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Gradient estimates for a class of quasilinear elliptic equations with measure data

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Abstract We study a class of non-homogeneous quasilinear elliptic equations with measure data to obtain an optimal regularity estimate. We prove that the gradient of a weak solution to the problem is as integrable as the first order maximal function of the associated measure in the Orlicz spaces up to a correct power.

Keywords quasilinear, elliptic, regularity, measure data, Orlicz space

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. In this paper we study the gradient estimates of weak solutions for the following non-homogeneous quasilinear elliptic equations with measure data:

$$-\operatorname{div}(a(|Du|)Du) = \mu \quad \text{in } \Omega, \tag{1.1}$$

where μ is a Borel measure with finite mass and $a \in C^1(0, +\infty)$ satisfies the structure assumption

$$0 \leqslant i_a := \inf_{t>0} \frac{ta'(t)}{a(t)} \leqslant \sup_{t>0} \frac{ta'(t)}{a(t)} =: s_a < +\infty. \tag{1.2}$$

Define

$$g(t) = ta(t) \tag{1.3}$$

and

$$G(t) = \int_0^t g(\tau)\, d\tau = \int_0^t \tau a(\tau)\, d\tau \quad \text{for } t \geqslant 0. \tag{1.4}$$

Thanks to (1.2) we know that $g(t)$ is strictly increasing and continuous over $[0, +\infty)$, and then $G(t)$ is increasing over $[0, +\infty)$ and strictly convex with $G(0) = 0$.

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It is easy to see that

$$G(t) = t^p \quad \text{and} \quad G(t) = t^q \log(1+t)$$

for any $p \geqslant 2, q > 2$ satisfy the condition (1.2). Especially when $a(t) = t^{p-2}$ (and then $G(t) = t^p/p$), (1.1) is reduced to the p-Laplacian equation

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu \quad \text{for } p \geqslant 2.$$

As usual, the solutions of (1.1) are taken in a weak sense.

Definition 1.1. A function $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) if for any $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega a(|Du|)Du \cdot D\varphi \, dx = \int_\Omega \varphi \, d\mu.$$

Here $W^{1,G}(\Omega)$ is the Orlicz-Sobolev spaces defined in Definition 2.5.

Elliptic equations of the type considered in (1.1) have been introduced by Lieberman [13], which can be seen as the natural generalization of the p-Laplace equations. The local boundedness and Hölder regularity both of solutions and their gradients, Harnack's inequalities and characterizations of De Giorgi classes were proved for this class of equations. Cianchi and Maz'ya in their series of papers [9–11] proved global Lipschitz regularity and obtained a sharp estimate for the decreasing rearrangement of the length of the gradient for the Dirichlet and Neumann elliptic boundary value problems of

$$-\operatorname{div}(a(|Du|)Du) = f \quad \text{in } \Omega$$

with the similar condition (1.2) and weak assumptions on the boundary, where f belongs to some Lorentz spaces or $f \in L^1(\Omega)$. Furthermore, Yao and Zhou [20] investigated the local L^p-type regularity estimates of weak solutions for the following quasilinear elliptic equations

$$\operatorname{div}(a(|Du|)Du) = \operatorname{div}(a(|\mathbf{f}|)\mathbf{f}) \quad \text{in } \Omega.$$

The elliptic and parabolic problems involving measure data naturally come from many interesting phenomena in the area of applied mathematics, for instance, the flow pattern of blood in the heart [15], and state-constrained optimal control problems [7,8]. For the p-Laplacian type elliptic equation

$$-\operatorname{div}\mathbf{a}(x, Du) = \mu \quad \text{in } \Omega,$$

under certain natural conditions on $\mathbf{a}$ and Ω, Phuc in [16] showed that the renormalized solutions satisfy the bound

$$\int_\Omega |Du|^q w \, dx \leqslant C \int_\Omega \mathcal{M}_1(\mu)^{\frac{q}{p-1}} w \, dx$$

for any $q \in (0, +\infty)$ and any weight w in the A_∞ class, where $\mathcal{M}_1$ is the fractional maximal function of order 1 for μ, defined as

$$\mathcal{M}_1(\mu)(x) := \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|}, \quad x \in \mathbb{R}^n.$$

Very recently, Byun and Park in [4] established the global gradient estimates for solutions of the elliptic equations with linear growth and measure data in the setting of variable exponent spaces. We also refer to [3,6] for the gradient estimates of more general $p(x)$-Laplacian type elliptic equations. On the other hand, Baroni [2], Yao and Zheng [19] proved the pointwise gradient estimates via the linear Riesz potential and the nonlinear Wolff potential for weak solution of (1.1), respectively.

