tau_u(t) \geq c_7 t^p - c_8 t^r \quad \text{for some } c_7, c_8 > 0.$$

Since $p < r$, we see that

$$\tau_u(t) > 0 \quad \text{for } t \in (0,1) \text{ small.} \tag{3.23}$$

Then from (3.22) and (3.23) we see that we can find $t_0(u) > 0$ local maximizer of $\tau_u(\cdot)$. Therefore $\tau_u'(t_0) = 0$. But from the uniqueness of the critical point of $\tau_u(\cdot)$ and (3.20), we have

$$t_0(u) = 1 \text{ and this is a global maximizer of } \tau_u(\cdot).$$

It follows that

$$\begin{aligned} \tau_u(t) &\leq \tau_u(1) \quad \text{for all } t > 0, \\ \Rightarrow \varphi(tu) &\leq \varphi(u) \quad \text{for all } t > 0. \end{aligned}$$

□

Next we show that N is bounded away from the origin.

Proposition 3.5. *If hypotheses* $\mathrm{H(a)}, \mathrm{H}_0, \mathrm{H}_1$ *hold, then there exists* $\hat{\delta} > 0$ *such that* $\hat{\delta} \leq \|u\|$ *for all* $u \in N$.

Proof. From hypotheses H_1(i), (iv) we know that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$f(z,x)x \leq \varepsilon|x|^p + c_\varepsilon|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.24}$$

Let $u \in W^{1,\theta}(\Omega)$, u is not a constant and $\|u\| = 1$. On account of Proposition 3.1, we can find a unique $t_u > 0$ such that $t_u u \in N$. Suppose $t_u < 1$. We have

$$\begin{aligned} & t_u^p \gamma_p(u) + t_u^q \|Du\|_q^q = \int_\Omega f(z, t_u u)(t_u u)\,dz, \\ \Rightarrow\, & t_u^p \left[\rho_\theta(|Du|) + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma\right] \leq \varepsilon c_9 t_u^p + c_{10} t_u^r \\ & \text{for some } c_9 > 0, c_{10} = c_{10}(\varepsilon) > 0 \quad (\text{recall } q < p, t_u < 1 \text{ and use (3.24)}), \\ \Rightarrow\, & t_u^p \left[\|Du\|_\theta^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma\right] \leq \varepsilon c_9 t_u^p + c_{10} t_u^r \\ & (\text{since } \|Du\|_\theta \leq \|u\| = 1 \text{ and using Proposotion 2.2}), \\ \Rightarrow\, & c_{11} t_u^p \leq \varepsilon c_9 t_u^p + c_{10} t_u^r \quad \text{for some } c_{11} > 0 \\ & (\text{see [25], Lemma 4.11 and [15], Proposition 2.4}). \end{aligned}$$

Choosing $\varepsilon > 0$ small, we have

$$\frac{1}{2} c_{11} t_u^p \leq c_{10} t_u^r,$$

$$\Rightarrow \left[\frac{c_{11}}{2c_{10}}\right]^{\frac{1}{r-p}} \leq t_u. \tag{3.25}$$

Let $\partial B_1 = \{u \in W^{1,\theta}(\Omega) : \|u\| = 1\}$. Following Szulkin-Weth [29] (see also Liu-Dai [17]), we introduce the maps $\hat{m} : W^{1,\theta}(\Omega)\backslash\{0\} \to N$ and $m : \partial B_1 \to N$ defined by

$$\hat{m}(u) = t_u u \quad \text{and} \quad m = \hat{m}\big|_{\partial B_1}.$$

Then $\hat{m}(\cdot)$ is continuous and $m(\cdot)$ is a homeomorphism beween ∂B_1 and N with $m^{-1}(u) = \frac{u}{\|u\|}$ (see Szulkin-Weth [29], Proposition 8). Therefore N is bounded away from the origin (see (3.25)), that is, there exists $\hat{\delta} > 0$ such that

$$\|u\| \geq \hat{\delta} > 0 \quad \text{for all } u \in N.$$

□

In a similar fashion we can establish analogous results for the Nehari manifolds $N_\pm$.

