WEIGHTED WEAK TYPE (1,1) ESTIMATE FOR THE CHRIST-JOURNÉ TYPE COMMUTATOR

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Abstract. Let K be the Calderón-Zygmund convolution kernel on $\mathbb{R}^d (d \geq 2)$. Christ and Journé defined the commutator associated with K and $a \in L^\infty(\mathbb{R}^d)$ by

$$T_a f(x) = \text{p.v.} \int_{\mathbb{R}^d} K(x-y) m_{x,y} a \cdot f(y) dy,$$

which is an extension of the classical Calderón commutator. In this paper, we show that T_a is weighted weak type (1,1) bounded with A_1 weight for $d \geq 2$.

1. Introduction

Assume that K is the Calderón-Zygmund convolution kernel on $\mathbb{R}^d \setminus \{0\}$ $(d \geq 2)$, which means that K satisfies the following three conditions:

$$|K(x)| \leq C|x|^{-d}, \tag{1.1}$$

$$\int_{R<|x|<2R} K(x)dx = 0, \text{ for all } R > 0, \tag{1.2}$$

$$|\nabla K(x)| \leq \frac{C}{|x|^{d+1}}. \tag{1.3}$$

In 1987, Christ and Journé [2] introduced a higher dimensional commutator associated with K and $a \in L^\infty(\mathbb{R}^d)$ by

$$T_a f(x) = \text{p.v.} \int_{\mathbb{R}^d} K(x-y) m_{x,y} a \cdot f(y) dy, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class and

$$m_{x,y} a = \int_0^1 a((1-t)x + ty)dt = \int_0^1 a(tx + (1-t)y)dt.$$

Note that T_a can be seen as a higher dimensional generalization of the Calderón commutator in [1]. In fact, when $d = 1$, let $A(x)$ be a Lipschitz function in $\mathbb{R}$ and denote $a(x) = A'(x) \in L^\infty(\mathbb{R})$. By using mean value formula, the Calderón commutator can be written as

$$\text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y)}{x-y} \frac{f(y)}{x-y} dy = \text{p.v.} \int_{\mathbb{R}} \frac{1}{x-y} \int_0^1 a(tx + (1-t)y)dt \cdot f(y)dy.$$

2010 *Mathematics Subject Classification.* 42B20.

Key words and phrases. Christ-Journé type commutator, weighted weak type (1,1), A_1 weight.

The work is supported by NSFC (No.11371057, No.11471033, No.11571160), SRFDP (No.20130003110003), the Fundamental Research Funds for the Central Universities (No.2014KJJCA10).

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Obviously, the right hand side of the equality above is just the Christ-Journé type commutator when $d = 1$.

Notice that for $d \geq 2$, the kernel $K(x-y)$ is smooth but $m_{x,y}a$ has no smoothness about variable x and y if $a \in L^\infty(\mathbb{R}^d)$. Therefore the standard Calderón-Zygmund theory cannot be applied directly. Christ and Journé [2] proved that T_a is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. In 1995, Hofmann [11] gave the weighted L^p $(1 < p < \infty)$ boundedness of T_a when the kernel $K(x) = \Omega(x/|x|)|x|^{-d}$ with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. In 2012, Grafakos and Honzík [9] proved that T_a is of weak type $(1,1)$ for $d = 2$. Later, Seeger [13] showed that T_a is still weak type $(1,1)$ bounded for all $d \geq 2$. In [3], we showed that T_a is bounded on $L^p(w)$ with $w \in A_p(1 < p < \infty)$ for $d \geq 2$ and T_a is weighted weak type (1,1) with power weight $|x|^{-\alpha}(-2 < \alpha < 0)$ in dimension $d = 2$.

In this paper, our goal is to show T_a is also weighted weak type (1,1) bounded in dimension $d \geq 2$. In the sequel, for $1 \leq p \leq \infty$, $A_p(\mathbb{R}^d)$ denotes the Muckenhoupt weight class and $L^p(w)$ denotes the weighted $L^p(\mathbb{R}^d)$ space with norm $\|\cdot\|_{p,w}$. We also denote $w(E) = \int_E w(x)dx$ for a measurable set E in $\mathbb{R}^d$.

Theorem 1.1. *Suppose K satisfies (1.1), (1.2) and (1.3) for $d \geq 2$. Let $a \in L^\infty(\mathbb{R}^d)$ and $w \in A_1(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that*

$$w\big(\{x \in \mathbb{R}^d : |T_a f(x)| > \lambda\}\big) \leq C\lambda^{-1}\|a\|_\infty \|f\|_{1,w}$$

for all $\lambda > 0$ and $f \in L^1(w)$.

Remark 1.2. In dimension $d = 2$, Grafakos and Honzík [9] used the TT^* method to obtain weak type (1,1) bound. In [3], we followed the idea in [9], using the weighted TT^* method to prove weighted weak (1,1) boundedness of T_a with power weight. The key point in [3] is to show the smoothness of the kernel of $(T_j^* T_j)_w$. However, the method in [3] can not be used to prove that T_a is of weighted weak type (1,1) for the general A_1 weight even when $d \geq 2$. Indeed, the reason why the smoothness of the kernel of $(T_j^* T_j)_w$ could be established in [3] is based on the fact that the power weight is smooth, but the general A_1 weight does not have smoothness.