Motivated by the works mentioned above, the aim of this paper is to establish an interior Calderón-Zygmund type estimate in the setting of Orlicz spaces for a weak solution u to the problem (1.1). More precisely, we want to prove that

$$\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}} \in L^\phi_{loc}(\Omega) \Longrightarrow |Du| \in L^\phi_{loc}(\Omega),$$

where i_a is the constant defined in (1.2), provided $\phi \in \Delta_2 \cap \nabla_2$ (see Definition 2.2). The technical approach in our proof is based on the comparison L^1-estimates with the homogeneous problems, the Vitali-type covering lemma, boundedness of the Hardy-Littlewood maximal operator in $L^\phi(\Omega)$ and estimates of the power decay of the upper level sets of the gradient $|Du|$.

The main result of this paper is stated as follows.

Theorem 1.2. *Let the structure assumption* (1.2) *be satisfied. If* $u \in W^{1,G}_{loc}(\Omega)$ *is a local weak solution to* (1.1)*, then for any Young function* ϕ *with* $\phi \in \Delta_2 \cap \nabla_2$*, we have*

$$\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}} \in L^\phi_{loc}(\Omega) \Longrightarrow |Du| \in L^\phi_{loc}(\Omega),$$

with the estimate

$$\int_{B_r} \phi(|Du|)\, dx \leqslant C\phi\left(\int_{B_{4R_0}} |Du|\, dx + 1\right) + C\int_{B_{4R_0}} \phi\left(\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x)\right) dx,$$

where i_a *is the constant defined in* (1.2)*,* $0 < r < R_0$ *with* $B_{4R_0} \subset\subset \Omega$ *and* C *is independent of* u *and* μ*.*

This paper is organized as follows. In Section 2, we state some preliminary tools and known results which will be used later. We will finish the proof of Theorem 1.2 in Section 3.

2 Preliminaries

2.1 Orlicz spaces

The theory of Orlicz spaces has been extensively studied in the area of analysis (see [1, 14]) and plays a crucial role in many fields of mathematics including geometry, probability, stochastic analysis, Fourier analysis and partial differential equations (see [17]). For the reader's convenience, we will give some definitions on the general Orlicz spaces. Denote by Φ the function class that consists of all functions $\phi : [0, +\infty) \to [0, +\infty)$ which are increasing and convex.

Definition 2.1. *A function* $\phi \in \Phi$ *is said to be a Young function if*

$$\lim_{t\to 0+} \frac{\phi(t)}{t} = \lim_{t\to +\infty} \frac{t}{\phi(t)} = 0.$$

Definition 2.2. A Young function $\phi \in \Phi$ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant M such that for every $t > 0$,

$$\phi(2t) \leqslant M\phi(t). \tag{2.1}$$

Moreover, A Young function $\phi \in \Phi$ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a number $a > 1$ such that for every $t > 0$,

$$\phi(t) \leqslant \frac{\phi(at)}{2a}. \tag{2.2}$$

Example 2.3.

(1) $\phi_1(t) = (1 + t)\log(1 + t) - t \in \Delta_2$, but $\phi_1(t) \notin \nabla_2$.

(2) $\phi_2(t) = e^t - t - 1 \in \nabla_2$, but $\phi_2(t) \notin \Delta_2$.

(3) $\phi_3(t) = t^p \log(1 + t) \in \Delta_2 \cap \nabla_2$, $p > 1$.

Remark 2.4. In fact, if a function ϕ satisfies (2.1) and (2.2), then

$$\phi(\theta_1 t) \leqslant K\theta_1^{\beta_1}\phi(t) \quad \text{and} \quad \phi(\theta_2 t) \leqslant 2a\theta_2^{\beta_2}\phi(t), \tag{2.3}$$

for every $t > 0$ and $0 < \theta_2 \leqslant 1 \leqslant \theta_1 < +\infty$, where $\beta_1 = \log_2 M$ and $\beta_2 = \log_a 2 + 1$. These doubling type conditions ensure that a Young function grows neither too slowly nor too fast.

Definition 2.5. Assume that ϕ is a Young function. Then the Orlicz class $K^{\phi}(\Omega)$ is the set of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\int_{\Omega} \phi(|g|)\, dx < +\infty.$$

The Orlicz space $L^{\phi}(\Omega)$ is the linear hull of $K^{\phi}(\Omega)$ endowed with the Luxemburg norm

$$\|g\|_{L^{\phi}(\Omega)} = \inf \left\{ k > 0 : \int_{\Omega} \phi\left(\frac{|g(x)|}{k}\right) dx \leqslant 1 \right\}.$$

Furthermore, the Orlicz-Sobolev space $W^{1,\phi}(\Omega) = \left\{ g \in L^{\phi}(\Omega) : \nabla g \in L^{\phi}(\Omega) \right\}$, endowed with the norm $\|g\|_{W^{1,\phi}(\Omega)} = \|g\|_{L^{\phi}(\Omega)} + \|\nabla g\|_{L^{\phi}(\Omega)}$.

In this work we need the following lemma. Here the $\Delta_2 \cap \nabla_2$ condition is crucial.