Proposition 3.6. *If hypotheses* $\mathrm{H(a)}, \mathrm{H}_0, \mathrm{H}_1$ *hold, then*

(a) *for every* $u \in W^{1,\theta}(\Omega)$, $u \geq 0$, u *not constant, there exists unique* $t_0^+ = t_0^+(u) > 0$ *such that* $t_0^+ u \in N_+$*;*
(b) *for every* $v \in W^{1,\theta}(\Omega)$, $v \leq 0$, v *not constant, there exists unique* $t_0^- = t_0^-(v) > 0$ *such that* $t_0^- v \in N_-$*;*
(c) *for every* $u \in N_+$ *and every* $v \in N_-$*, we have*

$$\varphi_+(tu) \leq \varphi_+(u) \quad \textit{and} \quad \varphi_-(tv) \leq \varphi_-(v) \quad \textit{for all } t > 0;$$

(d) *there exist* $\hat{\delta}^\pm > 0$ *such that*

$$\hat{\delta}^+ \leq \|u\| \quad \textit{for all } u \in N_+,$$
$$\hat{\delta}^- \leq \|v\| \quad \textit{for all } v \in N_-.$$

We set $m_\pm = \inf\limits_{N_\pm} \varphi_\pm$.

Proposition 3.7. *If hypotheses* $\mathrm{H(a)}, \mathrm{H}_0, \mathrm{H}_1$ *hold, then we can find* $\hat{u} \in N_+$ *and* $\hat{v} \in N_-$ *such that*

$$\varphi_+(\hat{u}) = m_+ \quad \textit{and} \quad \varphi_-(\hat{v}) = m_-.$$

Proof. Let $\{u_n\}_{n\geq 1} \subseteq N_+$ be a minimizing sequence for $\varphi_+\big|_{N_+}$, that is, $\varphi_+(u_n) \downarrow m_+$. From Corollary 3.3, we know that $\{u_n\}_{n\geq 1} \subseteq W^{1,\theta}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \hat{u} \text{ in } W^{1,\theta}(\Omega) \quad \text{and} \quad u_n \to \hat{u} \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.26}$$

Evidently $\hat{u} \geq 0$. We claim that $\hat{u} \neq 0$. Arguing by contradiction, suppose that $\hat{u} = 0$. Since $u_n \in N_+$, we have

$$\gamma_p(u_n) + \|Du_n\|_q^q = \int_\Omega f(z, u_n)u_n\, dz \quad (\text{recall } u_n \geq 0) \quad \text{for all } n \in \mathbb{N}, \tag{3.27}$$
$$\Rightarrow \rho_\theta(|Du_n|) \to 0 \quad \text{as } n \to \infty \quad (\text{see (3.26) and recall } \hat{u} = 0),$$
$$\Rightarrow \|Du_n\|_\theta \to 0 \quad \text{as } n \to \infty \quad (\text{see Proposition 2.2}),$$
$$\Rightarrow u_n \to 0 \quad \text{in } W^{1,\theta}(\Omega) \text{ as } n \to \infty.$$

This contradicts Proposition 3.6 (d). Therefore $\hat{u} \neq 0$, $\hat{u} \geq 0$.