The method presented in this paper is different from [3]. We will use some interpolation argument, which allows us to get weighted weak type (1,1) boundedness for the general A_1 weight. More precisely, we will obtain an unweighted weak type estimate with nice decay bound (see Lemma 2.3 below) and a weighted weak type estimate without decay bound (see (2.3) below). Interpolating these two estimate together, we may get a weighted weak type estimate with enough decay property. This kind of idea was first used to prove the weighted L^p boundedness. For more details, we refer to see [6], [16], [15] and [7].

Remark 1.3. In this paper, we have to get the unweighted weak type estimate with enough decay (i.e. Lemma 2.3 below). However, by using the original argument in [13], one may obtain $n^{-2}\log n$ as the decay bound in Lemma 2.3. This bound is not sufficient to prove weighted bound for A_1 weight. To obtain a enough decay bound, we should modify the whole progress of Seeger's argument and it is not trivial. Recall in [13], in proving the weak type (1,1) estimate of T_a, Seeger used a microlocal decomposition for the kernel and the Fourier transform of the function a, which involves the Littlewood-Paley decomposition and direction decomposition of the function a (see the proof of Proposition 2.3 in [13]). In this paper, we will follow some nice idea from Seeger [13] but there are some difference. First, we do not use this kind of decompositions on the function a because we observe that $m_{x,y}a$ has some smoothness in some sense. In fact, if writing $m_{x,y}a$ as

$$\frac{1}{|x-y|}\int_0^{|x-y|} a\Big(y+s\frac{x-y}{|x-y|}\Big)ds = \frac{1}{r}\int_0^r a(y+s\theta)ds$$

by making a polar transform $x-y=r\theta$, then even $a\in L^\infty(\mathbb{R}^d)$, $m_{x,y}a$ has some smoothness about r for a fixed y and θ. The proof of Lemma 2.3 is based on this observation. Secondly, we adopt a different method to use the cancelation of bad function (see subsection 5.3). It seems to be more direct though complicated. Here we also use some ideas from [12], [4] and [5].

This paper is organized as follows. In Section 2, we complete the proof of Theorem 1.1 based on Lemma 2.3. In Section 3, we prove Lemma 2.3 based on some lemmas, their proofs will be given in Section 4 and Section 5, respectively. Throughout this paper, the letter C stands for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. $A\lesssim B$ means $A\le CB$ for some constant C and $A\approx B$ means that $A\lesssim B$ and $B\lesssim A$. For a set $E\subset\mathbb{R}^d$, we denote by $|E|$ the Lebesgue measure of E. $\mathcal{F}f$ and $\hat{f}$ denote the Fourier transform of f defined by

$$\mathcal{F}f(\xi)=\int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x)dx.$$

$\mathbb{Z}_+$ denotes the set of all nonnegative integers and $\mathbb{Z}_+^d=\underbrace{\mathbb{Z}_+\times\cdots\times\mathbb{Z}_+}_{d}$. $[x]$ denotes the integer part of x.

2. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 based on a lemma, its proof will be given in Section 3. Let us begin with giving the definition of $A_1(\mathbb{R}^d)$ weight.

Definition 2.1 ($A_1(\mathbb{R}^d)$ weight)**.** A nonnegative local integrable function w on $\mathbb{R}^d$ is said to be a $A_1(\mathbb{R}^d)$ weight if there is a constant $C>0$ such that

$$Mw(x)\le Cw(x),$$

where M denotes the Hardy-Littlewood maximal operator defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f(y)|dy,$$

here $Q(x,r)$ denotes the cube with center at x and side length r and its sides parallel to the coordinate axes.

Fix $\lambda > 0$ and $f \in L^1(w)$, the following Calderón-Zygmund decomposition of $f \in L^1(w)$ is well known, see the proof of Theorem 3.5 in [10], for example.

Lemma 2.2. *Let $w \in A_1(\mathbb{R}^d)$ and $f \in L^1(w)$. Then for $a \in L^\infty(\mathbb{R}^d)$ and $\lambda > 0$, there exist functions g and b such that*

(i) $f = g + b$;

(ii) $\|g\|_{2,w}^2 \lesssim \frac{\lambda}{\|a\|_\infty}\|f\|_{1,w}$;

(iii) $b = \sum b_n$, supp$b_n \subset Q_n$, Q_n's are disjoint dyadic cubes, set $\mathcal{Q} = \{Q_n\}$;

(iv) $\int b_n = 0$, $\|b_n\|_1 \lesssim \frac{\lambda}{\|a\|_\infty}|Q_n|$, $\|b\|_{1,w} \lesssim \|f\|_{1,w}$;

(v) Each Q satisfies $|Q| \lesssim \frac{\|a\|_\infty}{\lambda}\int_Q |f(z)|dz$;

(vi) Set $E = \bigcup\limits_{Q\in\mathcal{Q}} Q$, then $w(E) \lesssim \frac{\|a\|_\infty}{\lambda}\|f\|_{1,w}$.