Lemma 2.6 ([1])**.** *Let ϕ be a Young function satisfying $\phi \in \Delta_2 \cap \nabla_2$. Then*

(1) $K^{\phi}(\Omega) = L^{\phi}(\Omega)$.

(2) $C_0^{\infty}(\Omega)$ is dense in $L^{\phi}(\Omega)$.

(3) $L^{\beta_1}(\Omega) \subset L^{\phi}(\Omega) \subset L^{\beta_2}(\Omega) \subset L^1(\Omega)$ with $\beta_1 \geqslant \beta_2 > 1$ as in (2.3)*.*

(4) If $g \in L^{\phi}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \phi(|g|)\ dx = \int_0^{\infty} \left|\left\{ x \in \mathbb{R}^N : |g| > \mu \right\}\right|\ d\left[\phi(\mu)\right]. \tag{2.4}$$

2.2 Maximal function

We use the Hardy-Littlewood maximal function, which controls the local behavior of a function. For a locally integrable function f defined on $\mathbb{R}^n$, we define its maximal function $\mathcal{M}(f)(x)$ as

$$\mathcal{M}(f)(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy.$$

If f is not defined outside a bounded domain Ω, then we define

$$\mathcal{M}_{\Omega} f = \mathcal{M}(f\chi_{\Omega})$$

for the standard characteristic function χ on Ω.

The basic properties for the Hardy-Littlewood maximal function are the followings.

Lemma 2.7 ([18], Chapter 1)**.** *If $f \in L^p(\mathbb{R}^n)$ with $1 < p \leqslant +\infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$ and*

$$\frac{1}{C}\|f\|_{L^p} \leqslant \|\mathcal{M}f\|_{L^p} \leqslant C\|f\|_{L^p}.$$

If $f \in L^1(\mathbb{R}^n)$, then

$$|\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > t\}| \leqslant \frac{C}{t} \int |f(x)|\, dx.$$

Lemma 2.8 ([12], Chapter 1)**.** *Let U be a bounded domain in $\mathbb{R}^n$ and ϕ be a Young function with $\phi \in \Delta_2 \cap \nabla_2$. If $f \in L^{\phi}(U)$, then $\mathcal{M}f \in L^{\phi}(U)$ and for $C = C(n, \phi) > 0$,*

$$\frac{1}{C} \int_U \phi(|f|)\, dx \leqslant \int_U \phi(\mathcal{M}f)\, dx \leqslant C \int_U \phi(|f|)\, dx.$$

2.3 Technical lemmas

In this paper, we use the following version of the Vitali covering lemma, which will be a crucial ingredient in obtaining our main result.

Lemma 2.9 ([5], Lemma 2.7). *Assume that C and D are measurable sets, $C \subset D \subset B_1$, and that there exists an $\varepsilon > 0$ such that $|C| < \varepsilon|B_1|$ and that for all $x \in B_1$ and for all $r \in (0, 1]$ with $|C \cap B_r(x)| \geqslant \varepsilon|B_r(x)|$ we have $B_r(x) \cap B_1 \subset D$. Then,*

$$|C| \leqslant 10^n \varepsilon |D|.$$

We next give the following important results.

Lemma 2.10 ([19], Lemmas 1.3, 1.4). *Assume that $a(t)$ satisfies (1.2) and $G(t)$ is defined in (1.4). Then we have*

(1) For any $t > 0$ we find that

$$\theta^{i_a} \leqslant \frac{a(\theta t)}{a(t)} \leqslant \theta^{s_a} \quad \text{and} \quad \theta^{2+i_a} \leqslant \frac{G(\theta t)}{G(t)} \leqslant \theta^{2+s_a} \quad \text{for any } \theta \geqslant 1. \tag{2.5}$$

(2) $G(t) \in \Delta_2 \cap \nabla_2$.

Lemma 2.11 ([19], Lemma 2.1). *Assume that $a(t)$ satisfies (1.2) and $G(t)$ is defined in (1.4). Then there exists $C = C(n, i_a, s_a) > 0$ we have*

$$[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \geqslant CG(|\xi - \eta|) \quad \text{for any } \xi, \eta \in \mathbb{R}^n.$$

In particular, we have

$$a(|\xi|)\xi \cdot \xi \geqslant CG(|\xi|) \quad \text{for any } \xi \in \mathbb{R}^n.$$

Furthermore, we derive comparison L^1-estimates for the gradient of the weak solution u to (1.1) in localized interior regions.