From (3.27) and (3.26), in the limit as $n \to \infty$, we obtain

$$\gamma_p(\hat{u}) + \|D\hat{u}\|_q^q \le \int_\Omega f(z,\hat{u})\hat{u}\,dz. \tag{3.28}$$

Assume that

$$\gamma_p(\hat{u}) + \|D\hat{u}\|_q^q < \int_\Omega f(z,\hat{u})\hat{u}\,dz. \tag{3.29}$$

Consider the function $\hat{k}_+ : (0,+\infty) \to \mathbb{R}$ defined by

$$\hat{k}_+(t) = t^{p-1}\gamma_p(\hat{u}) + t^{q-1}\|D\hat{u}\|_q^q - \int_\Omega f(z,t\hat{u})\hat{u}\,dz, \quad t>0.$$

As in the proof of Proposition 3.1, we have

$$\hat{k}_+(t) > 0 \text{ for } t \in (0,1) \text{ small (see (3.2))} \quad \text{and} \quad \hat{k}_+(1) < 0 \text{ (see (3.29))}.$$

Therefore $t_0^+ = t_0^+(\hat{u}) \in (0,1)$ (see Proposition 3.6 (a)). We have

$$\begin{aligned}
m_+ &\le \varphi_+(t_0^+\hat{u})\\
&= \frac{1}{p}\gamma_p(t_0^+\hat{u}) + \frac{1}{q}\|D(t_0^+\hat{u})\|_q^q - \int_\Omega F(z,t_0^+\hat{u})\,dz\\
&= \frac{1}{p}\left[\int_\Omega f(z,t_0^+\hat{u})(t_0^+\hat{u})\,dz - \|D(t_0^+\hat{u})\|_q^q\right] + \frac{1}{q}\|D(t_0^+\hat{u})\|_q^q\\
&\quad - \int_\Omega F(z,t_0^+\hat{u})\,dz \quad (\text{since } t_0^+\hat{u} \in N_+)\\
&= \int_\Omega \left[\frac{1}{p}f(z,t_0^+\hat{u})(t_0^+\hat{u}) - F(z,t_0^+\hat{u})\right]dz + \left[\frac{1}{q}-\frac{1}{p}\right]\|D(t_0^+\hat{u})\|_q^q\\
&< \int_\Omega \left[\frac{1}{p}f(z,\hat{u})\hat{u} - F(z,\hat{u})\right]dz + \left[\frac{1}{q}-\frac{1}{p}\right]\|D\hat{u}\|_q^q\\
&\qquad\qquad (\text{see hypothesis } \mathrm{H}_1(\mathrm{iv}) \text{ and recall that } q<p, t_0^+\hat{u} \in N_+)\\
&\le \liminf_{n\to\infty}\left[\int_\Omega\left(\frac{1}{p}f(z,u_n)u_n - F(z,u_n)\right)dz + \left(\frac{1}{q}-\frac{1}{p}\right)\|Du_n\|_q^q\right]\\
&= \liminf_{n\to\infty}\left[\frac{1}{p}\gamma_p(u_n) + \frac{1}{q}\|Du_n\|_q^q - \int_\Omega F(z,u_n)\,dz\right] \quad (\text{since } u_n \in N_+)\\
&= m_+,
\end{aligned}$$

a contradiction. Therefore (3.29) can not happen and finally we have

$$\varphi_+(\hat{u}) = m_+ \quad \text{with } \hat{u} \in N_+ \quad (\text{since } \theta(z,\cdot) \text{ is uniformly convex}).$$

Similarly, working this time with the functional φ_- and the Nehari manifold N_-, we show that there exists $\hat{v} \in W^{1,\theta}(\Omega)$ such that

$$\varphi_-(\hat{v}) = m_- \quad \text{with } \hat{v} \in N_-.$$

□

Proposition 3.8. *If hypotheses* $\mathrm{H}(a), \mathrm{H}_0, \mathrm{H}_1$ *hold and* $\hat{u} \in N_+, \hat{v} \in N_-$ *are as in Proposition 3.7, then* $\hat{u} \in K_{\varphi_+}, \hat{v} \in K_{\varphi_-}$.