We only focus on the dimension $d \geq 2$. By the property (i) in Lemma 2.2,

$$w\big(\{x \in \mathbb{R}^d : |T_af(x)| > \lambda\}\big) \leq w\Big(\Big\{x \in \mathbb{R}^d : |T_ag(x)| > \frac{\lambda}{2}\Big\}\Big) + w\Big(\Big\{x \in \mathbb{R}^d : |T_ab(x)| > \frac{\lambda}{2}\Big\}\Big).$$

Since $g \in L^2(w)$, by [3, Theorem 1.2], we have $\|T_ag\|_{2,w} \lesssim \|a\|_\infty\|g\|_{2,w}$. Hence, use Chebychev's inequality and the property (ii) in Lemma 2.2,

$$w(\{x \in \mathbb{R}^d : |T_ag(x)| > \lambda/2\}) \lesssim \frac{\|T_ag\|_{2,w}^2}{\lambda^2} \lesssim \|a\|_\infty\frac{\|f\|_{1,w}}{\lambda}.$$

For $Q \in \mathcal{Q}$, denote by $l(Q)$ the side length of cube Q. For $t > 0$, let tQ be the cube with the same center of Q and $l(tQ) = tl(Q)$. Let $E^* = \bigcup_{Q\in\mathcal{Q}} 2^{200}Q$. Then we have

$$w\big(\{|T_ab(x)| > \lambda/2\}\big) \leq w(E^*) + w\big(\{x \in (E^*)^c : |T_ab(x)| > \lambda/2\}\big).$$

Since w satisfies the doubling condition, by (vi) in Lemma 2.2, the set E^* satisfies

$$w(E^*) \lesssim w(E) \lesssim \frac{\|a\|_\infty}{\lambda}\|f\|_{1,w}.$$

Denote $\mathfrak{Q}_k = \{Q \in \mathcal{Q} : \ l(Q) = 2^k\}$ and let $\mathfrak{B}_k = \sum\limits_{Q\in\mathfrak{Q}_k} b_Q$. Then b can be rewritten as $b = \sum\limits_{j\in\mathbb{Z}} \mathfrak{B}_j$. Let ψ be a radial $C^\infty(\mathbb{R}^d)$ function such that $\psi(\xi) = 1$ for $|\xi| \leq 1$, $\psi(\xi) = 0$ for $|\xi| \geq 2$ and $0 \leq \psi(\xi) \leq 1$ for all $\xi \in \mathbb{R}^d$. Define $\phi(x) = \psi(x) - \psi(2x)$. Then supp $\phi \subset \{x \in \mathbb{R}^d : \frac{1}{2} \leq |x| \leq 2\}$ and $\sum_j \phi_j(x) = 1$ for all $x \in \mathbb{R}^d\backslash\{0\}$, where $\phi_j(x) = \phi(2^{-j}x)$. Now we define the operator T_j as

$$T_jf(x) = \int_{\mathbb{R}^d} \phi_j(x-y)K(x-y)m_{x,y}a \cdot f(y)dy. \tag{2.1}$$

Then $T_a = \sum\limits_j T_j$. For simplicity, we set $K_j(x) = \phi_j(x)K(x)$. We write

$$T_ab(x) = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} T_j\mathfrak{B}_{j-n}.$$

Note that $T_j\mathfrak{B}_{j-n}(x) = 0$, for $x \in (E^*)^c$ and $n < 100$. Therefore

$$w\bigg(\bigg\{x \in (E^*)^c : |T_ab(x)| > \frac{\lambda}{2}\bigg\}\bigg) = w\bigg(\bigg\{x \in (E^*)^c : \bigg|\sum_{n\geq100}\sum_{j\in\mathbb{Z}} T_j\mathfrak{B}_{j-n}(x)\bigg| > \frac{\lambda}{2}\bigg\}\bigg).$$

To finish the proof of Theorem 1.1, it is enough to show

$$w\Big(\Big\{x\in\mathbb{R}^d:\Big|\sum_{n\ge100}\sum_{j\in\mathbb{Z}}T_j\mathfrak{B}_{j-n}(x)\Big|>\frac{\lambda}{2}\Big\}\Big)\lesssim\frac{1}{\lambda}\|a\|_\infty\|f\|_{1,w}. \tag{2.2}$$

Fix $n\ge 100$. By using Fubini's theorem and (iv) in Lemma 2.2, for any nonnegative function $\mu(x)$, we get

$$\begin{aligned}\Big\|\sum_j T_j\mathfrak{B}_{j-n}\Big\|_{1,\mu}&\lesssim\|a\|_\infty\sum_j\int|\mathfrak{B}_{j-n}(y)|\Big(\int|K_j(x-y)|\mu(x)dx\Big)dy\\&\lesssim\|a\|_\infty\sum_j\sum_{Q\in\mathfrak{Q}_{j-n}}\int|b_Q(y)|\frac{1}{2^{jn}}\int_{Q(y,2^{j+2})}\mu(x)dxdy\\&\lesssim\|a\|_\infty\sum_j\sum_{Q\in\mathfrak{Q}_{j-n}}\int|b_Q(y)|\inf_{z\in Q}\frac{1}{2^{jn}}\int_{Q(z,2^{j+3})}\mu(x)dxdy\\&\lesssim\|a\|_\infty\sum_j\sum_{Q\in\mathfrak{Q}_{j-n}}\int|b_Q(y)|\inf_{z\in Q}M(\mu)(z)dy\\&\lesssim\lambda\sum_{Q\in\mathcal{Q}}|Q|\inf_Q M(\mu),\end{aligned}$$

where in the third inequality we use the fact that $Q\in\mathfrak{Q}_{j-n}$ has side length 2^{j-n} and $\inf_Q M(\mu)=\inf_{z\in Q}M(\mu)(z)$. Then by Chebyshev's inequality,

$$\mu\Big(\Big\{x\in\mathbb{R}^d:\Big|\sum_j T_j\mathfrak{B}_{j-n}(x)\Big|>\lambda\Big\}\Big)\lesssim\sum_Q|Q|\inf_{Q\in\mathcal{Q}}M(\mu). \tag{2.3}$$

Here we should point out that the right side of (2.3) is bounded by $\frac{\|a\|_\infty}{\lambda}\|f\|_{1,w}$ by using the property (v) in Lemma 2.2 if we set $\mu=w\in A_1(\mathbb{R}^d)$. So to prove (2.2), we need to get a estimate better than (2.3) with a decay bound like $2^{-n\varepsilon}$ with $\varepsilon>0$. It may be difficult to obtain this kind of estimate directly for higher dimension (in [3], we got similar estimates with enough decay bounds directly when $d=2$). However, we will use an interpolation arguments between the estimate (2.3) and the following lemma.