Lemma 2.12. *Assume that $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) with $B_{2R} \subset \Omega$ and (1.2). If $v \in W^{1,G}(B_R)$ is the weak solution of*

$$\begin{cases} \operatorname{div}\,(a(|Dv|)\,Dv) = 0 & \text{in } B_R, \\ v = u & \text{on } \partial B_R, \end{cases} \tag{2.6}$$

then there exists a constant $C_1 = C_1(n, i_a, s_a) > 1$ such that

$$\fint_{B_R} |Du - Dv|\, dx \leqslant C_1 \left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{1+i_a}}.$$

Proof. Without loss of generality we may as well assume that $R = 1$ by defining

$$\tilde{u}(x) = R^{-1}u(Rx), \quad \tilde{v}(x) = R^{-1}v(Rx) \quad \text{and} \quad \tilde{\mu}(x) = R\mu(Rx).$$

For $k \geqslant 1$ we define the following truncation operators

$$T_k(s) := \max\{-k, \min\{k, s\}\} \quad \text{and} \quad \Phi_k(s) := T_1(s - T_k(s)),\ s \in \mathbb{R}.$$

Since u and v are weak solutions of (1.1) and (2.6) respectively, then by approximation we have

$$\int_{B_1} [a(|Du|)\,Du - a(|Dv|)\,Dv] \cdot D\varphi\, dx = \int_{B_1} \varphi\, d\mu \tag{2.7}$$

for any $\varphi \in L^\infty(B_1) \cap W^{1,G}_0(B_1)$. We divide into two cases.

Case 1: $|\mu|(B_1) \leqslant 1$.

If $2+i_a > n$ (recall that $u-v \in W_0^{1,G}(B_1)$ and then $u-v \in W_0^{1,2+i_a}(B_1)$), then from Sobolev's inequality we have $u-v \in L^\infty(B_1)$. By selecting $\varphi = u-v \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$, from Lemma 2.11 and Sobolev's inequality we find that

$$\int_{B_1} |G(Du-Dv)|\, dx \leqslant C\|u-v\|_{L^\infty(B_1)}|\mu|(B_1) \leqslant C\|Du-Dv\|_{L^{2+i_a}(B_1)}.$$

Then from (2.5) we have

$$\begin{aligned}\|Du-Dv\|_{L^{2+i_a}(B_1)}^{2+i_a} &= \int_{B_1} |Du-Dv|^{2+i_a}\, dx \\ &\leqslant C\int_{B_1} |G(Du-Dv)|+1\, dx \\ &\leqslant C\|Du-Dv\|_{L^{2+i_a}(B_1)} + C,\end{aligned}$$

which implies that

$$\|Du-Dv\|_{L^{2+i_a}(B_1)} \leqslant C \quad \text{and then} \quad \int_{B_1} |Du-Dv|\, dx \leqslant C$$

by using Hölder's inequality. Therefore, we may as well assume that $2+i_a \leqslant n$. Then by selecting the test function $\varphi = T_k(u-v) \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$, from (2.7) and Lemma 2.11 we have

$$\int_{D_k} |G(Du-Dv)|\, dx \leqslant Ck|\mu|(B_1) \leqslant Ck,$$

where $D_k := \{x \in B_1 : |u(x)-v(x)| \leqslant k\}$, which implies that

$$\int_{D_k} |Du-Dv|^{2+i_a}\, dx \leqslant \int_{D_k} |G(Du-Dv)|+1\, dx \leqslant Ck$$

and then

$$\int_{D_k} |Du-Dv|\, dx \leqslant Ck \tag{2.8}$$

by using Young's inequality. Moreover, testing (2.7) again with $\varphi = \Phi_k(u-v) \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$ and using Lemma 2.11, we find that

$$\int_{C_k} |G(Du-Dv)|\, dx \leqslant C\int_{B_1} |\mu|\, dx \leqslant C,$$

where $C_k := \{x \in B_1 : k < |u(x)-v(x)| \leqslant k+1\}$, which implies that

$$\int_{C_k} |Du-Dv|^{2+i_a}\, dx \leqslant \int_{C_k} |G(Du-Dv)|+1\, dx \leqslant C. \tag{2.9}$$

From (2.5), Hölder's inequality, (2.8), (2.9) and the definition of C_k we find that

$$\begin{aligned}\int_{C_k} |Du-Dv|\, dx &\leqslant C\,|C_k|^{1-\frac{1}{2+i_a}} \left(\int_{C_k} |Du-Dv|^{2+i_a}\, dx\right)^{\frac{1}{2+i_a}} \\ &\leqslant C\,|C_k|^{1-\frac{1}{2+i_a}} \leqslant \frac{C}{k^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}} \left(\int_{C_k} |u-v|^{\frac{n}{n-1}}\, dx\right)^{1-\frac{1}{2+i_a}},\end{aligned}$$

which implies that

$$\int_{B_1} |Du-Dv|\, dx = \int_{D_{k_0}} |Du-Dv|\, dx + \sum_{k=k_0}^{\infty} \int_{C_k} |Du-Dv|\, dx$$

$$
\begin{aligned}
&\leqslant Ck_0 + C\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}} \left(\int_{C_k} |Du-Dv|\,dx\right)^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}\\
&\leqslant Ck_0 + C\left[\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-1}}}\right]^{\frac{1}{2+i_a}} \left(\sum_{k=k_0}^{\infty}\int_{C_k} |Du-Dv|\,dx\right)^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}\\
&\leqslant Ck_0 + C\left[\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-1}}}\right]^{\frac{1}{2+i_a}} \left(\int_{B_1} |Du-Dv|\,dx\right)^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}
\end{aligned}
$$