Proof. Consider the function $e_+ : W^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$e_+(u) = \gamma_p(u) + \|Du\|_q^q - \int_\Omega f(z,u^+)u^+\,dz \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

Evidently $e_+(\cdot)$ is locally Lipschitz. According to Proposition 3.7, we have

$$\varphi_+(\hat{u}) = m_+ = \inf\left[\varphi_+(u) : e_+(u) = 0, \|u\| > 0, u \geq 0\right].$$

Invoking the nonsmooth multiplier rule of Clarke [6] (Theorem 10.47, p. 221), we can find $\mu \geq 0$ and $h^* \in -W_+^* =$ the negative dual cone of $W_+ = \{y \in W^{1,\theta}(\Omega) : y(z) \geq 0 \quad \text{for all } z \in \Omega\}$ such that

$$\begin{aligned} &0 \in \partial\left[\varphi_+ + \mu e_+\right](\hat{u}) + h^*, \\ \Rightarrow\, &0 \in \varphi_+'(\hat{u}) + \mu \partial\, e_+(\hat{u}) + h^*, \\ \Rightarrow\, &0 = \varphi_+'(\hat{u}) + \mu u^* + h^* \quad \text{for some } u^* \in \partial\, e_+(\hat{u}). \end{aligned}$$

Acting with $\hat{u} \in N_+$ we obtain

$$\begin{aligned} &\mu\langle u^*, \hat{u}\rangle + \langle h^*, \hat{u}\rangle = 0, \\ \Rightarrow\, &\mu\left[p\gamma_p(\hat{u}) + q\|D\hat{u}\|_q^q - \int_\Omega f(z,\hat{u})\hat{u}\,dz - \int_\Omega f_x'(z,\hat{u})\hat{u}^2\,dz\right] + \langle h^*, \hat{u}\rangle = 0 \end{aligned}$$

(see Clarke [6], p. 202).

On account of hypothesis H_1(iii) and Theorem 3.3 of Liu-Dai [18], we infer that $\mu = 0$ and $h^* = 0$. Therefore we have

$$\begin{aligned} &0 \in \partial\,\varphi_+(\hat{u}), \\ \Rightarrow\, &\varphi_+'(\hat{u}) = 0 \quad (\text{since } \varphi_+ \in C^1(W^{1,\theta}(\Omega)), \\ \Rightarrow\, &\hat{u} \in K_{\varphi_+}. \end{aligned}$$

In a similar fashion using this time the functional φ_- and the Nehari manifold N_-, we show that $\hat{v} \in K_{\varphi_-}$. □

Now we are ready to produce two constant sign ground state solutions for problem (1.1).

Proposition 3.9. *If hypotheses* $H(a), H_0, H_1$ *hold, then problem* (1.1) *has two nontrivial ground state solutions*

$$\begin{aligned} &\hat{u} \in W^{1,\theta}(\Omega), \quad \hat{u}(z) > 0 \quad \text{for a.a. } z \in \Omega, \\ &\hat{v} \in W^{1,\theta}(\Omega), \quad \hat{v}(z) < 0 \quad \text{for a.a. } z \in \Omega. \end{aligned}$$

Proof. Let $\hat{u} \in N_+$ be such that $\varphi_+(\hat{u}) = m_+$ (see Proposition 3.7). From Proposition 3.8 we know that $\hat{u} \in K_{\varphi_+}$. Hence we have

$$\begin{aligned} &\varphi_+'(\hat{u}) = 0, \\ \Rightarrow\, &\langle \varphi_+'(\hat{u}), h\rangle = 0 \quad \text{for all } h \in W^{1,\theta}(\Omega), \\ \Rightarrow\, &\int_\Omega a(z)|D\hat{u}|^{p-2}(D\hat{u}, Dh)_{\mathbb{R}^N}\,dz + \int_\Omega |D\hat{u}|^{q-2}(D\hat{u}, Dh)_{\mathbb{R}^N}\,dz + \int_\Omega \xi(z)\hat{u}^{p-1}h\,dz \\ &\quad + \int_{\partial\Omega} \beta(z)\hat{u}^{p-1}h\,d\sigma = \int_\Omega f(z,\hat{u})h\,dz \quad \text{for all } h \in W^{1,\theta}(\Omega) \quad (\text{since } \hat{u} \geq 0), \\ \Rightarrow\, &\hat{u} \text{ is a solution of (1.1).} \end{aligned}$$

Finally using Theorem 3.3 of Liu-Dai [18], we conclude that $\hat{u}(z) > 0$ for a.a. $z \in \Omega$.