Lemma 2.3. *There exists an $\varepsilon>0$ such that for any integer $n\ge100$,*

$$\Big|\Big\{x\in\mathbb{R}^d:\Big|\sum_j T_j\mathfrak{B}_{j-n}(x)\Big|>\lambda\Big\}\Big|\lesssim n2^{-\varepsilon n}\sum_{Q\in\mathcal{Q}}|Q|. \tag{2.4}$$

The proof of Lemma 2.3 will be given in Section 3. We continue the interpolation arguments. For convenience, define

$$E_\lambda^n=\Big\{x\in\mathbb{R}^d:\Big|\sum_j T_j\mathfrak{B}_{j-n}(x)\Big|>\lambda\Big\}.$$

Lemma 2.4. *For any constant $u>0$ and nonnegative function $\mu(x)$,*

$$\int_{E_\lambda^n}\min\{\mu(x),u\}dx\lesssim\sum_{Q\in\mathcal{Q}}|Q|\min\{un2^{-\varepsilon n},\inf_Q M(\mu)\}, \tag{2.5}$$

where ε is determined by Lemma 2.3.

Proof. For $u>0$, we set

$$\mathfrak{C}_u=\{Q\in\mathcal{Q}:\inf_Q M(\mu)<un2^{-\varepsilon n}\}$$

and $\mathfrak{C}_u^c=\mathcal{Q}\setminus\mathfrak{C}_u$. For each j, split $\mathfrak{B}_j$ into two parts $\mathfrak{B}_j=\mathfrak{B}_j^{'}+\mathfrak{B}_j^{''}$, where

$$\mathfrak{B}_j^{\prime}=\sum_{l(Q)=2^j,Q\in\mathfrak{C}_u}b_Q\quad\text{and}\quad\mathfrak{B}_j^{''}=\sum_{l(Q)=2^j,Q\in\mathfrak{C}_u^c}b_Q.$$

Define

$$E_\lambda^{n'}=\Big\{x\in\mathbb{R}^d:\Big|\sum_j T_j\mathfrak{B}_{j-n}^{\prime}(x)\Big|>\lambda\Big\},$$
$$E_\lambda^{n''}=\Big\{x\in\mathbb{R}^d:\Big|\sum_j T_j\mathfrak{B}_{j-n}^{''}(x)\Big|>\lambda\Big\}.$$

By the linearity of T_j, $E_\lambda^n\subset E_{\lambda/2}^{n'}\bigcup E_{\lambda/2}^{n''}$. Therefore

$$\begin{aligned}\int_{E_\lambda^n}\min\{\mu(x),u\}dx&\le\Big(\int_{E_{\lambda/2}^{n'}}+\int_{E_{\lambda/2}^{n''}}\Big)\min\{\mu(x),u\}dx\\&\le\int_{E_{\lambda/2}^{n'}}\mu(x)dx+\int_{E_{\lambda/2}^{n''}}udx\\&=:I+II.\end{aligned}$$

By using (2.3) and Lemma 2.3, we get the estimates of I and II, respectively,

$$I\lesssim\sum_{Q\in\mathfrak{C}_u}|Q|\inf_Q M(w)=\sum_{Q\in\mathfrak{C}_u}|Q|\min\{un2^{-\varepsilon n},\inf M(\mu)\},$$
$$II\lesssim un2^{-\varepsilon n}\sum_{Q\in\mathfrak{C}_u^c}|Q|=\sum_{Q\in\mathfrak{C}_u^c}|Q|\min\{un2^{-\varepsilon n},\inf_Q M(\mu)\}.$$

Combining these estimates for I and II, we finish the proof. □

Now we return to the proof of Theorem 1.1. Multiply both sides of the inequality (2.5) by $u^{-1+\theta}(\theta\in(0,1))$, and integrate them on $(0,\infty)$ with respect to the measure du/u. Using the following formula

$$\int_0^\infty\min\{A,u\}u^{-1+\theta}\frac{du}{u}=C_\theta A^\theta,$$

then by Fubini's theorem , we could get

$$\begin{aligned}\int_{E_\lambda^n}\mu(x)^\theta dx&\lesssim\sum_{Q\in\mathcal{Q}}|Q|(n2^{-\varepsilon n})^{1-\theta}\inf_Q M(\mu)^\theta\\&\lesssim\lambda^{-1}(n2^{-\varepsilon n})^{1-\theta}\|a\|_\infty\sum_{Q\in\mathcal{Q}}\inf_Q M(\mu)^\theta\int_Q|f(x)|dx\\&\le\lambda^{-1}(n2^{-\varepsilon n})^{1-\theta}\|a\|_\infty\int_{\mathbb{R}^d}|f(x)|M(\mu)^\theta(x)dx\end{aligned}\tag{2.6}$$

where the second inequality follows from the property (v) in Lemma 2.2. Since $w\in A_1(\mathbb{R}^d)$ which means $w(x)$ is nonnegative, we can substitute $w^{1/\theta}$ for μ in (2.6). So we obtain

$$w(E_\lambda^n)=\int_{E_\lambda^n}w(x)dx\lesssim\lambda^{-1}(n2^{-\varepsilon n})^{1-\theta}\|a\|_\infty\int_{\mathbb{R}^d}|f(x)|M_{\frac{1}{\theta}}(w)(x)dx,$$

where M_q is the Hardy-Littlewood maximal operator of order q defined by

$$M_q f(x) = \sup_{s>0} \left(\frac{1}{|Q(x,s)|} \int_{Q(x,s)} |f(y)|^q dy \right)^{1/q}.$$