in view of Sobolev's inequality. Considering the fact that

$$
\frac{n(1+i_a)}{n-1} > 1 \quad \text{and} \quad \frac{n}{n-1}\left(1-\frac{1}{2+i_a}\right) \leqslant 1
$$

in view of $2+i_a \leqslant n$ and then choosing $k_0 \in \mathbb{N}$ large enough such that

$$
C\left[\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-1}}}\right]^{\frac{1}{2+i_a}} \leqslant \frac{1}{2},
$$

we obtain

$$
\int_{B_1} |Du-Dv|\,dx \leqslant C.
$$

Case 2: $|\mu|(B_1) > 1$.

Let

$$
\tilde{u}(x) = A^{-1}u(x), \quad \tilde{v}(x) = A^{-1}v(x),
$$

and

$$
\tilde{\mu}(x) = A^{-1-i_a}\mu(x) \quad \text{and} \quad \tilde{a}\,(t) = A^{-i_a}a(At),
$$

where

$$
A = (|\mu|(B_1))^{\frac{1}{1+i_a}} > 1. \tag{2.10}
$$

Then it is easy to check that

$$
|\tilde{\mu}|(B_1) = 1.
$$

Moreover, $\tilde{a}\,(t)$ satisfies (1.2) and $\tilde{u}(x) \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of

$$
\operatorname{div}\,(\tilde{a}\,(|D\tilde{u}|)\,D\tilde{u}) = \tilde{\mu}
$$

If $2+i_a > n$ (recall that $u-v \in W^{1,G}_0(B_1)$ and then $\tilde{u}-\tilde{v} \in W^{1,2+i_a}_0(B_1)$), then from Sobolev's inequality we have $\tilde{u}-\tilde{v} \in L^\infty$. Moreover, testing (2.7) again with $\varphi = \tilde{u}-\tilde{v} \in L^\infty(B_1) \cap W^{1,G}_0(B_1)$, and using Lemma 2.11 and Sobolev's inequality, we find that

$$
\begin{aligned}
A^{-2-i_a}\int_{B_1} |G\,(Du-Dv)|\,dx &\leqslant C\|\tilde{u}-\tilde{v}\|_{L^\infty(B_1)}|\tilde{\mu}|(B_1)\\
&\leqslant C\|D\tilde{u}-D\tilde{v}\|_{L^{2+i_a}(B_1)}.
\end{aligned} \tag{2.11}
$$

Then from (1.2), (2.5), (2.10) and (2.11) we have

$$
\begin{aligned}
\|D\tilde{u}-D\tilde{v}\|^{2+i_a}_{L^{2+i_a}(B_1)} &= \int_{B_1} |D\tilde{u}-D\tilde{v}|^{2+i_a}\,dx\\
&\leqslant CA^{-2-i_a}\int_{B_1} |G\,(Du-Dv)| + 1\,dx
\end{aligned}
$$

$$\leqslant C\|D\tilde{u}-D\tilde{v}\|_{L^{2+i_a}(B_1)}+C,$$

which implies that

$$\|D\tilde{u}-D\tilde{v}\|_{L^{2+i_a}(B_1)}\leqslant C \quad \text{and then} \quad \int_{B_1}|D\tilde{u}-D\tilde{v}|\,dx\leqslant C$$

by using Hölder's inequality. Therefore, we may as well assume that $2+i_a\leqslant n$. Furthermore, similarly to Case 1 we find that

$$\int_{B_1}|D\tilde{u}-D\tilde{v}|\ dx\leqslant C,$$

which finishes our proof. □

The following lemma is indeed essentially a little variation of Lemma 5.1 in [13].

Lemma 2.13 ([2], Lemma 4.1)**.** *Let $v\in W^{1,G}(A)$ be a solution to*

$$\operatorname{div}(a(|Dv|)Dv)=0 \quad \text{on } A\subset\mathbb{R}^n.$$

Then for every ball $B_R\equiv B_R(x_0)\subset A$ the following De Giorgi type estimate holds:

$$\sup_{B_{R/2}}|Dv|\leqslant C\fint_{B_R}|Dv|\,dx.$$

3 Proof of the main result

In this section we will finish the proof of Theorem 1.2. Hereafter we set $0<r<R_0$ with $B_{4R_0}\subset\subset\Omega$.