Similarly, since $\hat{v} \in K_{\varphi_-}$, we conclude that $\hat{v}$ is a negative solution of (1.1) such that $\hat{v}(z) < 0$ for a.a. $z \in \Omega$. □

Remark 3.10. *We would like to mention that the regularity issues for nonhomogeneous problems with double phase structure have been treated for instance in papers* [3, 4, 10, 11, 12]. *It will be interesting to know that if these solutions obtained in Proposition above have additional regularity properties and in particular global regularity properties (that is, up to the boundary).*

4. Nodal solutions

In this section we produce a nodal solution for problem (1.1). To this end we will use the set

$$N_0 = \{u \in W^{1,\theta}(\Omega) : u^+ \in N, -u^- \in N\}.$$

We set $m_0 = \inf_{N_0} \varphi$.

Proposition 4.1. *If hypotheses* $\mathrm{H(a)}, \mathrm{H}_0, \mathrm{H}_1$ *hold, then there exists* $\hat{y} \in N_0$ *such that* $\varphi(\hat{y}) = m_0$.

Proof. Let $\{y_n\}_{n\geq 1} \subseteq N_0$ be a minimizing sequence for $\varphi\big|_{N_0}$, that is, $\varphi(y_n) \downarrow m_0$. We have

$$\varphi(y_n) = \varphi(y_n^+) + \varphi(-y_n^-) \quad \text{and} \quad y_n^+ \in N, -y_n^- \in N \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

From Proposition 3.2 we know that

$$\{y_n^+\}_{n\geq 1} \subseteq W^{1,\theta}(\Omega) \quad \text{and} \quad \{y_n^-\}_{n\geq 1} \subseteq W^{1,\theta}(\Omega) \text{ are bounded.}$$

So, we may assume that

$$y_n^+ \xrightarrow{w} v_1 \quad \text{in } W^{1,\theta}(\Omega), v_1 \geq 0 \quad \text{and} \quad y_n^- \xrightarrow{w} v_2 \quad \text{in } W^{1,\theta}(\Omega), v_2 \geq 0. \tag{4.2}$$

Assume that $v_1 = 0$. Since $y_n^+ \in N$ (see (4.1)), we have

$$\begin{aligned} 0 &= \langle \varphi'(y_n^+), y_n^+ \rangle \\ &= \gamma_p(y_n^+) + \|Dy_n^+\|_q^q - \int_\Omega f(z, y_n^+) y_n^+ \, dz \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow\ & \rho_\theta(|Dy_n^+|) \to 0 \quad \text{as } n \to \infty, \\ \Rightarrow\ & y_n^+ \to 0 \quad \text{in } W^{1,\theta}(\Omega). \end{aligned} \tag{4.3}$$

But from Proposition 3.5, we know that $\hat{\delta} \leq \|y_n^+\|$ for all $n \in \mathbb{N}$, which contradicts (4.3). Therefore $v_1 \neq 0$. Similarly we show that $v_2 \neq 0$.