Choose $\sigma > 1$ and $C_\sigma > 0$ such that $C_\sigma \sum_{n\geq 100} n^{-\sigma} = \frac{1}{2}$. Then we have

$$w(E^n_{C_\sigma n^{-\sigma}\frac{\lambda}{2}}) \lesssim n^{\sigma+1-\theta} 2^{-n\varepsilon(1-\theta)} \lambda^{-1} \|a\|_\infty \|f\|_{1,M_{\frac{1}{\theta}}(w)}. \tag{2.7}$$

By using the pigeonhole principle, it is easy to see that

$$\Big\{x : \sum_i f_i(x) > \sum_i \lambda_i \Big\} \subseteq \bigcup_i \Big\{ x : f_i(x) > \lambda_i \Big\}. \tag{2.8}$$

Since $\sum_{n\geq 100} C_\sigma n^{-\sigma} = \frac{1}{2}$, we may get

$$\Big\{ x \in \mathbb{R}^d : \sum_{n\geq 100} \Big| \sum_{j\in\mathbb{Z}} T_j \mathfrak{B}_{j-n}(x) \Big| > \frac{\lambda}{2} \Big\} \subset \bigcup_{n\geq 100} E^n_{C_\sigma n^{-\sigma}\lambda}.$$

By (2.7), we obtain

$$w\Big(\Big\{ x \in \mathbb{R}^d : \sum_{n\geq 100} \Big| \sum_{j\in\mathbb{Z}} T_j \mathfrak{B}_{j-n} \Big| > \lambda \Big\}\Big) \leq \sum_{n\geq 100} w(E^n_{C_\sigma n^{-\sigma}\lambda}) \lesssim \frac{1}{\lambda} \|a\|_\infty \|f\|_{1,M_{\frac{1}{\theta}}(w)}. \tag{2.9}$$

Since w is an $A_1(\mathbb{R}^d)$ weight, there exists $r > 1$ such that $w^r \in A_1(\mathbb{R}^d)$ (see [8, Theorem 7.2.5]). Thus, we take $\theta = \frac{1}{r}$ in (2.9). By the definition of $A_1(\mathbb{R}^d)$ weight, we have $M_r(w)(x) \lesssim w(x)$. Hence we get (2.2) by (2.9) and finish the proof of Theorem 1.1 once we show Lemma 2.3.

3. Proof of Lemma 2.3

To prove Lemma 2.3, we will make a series of decomposition of T_j. Some important estimates play a key role in the proof. We present them by some lemmas, which will be proved in Section 4 and Section 5, respectively. The first estimate tells us that the operator T_j can be approximated by an operator T^n_j in measure, which is defined below.

Let $l_\tau(n) = \tau n + 2$, where $0 < \tau < 1$ will be chosen later. Let η be a nonnegative, radial $C^\infty_c(\mathbb{R}^d)$ function which is supported in $\{|x| \leq 1\}$ and satisfies $\int_{\mathbb{R}^d} \eta(x)dx = 1$. Set $\eta_i(x) = 2^{-id}\eta(2^{-i}x)$. Define

$$K^n_j(x) = \eta_{j-l_\tau(n)} * K_j(x).$$

Since $K_j(x)$ is supported in $\{2^{j-1} \leq |x| \leq 2^{j+1}\}$ and $\eta_{j-l_\tau(n)}(x)$ is supported in $\{|x| \leq 2^{j-l_\tau(n)}\}$, we see that $K^n_j(x)$ is supported in $\{2^{j-2} \leq |x| \leq 2^{j+2}\}$. Therefore

$$|K^n_j(x)| \lesssim 2^{-jd} \chi_{\{2^{j-2}\leq|x|\leq 2^{j+2}\}} \tag{3.1}$$

and similarly for multi-indices α

$$|\partial^\alpha K^n_j(x)| \lesssim 2^{-jd+(l_\tau(n)-j)|\alpha|} \chi_{\{2^{j-2}\leq|x|\leq 2^{j+2}\}}. \tag{3.2}$$

Let ρ_n be a smooth, nonnegative function defined on $\mathbb{R}$ such that $\rho_n(s)=1$ on $[2^{-\tau n},1-2^{-\tau n}]$, supp $\rho_n\subset(2^{-\tau n-1},1-2^{-\tau n-1})$, and the derivatives of ρ_n satisfy the natural estimates

$$\Big|\frac{\mathrm{d}^k}{\mathrm{d}s^k}\rho_n(s)\Big|\lesssim 2^{k\tau n}\quad\text{for all } k\in\mathbb{Z}_+.$$

Let

$$m^n_{x,y}a=\int_0^1\rho_n(s)a(sx+(1-s)y)ds.$$

Define the operator T^n_j by

$$T^n_jh(x)=\int_{\mathbb{R}^d}K^n_j(x-y)m^n_{x,y}a\cdot h(y)dy.$$

Lemma 3.1. *With the definitions above, for $n\geq 100$,*

$$\Big|\Big\{x\in\mathbb{R}^d:\Big|\sum_j\big(T_j\mathfrak{B}_{j-n}(x)-T^n_j\mathfrak{B}_{j-n}(x)\big)\Big|>\frac{\lambda}{4}\Big\}\Big|\lesssim 2^{-n\tau}\sum_Q|Q|.$$