Lemma 3.1. *There is a constant $N=N(n,i_a,s_a)>0$ so that for any $\varepsilon>0$, there exists a small $\delta=\delta(\varepsilon)>0$ such that if $u\in W^{1,G}_{loc}(\Omega)$ is a local weak solution of* (1.1) *in $B_{6r}\subset\Omega$ with*

$$B_r\cap\{x:\mathcal{M}(|Du|)(x)\leqslant\lambda\}\cap\{x:\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x)\leqslant\delta\lambda\}\neq\emptyset, \tag{3.1}$$

then we have

$$|\{x\in B_1:\mathcal{M}(|Du|)(x)>N\lambda\}\cap B_r|<\varepsilon|B_r|. \tag{3.2}$$

Proof. From (3.1), there exists a point $x_0\in B_r$ such that

$$\fint_{B_\rho(x_0)}|Du|\,dx\leqslant\lambda \quad \text{and} \quad \left(r\fint_{B_\rho(x_0)}d|\mu|\right)^{\frac{1}{1+i_a}}\leqslant\delta\lambda \tag{3.3}$$

for all $\rho>0$.

Since $B_{4r}\subset B_{5r}(x_0)$, it follows from (3.3) that

$$\begin{aligned}\fint_{B_{4r}}|Du|\,dx&\leqslant\frac{|B_{5r}(x_0)|}{|B_{4r}|}\cdot\frac{1}{|B_{5r}(x_0)|}\int_{|B_{5r}(x_0)|}|Du|\,dx\\&\leqslant 2^n\lambda.\end{aligned}$$

Likewise, we also have

$$\left(r\fint_{B_{4r}}d|\mu|\right)^{\frac{1}{1+i_a}}\leqslant 2^n\delta\lambda.$$

Then we are under the hypotheses of Lemma 2.12. This implies that there exists a constant $C=C(n,i_a,s_a)>1$ such that

$$\fint_{B_{4r}}|Du-Dv|\,dx\leqslant C2^n\delta\lambda,$$

where $v \in W^{1,G}(B_{4r})$ is the weak solution of (2.6) in B_{4r}. If follows from Lemma 2.13 that

$$\begin{aligned}\|Dv\|_{L^\infty(B_{2r})} &\leqslant C\fint_{B_{4r}} |Dv|\,dx \\ &\leqslant C\fint_{B_{4r}} |Du|\,dx + C\fint_{B_{4r}} |Du-Dv|\,dx \\ &\leqslant C2^n(1+\delta)\lambda \leqslant C2^{n+1}\lambda := N_0\lambda.\end{aligned}$$

Write $N=\max\{3^n, 2N_0\}$. We claim that

$$\{x\in B_r : \mathcal{M}(|Du|)(x) > N\lambda\} \subset \{x \in B_r : \mathcal{M}(|Du-Dv|)(x) > N_0\lambda\}. \tag{3.4}$$

To show this, we take $x_1 \in \{x \in B_r : \mathcal{M}(|Du-Dv|)(x) \leqslant N_0\lambda\}$. If $0<\rho<r$, then $B_\rho(x_1)\subset B_{2r}$, and so we have

$$\begin{aligned}\fint_{B_\rho(x_1)} |Du|\,dx &\leqslant \fint_{B_\rho(x_1)} (|Dv| + |Du-Dv|)\,dx \\ &\leqslant 2N_0\lambda \leqslant N\lambda.\end{aligned}$$

On the other hand, if $\rho \geqslant r$, then $B_\rho(x_1)\subset B_{3\rho(x_0)}$. From (3.3), we find

$$\begin{aligned}\fint_{B_\rho(x_1)} |Du|\,dx &\leqslant \frac{|B_{3\rho}(x_0)|}{|B_\rho(x_1)|}\fint_{B_{3\rho}(x_0)} |Du|\,dx \\ &\leqslant 3^n\lambda \leqslant N\lambda.\end{aligned}$$

Thus (3.4) now follows. Then we estimate

$$\begin{aligned}&\frac{1}{|B_r|}|\{x\in B_r : \mathcal{M}_{B_{4r}}(|Du|) > N\lambda\}| \\ &\leqslant \frac{1}{|B_r|}|\{x\in B_r : \mathcal{M}_{B_{4r}}(|Du-Dv|) > N_0\lambda\}| \\ &\leqslant \frac{C}{N_0\lambda}\fint_{B_{4r}} |Du-Dv|\,dx \leqslant C\delta \leqslant \varepsilon,\end{aligned}$$

by taking δ such that the last inequality holds. □

The contraposition of Lemma 3.1 can be stated as follows.

Lemma 3.2. *There is a constant $N>0$ such that for any $\varepsilon>0$, there exists a small $\delta=\delta(\varepsilon)>0$ so that if*

$$|\{x : \mathcal{M}(|Du|)(x) > N\lambda\} \cap B_r| \geqslant \varepsilon|B_r|,$$

then we have

$$B_r \subset \{x : \mathcal{M}(|Du|) > \lambda\} \cup \{x : \mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x) > \delta\lambda\}.$$