According to Proposition 3.1, we can find unique $t_1, t_2 > 0$ such that

$$t_1 v_1 \in N \quad \text{and} \quad t_2 v_2 \in N.$$

Let $y_0 = t_1 v_1 - t_2 v_2 \in W^{1,\theta}(\Omega)$. Then $y_0^+ = t_1 v_1, y_0^- = t_2 v_2$ and so $y_0 \in N_0$. We have

$$\begin{aligned} m_0 &= \lim_{n\to\infty} \varphi(y_n) \\ &= \lim_{n\to\infty} [\varphi(y_n^+) + \varphi(-y_n^-)] \\ &\geq \liminf_{n\to\infty} [\varphi(t_1 y_n^+) + \varphi(-t_2 y_n^-)] \quad \text{(see Proposition 3.4)} \\ &\geq \varphi(t_1 v_1) + \varphi(-t_2 v_2) \quad \text{(see (4.2))} \\ &= \varphi(y_0) \quad \text{(recall that } y_0 = t_1 v_1 - t_2 v_2) \\ &\geq m_0 \quad \text{(see } y_0 \in N_0), \\ \Rightarrow\ & \varphi(y_0) = m_0 \quad \text{with } y_0 \in N_0. \end{aligned}$$

□

Proposition 4.2. *If hypotheses* H(a), H_0, H_1 *hold and* $y_0 \in N_0$ *is as in Proposition 4.1, then* $y_0 \in K_\varphi$.

Proof. Let $e : W^{1,\theta}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$e(u) = \gamma_p(u) + \|Du\|_q^q - \int_\Omega f(z,u)u\,dz \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

We have

$$\varphi(y_0) = m_0 = \inf\,[\varphi(y) : e(y^+) = 0, e(-y^-) = 0, y^\pm \neq 0].$$

The two constraint functions

$$y \to e_1(y) = e(y^+) \quad \text{and} \quad y \to e_2(y) = e(-y^-)$$

are locally Lipschitz. So, we can apply the nonsmooth multiplier rule of Clarke [6] (Theorem 10.47, p. 221) and so we can find $\mu_1, \mu_2 \geq 0$ such that

$$\begin{aligned} & 0 \in \partial\,[\varphi + \mu_1 e_1 + \mu_2 e_2](y_0), \\ \Rightarrow\, & 0 \in \varphi'(y_0) + \mu_1 \partial\, e_1(y_0) + \mu_2 \partial\, e_2(y_0), \\ \Rightarrow\, & 0 = \varphi'(y_0) + \mu_1 \eta_1^* + \mu_2 \eta_2^* \quad \text{with } \eta_1^* \in \partial\, e_1(y_0), \eta_2^* \in \partial\, e_2(y_0). \end{aligned} \tag{4.4}$$

Since $y_0 \in N_0$, acting on (4.4) with $y_0^+ \in N$ and with $-y_0^- \in N$, we obatin

$$\mu_1 \langle \eta_1^*, y_0^+ \rangle = 0 \quad \text{and} \quad \mu_2 \langle \eta_2^*, -y_0^- \rangle = 0.$$

If $\mu_1 \neq 0$, then

$$\begin{aligned} & \langle \eta_1^*, y_0^+ \rangle = 0, \\ \Rightarrow\, & p\gamma_p(y_0^+) + q\|Dy_0^+\|_q^q - \int_\Omega f(z, y_0^+)y_0^+\,dz - \int_\Omega f'_x(z, y_0^+)(y_0^+)^2\,dz = 0 \\ & \quad \text{(using the nonsmooth chain rule of Clarke [6], p. 202)}, \\ \Rightarrow\, & p\left[\gamma_p(y_0^+) + \|Dy_0^+\|_q^q\right] - (p-q)\|Dy_0^+\|_q^q - \int_\Omega f(z, y_0^+)y_0^+\,dz \\ & \quad - \int_\Omega f'_x(z, y_0^+)(y_0^+)^2\,dz = 0, \\ \Rightarrow\, & \int_\Omega \left[(p-1)f(z, y_0^+)y_0^+ - f'_x(z, y_0^+)(y_0^+)^2\right]\,dz - (p-q)\|Dy_0^+\|_q^q = 0 \\ & \quad \text{(since } y_0^+ \in N\text{)}. \end{aligned}$$

But this is a contradiction since on account of hypothesis H_1(iii) and since $y_0^+ \in N$ and $q < p$, the left hand side is strictly negative. Therefore $\mu_1 = 0$.