By Lemma 3.1, it is easy to see that the proof of Lemma 2.3 now is reduced to find $\varepsilon>0$ such that

$$\Big|\Big\{x\in\mathbb{R}^d:\Big|\sum_{j\in\mathbb{Z}}T^n_j\mathfrak{B}_{j-n}(x)\Big|>\frac{\lambda}{4}\Big\}\Big|\lesssim n2^{-n\varepsilon}\sum_Q|Q|. \tag{3.3}$$

In the following, we need to make a microlocal decomposition of the kernel. To do this, we give a partition of unity on the unit surface $\mathbb{S}^{d-1}$. Choose $n\geq 100$. Let $\Theta_n=\{e^n_v\}_v$ be a collection of unit vectors on $\mathbb{S}^{d-1}$ which satisfies the following two conditions:

(a) $|e^n_v-e^n_{v'}|\geq 2^{-n\gamma-4}$, if $v\neq v'$;

(b) If $\theta\in\mathbb{S}^{d-1}$, there exists an e^n_v such that $|e^n_v-\theta|\leq 2^{-n\gamma-4}$.

The constant γ in (a) and (b) satisfying $\tau<\gamma<1$ will be chosen later. In fact, we may simply take a maximal collection $\{e^n_v\}_v$ for which (a) holds. Notice that there are $C2^{n\gamma(d-1)}$ elements in the collection $\{e^n_v\}_v$. For every $\theta\in\mathbb{S}^{d-1}$, there only exists finite e^n_v such that $|e^n_v-\theta|\leq 2^{-n\gamma-4}$. Now we can construct an associated partition of unity on the unit surface $\mathbb{S}^{d-1}$. Let ζ be a smooth, nonnegative, radial function on $\mathbb{R}^d$ with $\zeta(u)=1$ for $|u|\leq\frac12$ and $\zeta(u)=0$ for $|u|>1$. Set

$$\tilde\Gamma^n_v(\xi)=\zeta\Big(2^{n\gamma}(\frac{\xi}{|\xi|}-e^n_v)\Big)$$

and define

$$\Gamma^n_v(\xi)=\tilde\Gamma^n_v(\xi)\Big(\sum_{e^n_v\in\Theta_n}\tilde\Gamma^n_v(\xi)\Big)^{-1}.$$

Then it is easy to see that Γ^n_v is homogeneous of degree 0 with $\sum_v\Gamma^n_v(\xi)=1$, for all $\xi\neq0$ and all n. In addition, the following estimate holds for multi-indices α and $\xi\neq0$,

$$|\partial^\alpha_\xi\Gamma^n_v(\xi)|\lesssim 2^{n\gamma|\alpha|}|\xi|^{-|\alpha|}. \tag{3.4}$$

Now we define operator $T^{n,v}_j$ by

$$T^{n,v}_jh(x)=\int_{\mathbb{R}^d}K^{n,v}_j(x-y)m^n_{x,y}a\cdot h(y)dy, \tag{3.5}$$

where $K_j^{n,v}(x) = K_j^n(x)\Gamma_v^n(x)$. Therefore, we have

$$T_j^n = \sum_v T_j^{n,v}.$$

In the sequel, we need to separate the phase into different directions. Hence we define a multiplier operator by

$$\widehat{G_{n,v}h}(\xi) = \Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle)\hat{h}(\xi),$$

where h is a Schwartz function and Φ is a smooth, nonnegative, radial function such that $0 \le \Phi(x) \le 1$ and $\Phi(x) = 1$ on $|x| \le 2$, $\Phi(x) = 0$ on $|x| > 4$. Now we can split $T_j^{n,v}$ into two parts:

$$T_j^{n,v} = G_{n,v}T_j^{n,v} + (I - G_{n,v})T_j^{n,v},$$

where I denotes the identity operator. The following lemma gives the L^2 estimate involving $G_{n,v}T_j^{n,v}$, which will be proved in next section.

Lemma 3.2. *With the definitions above, for $n \ge 100$,*

$$\Big\| \sum_j \sum_v G_{n,v}T_j^{n,v}\mathfrak{B}_{j-n}\Big\|_2^2 \lesssim 2^{-n\gamma}\lambda^2 \sum_Q |Q|.$$

The estimates of the terms involving $(I - G_{n,v})T_j^{n,v}$ are more complicated. In Section 5, we shall prove the following lemma.

Lemma 3.3. *For any $n \ge 100$, there exists a $\varepsilon > 0$ such that*

$$\Big|\Big\{x \in \mathbb{R}^d : \Big|\sum_j \sum_v (I - G_{n,v})T_j^{n,v}\mathfrak{B}_{j-n}(x)\Big| > \lambda\Big\}\Big| \lesssim n2^{-n\varepsilon}\sum_Q |Q|.$$

Now we can finish the proof of Lemma 2.3. It suffices to consider (3.3). By Chebychev's inequality,

$$\begin{aligned}
&\Big|\Big\{x \in \mathbb{R}^d : \Big|\sum_j T_j^n\mathfrak{B}_{j-n}(x)\Big| > \frac{\lambda}{4}\Big\}\Big| \\
&\lesssim \lambda^{-2}\Big\|\sum_j \sum_v G_{n,v}T_j^{n,v}\mathfrak{B}_{j-n}\Big\|_2^2 + \Big|\Big\{x \in \mathbb{R}^d : \Big|\sum_j \sum_v (I - G_{n,v})T_j^{n,v}\mathfrak{B}_{j-n}(x)\Big| > \frac{\lambda}{8}\Big\}\Big|
\end{aligned}$$

By Lemma 3.2 and Lemma 3.3, we can get the estimates for the first term and the second term above, respectively. We hence complete the proof of Lemma 2.3 once Lemmas 3.1-3.3 hold.