Now fix ε, which will be determined later and take the corresponding $\delta>0$ given by Lemma 3.2 and set

$$\varepsilon_1 = 10^n\varepsilon.$$

Denote

$$\begin{aligned}\mu_1(t) &= |\{x\in B_r : \mathcal{M}(|Du|)(x) > t\}|, \\ \mu_2(t) &= |\{x\in B_r : \mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x) > t\}|\end{aligned}$$

and

$$\lambda_0 = \frac{2C}{N|B_1|\varepsilon}\left(\int_{B_{4R_0}} |Du|\,dx + 1\right). \tag{3.5}$$

Let $u \in W^{1,G}_{loc}(\Omega)$ be the local weak solution of (1.1). It follows from the weak $(1-1)$ estimate and (3.5) that

$$\begin{aligned}\mu_1(N\lambda_0) &= |\{x \in B_r : \mathcal{M}(|Du|)(x) > N\lambda_0\}| \\ &\leqslant \frac{C}{N\lambda_0}\int_{B_r} |Du|\,dx \leqslant \frac{\varepsilon|B_1|}{2} < \varepsilon|B_1|.\end{aligned}$$

Using Lemma 2.9 and Lemma 3.2, we can get

$$\mu_1(N_1\lambda_0) \leqslant \varepsilon_1[\mu_1(\lambda_0) + \mu_2(\delta\lambda_0)] \leqslant \varepsilon_1\mu_1(\lambda_0) + \mu_2(\delta\lambda_0),$$

where $\varepsilon_1 = 10^n\varepsilon < 1$. By induction, we deduce

$$\mu_1(N^{m+1}\lambda_0) \leqslant \varepsilon_1^{m+1}\mu_1(\lambda_0) + \sum_{i=0}^{m}\varepsilon_1^{m-i}\mu_2(N^i\delta\lambda_0) \tag{3.6}$$

for every integer $m \geqslant 0$.

Now we can prove our main result.

Proof of Theorem 1.2. According to the standard arguments of measure theory and Lemma 2.8, we observe that

$$\begin{aligned}\int_{B_r}\phi(|Du|)\,dx &\leqslant \int_{B_r}\phi(\mathcal{M}(|Du|))\,dx \\ &= \int_0^\infty \mu_1(\lambda)\,d[\phi(\lambda)] \\ &= \int_0^{\lambda_0}\mu_1(\lambda)\,d[\phi(\lambda)] + \int_{\lambda_0}^\infty \mu_1(\lambda)\,d[\phi(\lambda)].\end{aligned}$$

Moreover, we find from (2.3) that

$$\begin{aligned}\int_0^{\lambda_0}\mu_1(\lambda)\,d[\phi(\lambda)] \leqslant \phi(\lambda_0)|\Omega| &\leqslant |\Omega|\phi\left[\frac{C}{N|B_1|\varepsilon}\left(\int_{B_{4R_0}}|Du|\,dx + 1\right)\right] \\ &\leqslant C(\varepsilon)\phi\left(\int_{B_{4R_0}}|Du|\,dx + 1\right) \\ &\leqslant C(\varepsilon)\phi\left(\int_{B_{4R_0}}|Du|\,dx + 1\right).\end{aligned}$$

In a similar way we have

$$\begin{aligned}\int_{\lambda_0}^\infty \mu_1(\lambda)\,d[\phi(\lambda)] &= \sum_{m=0}^\infty \int_{N^m\lambda_0}^{N^{m+1}\lambda_0}\mu_1(\lambda)\,d[\phi(\lambda)] \\ &\leqslant \sum_{m=0}^\infty [\phi(N^{m+1}\lambda_0) - \phi(N^m\lambda_0)]\mu_1(N^m\lambda_0) \\ &\leqslant \sum_{m=0}^\infty \phi(N^{m+1}\lambda_0)\mu_1(N^m\lambda_0).\end{aligned}$$

Since

$$\begin{aligned}\phi(N\lambda_0)\mu_1(\lambda_0) &\leqslant |\Omega|\phi\left(\frac{C}{\varepsilon|B_1|}\int_{B_{4R_0}}|Du|\,dx + 1\right) \\ &\leqslant C(\varepsilon)\phi\left(\int_{B_{4R_0}}|Du|\,dx + 1\right),\end{aligned}$$

we obtain from (3.6) that

$$
\begin{aligned}
\int_{B_r} \phi(|Du|)\,dx &\leqslant C\phi\left(\int_{B_{4R_0}} |Du|\,dx+1\right)+\sum_{m=1}^{\infty}\phi(N^{m+1}\lambda_0)\mu_1(N^m\lambda_0)\\
&\leqslant C\phi\left(\int_{B_{4R_0}} |Du|\,dx+1\right)+\sum_{m=1}^{\infty}\phi(N^{m+1}\lambda_0)\varepsilon_1^m\mu_1(\lambda_0)\\
&\quad+\sum_{m=1}^{\infty}\phi(N^{m+1}\lambda_0)\sum_{i=0}^{m-1}\varepsilon_1^{m-1-i}\mu_2(N^i\delta\lambda_0)\\
&:= I_1+I_2+I_3.
\end{aligned}
$$

Estimate of I_2. Observe that

$$
\begin{aligned}
I_2 &\leqslant C\phi(\lambda_0)|\Omega|\sum_{m=1}^{\infty} MN^{(m+1)\beta_1}\varepsilon_1^m\\
&\leqslant C\phi(\lambda_0)|\Omega|MN^{\beta_1}\cdot\sum_{m=1}^{\infty}(N^{\beta_1}\varepsilon_1)^m\\
&\leqslant C\phi\left(\int_{B_{4R_0}} |Du|\,dx+1\right),
\end{aligned}
$$

by choosing ε_1 small enough satisfying $N^{\beta_1}\varepsilon_1<1$.