Similarly we show that $\mu_2 = 0$. Therefore finally we have

$$0 \in \partial\,\varphi(y_0).$$

Since $\varphi \in C^1(W^{1,\theta}(\Omega))$, we have $\partial\,\varphi(y_0) = \{\varphi'(y_0)\}$. Thus we conclude that

$$\begin{aligned} & \varphi'(y_0) = 0, \\ \Rightarrow\, & u_0 \in K_\varphi. \end{aligned}$$

□

Then we can state the following result about nodal solutions.

Proposition 4.3. *If hypotheses* $\mathrm{H}(\mathrm{a}), \mathrm{H}_0, \mathrm{H}_1$ *hold, then problem* (1.1) *admits a ground state nodal solution* $y_0 \in W^{1,\theta}(\Omega)$ *which changes sign only once.*

Proof. Let $y_0 \in N_0$ be such that $\varphi(y_0) = m_0$ (see Proposition 4.1). Then on account of Proposition 4.2 and the definition of N_0, y_0 is a nodal solution of (1.1). Next we show that y_0 has only two nodal domains. Suppose $\Omega_k, k = 1, 2, 3$ are disjoint open, connected subsets of Ω on which y_0 has fixed sign. Let $y_k = y\big|_{\Omega_k}, k = 1, 2, 3$. We have

$$y_1\big|_{\Omega\setminus(\Omega_1\cup\Omega_2)} = y_2\big|_{\Omega\setminus(\Omega_1\cup\Omega_2)} = y_3\big|_{\Omega_1\cup\Omega_2}$$

and without loss of generality we may assume that

$$y_1 > 0, \quad y_2 < 0 \quad \text{and} \quad y_3 < 0.$$

We set $\hat{y} = y_1 + y_2$. Evidently $\hat{y}^+ = y_1, \hat{y}^- = -y_2$ and $y_0 = y_1 + y_2 + y_3 = \hat{y} + y_3$. Recall that $\varphi'(y_0) = 0$ (see Proposition 4.2). Then

$$\begin{aligned} &\langle \varphi'(y_0), \hat{y}^+ \rangle = \langle \varphi'(\hat{y}) + \varphi'(y_3), \hat{y}^+ \rangle = \langle \varphi'(\hat{y}), \hat{y}^+ \rangle, \\ \Rightarrow &\langle \varphi'(\hat{y}), \hat{y}^+ \rangle = 0. \end{aligned}$$

Similarly we obtain

$$\langle \varphi'(\hat{y}), \hat{y}^- \rangle = 0.$$

It follows that $\hat{y}^+, -\hat{y}^- \in N$ and so $\hat{y} \in N_0$. Then we have

$$\begin{aligned} m_0 &= \varphi(y_0) \\ &= \varphi(y_0) - \frac{1}{p}\langle \varphi'(y_0), y_0 \rangle \\ &= \varphi(\hat{y}) + \varphi(y_3) - \frac{1}{p}\left[\langle \varphi'(\hat{y}), \hat{y} \rangle + \langle \varphi'(y_3), y_3 \rangle\right] \\ &= \varphi(\hat{y}) + \varphi(y_3) - \frac{1}{p}\langle \varphi'(y_3), y_3 \rangle \\ &= \varphi(\hat{y}) + \left[\frac{1}{q} - \frac{1}{p}\right] \|Dy_3\|_q^q + \int_\Omega \left[\frac{1}{p} f(z, y_3) y_3 - F(z, y_3)\right] dz \\ &\geq m_0 + \left[\frac{1}{q} - \frac{1}{p}\right] \|Dy_3\|_q^q \quad \text{(recall } e(z, \cdot) \text{ is nondecreasing)}, \\ \Rightarrow \Omega_3 &= \emptyset. \end{aligned}$$

□

Summarizing the results in Section 3 and Section 4, we finally finish the proof of Theorem 1.1.

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