4. PROOFS OF LEMMAS 3.1-3.2

4.1. **Proof of Lemma 3.1.**

We first focus on the proof of Lemma 3.1. By the definitions of T_j and T_j^n, we see

$$\begin{aligned}
\|T_jf - T_j^nf\|_1 &= \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\Big(K_j(x-y)m_{x,y}a - K_j^n(x-y)m_{x,y}^n a\Big)f(y)dy\Big|dx \\
&\le I + II,
\end{aligned}$$

where

$$I=\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}(K_j(x-y)-K_j^n(x-y))m_{x,y}a\cdot f(y)dy\Big|dx,$$

$$II=\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}K_j^n(x-y)(m_{x,y}a-m^n_{x,y}a)f(y)dy\Big|dx.$$

Consider I firstly. By the definition of $K_j^n(x)$, we see

$$K_j(x-y)-K_j^n(x-y)=\int_{\mathbb{R}^d}\eta_{j-l_\tau(n)}(z)(K_j(x-y)-K_j(x-y-z))dz.$$

Notice that

$$\begin{aligned}|K_j(x-y)-K_j(x-y-z)|\le{}&|\phi_j(x-y)(K(x-y)-K(x-y-z))|\\&+|\phi_j(x-y)-\phi_j(x-y-z)||K(x-y-z)|.\end{aligned}$$

Consider the first term. Note that $|z|\le 2^{j-l_\tau(n)}$ and $2^{j-1}\le|x-y|\le 2^{j+1}$, then we have $2|z|<|x-y|$. By the regularity condition (1.3), the first term above is bounded by

$$\frac{|z|}{|x-y|^{d+1}}\chi_{\{2^{j-1}\le|x-y|\le 2^{j+1}\}}\lesssim 2^{-\tau n}2^{-jd}\chi_{\{2^{j-1}\le|x-y|\le 2^{j+1}\}}.$$

For the second term, by the fact $|z|\le 2^{j-l_\tau(n)}$ and the support of ϕ_j, we have $|x-y|\approx|x-z-y|$ and $2^{j-2}\le|x-y|\le 2^{j+2}$. By (1.1), the second term is controlled by

$$\frac{2^{-j}|z|}{|x-z-y|^{d}}\chi_{\{2^{j-2}\le|x-y|\le 2^{j+2}\}}\lesssim 2^{-\tau n}2^{-jd}\chi_{\{2^{j-2}\le|x-y|\le 2^{j+2}\}}.$$

Consider II, we get

$$|m_{x,y}a-m^n_{x,y}a|=\Big|\int_0^1(1-\rho_n(s))a(sx+(1-s)y)ds\Big|\lesssim 2^{-\tau n}\|a\|_\infty.$$

Combining the above three estimates and applying Minkowski's inequality, we obtain

$$\begin{aligned}\|T_jf-T_j^nf\|_1&\lesssim 2^{-\tau n}\|a\|_\infty\int_{\mathbb{R}^d}\int_{2^{j-2}\le|x-y|\le 2^{j+2}}2^{-jd}\int_{\mathbb{R}^d}\eta_{j-l_\tau(n)}(z)dz|f(y)|dydx\\&\lesssim 2^{-\tau n}\|a\|_\infty 2^{-jd}\int_{\mathbb{R}^d}\int_{2^{j-2}\le|x-y|\le 2^{j+2}}|f(y)|dydx\lesssim 2^{-\tau n}\|a\|_\infty\|f\|_1.\end{aligned}\tag{4.1}$$

By Chebychev's inequality, Minkowski's inequality, (4.1) and the property (iv) in Lemma 2.2, we get the bound

$$\begin{aligned}&\Big|\Big\{x\in\mathbb{R}^d:\Big|\sum_j T_j\mathfrak{B}_{j-n}(x)-T_j^n\mathfrak{B}_{j-n}(x)\Big|>\frac{\lambda}{4}\Big\}\Big|\\&\lesssim\lambda^{-1}\sum_j\Big\|T_j\mathfrak{B}_{j-n}-T_j^n\mathfrak{B}_{j-n}\Big\|_1\\&\lesssim\lambda^{-1}\|a\|_\infty 2^{-\tau n}\sum_j\|\mathfrak{B}_{j-n}\|_1\lesssim 2^{-n\tau}\sum_Q|Q|,\end{aligned}$$

which is the required estimate. □

4.2. **Proof of Lemma 3.2.**

The proof of this part is similar to [13]. For completeness, we still give a proof here. As usually, we adopt the TT^* method in the L^2 estimate. Moreover, we also use some orthogonality argument based on the following observation of the support of $\mathcal{F}(G_{n,v}T_j^{n,v})$: For a fixed $n \geq 100$, we have

$$\sup_{\xi\neq 0}\sum_v |\Phi^2(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle)| \lesssim 2^{n\gamma(d-2)}. \tag{4.2}$$

In fact, by the homogeneousness of Φ, it suffices to take the supremum over the surface $\mathbb{S}^{d-1}$. For $|\xi| = 1$ and $\xi \in \text{supp } \Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle)$, denote by $\xi^\perp$ the hyperplane perpendicular to ξ. Thus

$$\text{dist}(e_v^n, \xi^\perp) \lesssim 2^{-n\gamma}. \tag{4.3}$$

Since the mutual distance of e_v^n's is bounded by $2^{-n\gamma-4}$, there are at most $2^{n\gamma(d-2)}$ vectors satisfy (4.3). We hence get (4.2).