Estimate of I_3. From (2.3) we have

$$
\begin{aligned}
I_3 &= \sum_{m=1}^{\infty}\phi(N^{m+1}\lambda_0)\sum_{i=0}^{m-1}\varepsilon_1^{m-1-i}\mu_2(N^i\delta\lambda_0)\\
&\leqslant \sum_{m=1}^{\infty}\sum_{i=0}^{m-1} MN^{(m-i+1)\beta_1}\phi(N^i\lambda_0)\varepsilon_1^{m-1-i}\mu_2(N^i\delta\lambda_0)\\
&\leqslant \sum_{i=0}^{\infty}\phi(N^i\lambda_0)\mu_2(N^i\delta\lambda_0)\sum_{m=i+1}^{\infty} MN^{(m-i+1)\beta_1}\varepsilon_1^{m-1-i}\\
&\leqslant KN^{2\beta_1}\sum_{i=0}^{\infty}\phi(N^i\lambda_0)\mu_2(N^i\delta\lambda_0)\sum_{m=i+1}^{\infty}\left(N^{\beta_1}\varepsilon_1\right)^{m-1-i}\\
&\leqslant C\sum_{i=0}^{\infty}\phi(N^i\lambda_0)\mu_2(N^i\delta\lambda_0).
\end{aligned}
$$

Moreover, we observe that

$$
\begin{aligned}
\int_{B_r}\phi\left(\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x)\right)dx &= \int_0^{\infty}\mu_2(\lambda)\,d\left[\phi(\lambda)\right]\\
&= \int_0^{\delta\lambda_0}\mu_2(\lambda)\,d\left[\phi(\lambda)\right]+\int_{\delta\lambda_0}^{\infty}\mu_2(\lambda)\,d\left[\phi(\lambda)\right]\\
&\geqslant \mu_2(\delta\lambda_0)\phi(\delta\lambda_0)+\sum_{k=0}^{\infty}\int_{N^k\delta\lambda_0}^{N^{k+1}\delta\lambda_0}\mu_2(\lambda)\,d\left[\phi(\lambda)\right].
\end{aligned}
$$

Choosing $N=\max\{2N_0,3^n,2a\}$ and using (2.3) again we obtain that

$$
\int_{B_r}\phi\left(\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x)\right)dx
$$

$$
\begin{aligned}
&\geqslant \mu_2(\delta\lambda_0)\phi(\delta\lambda_0) + \sum_{k=0}^{\infty} \mu_2(N^{k+1}\delta\lambda_0)\left[\phi\left(N^{k+1}\delta\lambda_0\right) - \phi\left(N^k\delta\lambda_0\right)\right] \\
&\geqslant \mu_2(\delta\lambda_0)\phi(\delta\lambda_0) + \left[1 - 2a/N^{\beta_2}\right] \sum_{k=0}^{\infty} \mu_2(N^{k+1}\delta\lambda_0)\phi\left(N^{k+1}\delta\lambda_0\right) \\
&\geqslant \frac{\delta^{\beta_1}}{M}\mu_2(\delta\lambda_0)\phi(\lambda_0) + \frac{C\delta^{\beta_1}}{M} \sum_{k=0}^{\infty} \mu_2(N^{k+1}\delta\lambda_0)\phi\left(N^{k+1}\lambda_0\right) \\
&\geqslant \frac{C\delta^{\beta_1}}{M} \sum_{k=0}^{\infty} \mu_2(N^k\delta\lambda_0)\phi\left(N^k\lambda_0\right),
\end{aligned}
$$

where M is the constant defined in (2.1). Combining the above estimates, we get

$$
\begin{aligned}
I_3 &\leqslant C \sum_{i=0}^{\infty} \phi(N^i\lambda_0)\mu_2(N^i\delta\lambda_0) \\
&\leqslant C \int_{B_{4R_0}} \phi\left(\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x)\right) dx.
\end{aligned}
$$

Therefore, we conclude that

$$
\int_{B_r} \phi(|Du|)\, dx \leqslant C\phi\left(\int_{B_{4R_0}} |Du|\, dx + 1\right) + C\int_{B_{4R_0}} \phi\left(\mathcal{M}_1(\mu)^{\frac{1}{1+i_a}}(x)\right) dx.
$$

This finishes the proof. □

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