By applying Plancherel's theorem and Cauchy-Schwartz inequality, we have

$$\begin{aligned}\Big\|\sum_v\sum_j G_{n,v}T_j^{n,v}\mathfrak{B}_{j-n}\Big\|_2^2 &= \Big\|\sum_v \Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle)\mathcal{F}\Big(\sum_j T_j^{n,v}\mathfrak{B}_{j-n}\Big)(\xi)\Big\|_2^2\\ &\lesssim 2^{n\gamma(d-2)}\Big\|\sum_v\Big|\mathcal{F}\Big(\sum_j T_j^{n,v}\mathfrak{B}_{j-n}\Big)\Big|^2\Big\|_1\\ &\lesssim 2^{n\gamma(d-2)}\sum_v\Big\|\sum_j T_j^{n,v}\mathfrak{B}_{j-n}\Big\|_2^2.\end{aligned} \tag{4.4}$$

Once it is showed that for a fixed e_v^n,

$$\Big\|\sum_j T_j^{n,v}\mathfrak{B}_{j-n}\Big\|_2^2 \lesssim 2^{-2n\gamma(d-1)}\lambda\|a\|_\infty\sum_j\|\mathfrak{B}_{j-n}\|_1, \tag{4.5}$$

then by $\text{card}(\Theta_n) \lesssim 2^{n\gamma(d-1)}$, and applying (4.4), (4.5) and the property (iv) in Lemma 2.2 we get

$$\Big\|\sum_v\sum_j G_{n,v}T_j^{n,v}\mathfrak{B}_{j-n}\Big\|_2^2 \lesssim 2^{-n\gamma}\lambda\|a\|_\infty\sum_j\|\mathfrak{B}_{j-n}\|_1 \lesssim 2^{-n\gamma}\lambda^2\sum_Q|Q|,$$

which is just desired bound of Lemma 3.2. Thus, to finish the proof of Lemma 3.2, it is enough to prove (4.5). By applying (3.1) and the support of Γ_v^n,

$$\begin{aligned}|T_j^{n,v}\mathfrak{B}_{j-n}(x)| &\lesssim \|a\|_\infty\int_{\mathbb{R}^d}\Gamma_v^n(x-y)|K_j^n(x,y)||\mathfrak{B}_{j-n}(y)|dy\\ &\lesssim \|a\|_\infty H_j^{n,v} * |\mathfrak{B}_{j-n}|(x),\end{aligned}$$

where $H_j^{n,v}(x) := 2^{-jd}\chi_{E_j^{n,v}}(x)$ and $\chi_{E_j^{n,v}}(x)$ is a characteristic function of the set

$$E_j^{n,v} := \big\{x \in \mathbb{R}^d : |\langle x, e_v^n\rangle| \leq 2^{j+2} \quad \text{and} \quad |x - \langle x, e_v^n\rangle e_v^n| \leq 2^{j+2-n\gamma}\big\}.$$

For a fixed e_v^n, we write

$$
\begin{aligned}
\Big\| \sum_j T_j^{n,v} \mathfrak{B}_{j-n} \Big\|_2^2 &\lesssim \|a\|_\infty^2 \sum_j \int_{\mathbb{R}^d} H_j^{n,v} * H_j^{n,v} * |\mathfrak{B}_{j-n}|(x) \cdot |\mathfrak{B}_{j-n}(x)| dx \\
&\quad + \|a\|_\infty^2 \sum_j \sum_{i=-\infty}^{j-1} \int_{\mathbb{R}^d} H_j^{n,v} * H_i^{n,v} * |\mathfrak{B}_{i-n}|(x) \cdot |\mathfrak{B}_{j-n}(x)| dx.
\end{aligned} \tag{4.6}
$$

Observe that $\|H_i^{n,v}\|_1 \lesssim 2^{-id} m(E_i^{n,v}) \lesssim 2^{-n\gamma(d-1)}$, therefore for any $i \le j$,

$$
H_j^{n,v} * H_i^{n,v}(x) \le 2^{-n\gamma(d-1)} 2^{-jd} \chi_{\widetilde{E}_j^{n,v}},
$$

where $\widetilde{E}_j^{n,v} = E_j^{n,v} + E_j^{n,v}$. Hence for a fixed j, n, e_v^n and x, we get

$$
\begin{aligned}
&H_j^{n,v} * H_j^{n,v} * |\mathfrak{B}_{j-n}|(x) + \sum_{i=-\infty}^{j-1} H_j^{n,v} * H_i^{n,v} * |\mathfrak{B}_{i-n}|(x) \\
&\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i \le j} \int_{x+\widetilde{E}_j^{n,v}} |\mathfrak{B}_{i-n}(y)| dy \\
&\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i \le j} \sum_{\substack{Q \in \mathfrak{Q}_{i-n} \\ Q \cap \{x+\widetilde{E}_j^{n,v}\} \neq \emptyset}} \int_{\mathbb{R}^d} |b_Q(y)| dy \\
&\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i \le j} \sum_{\substack{Q \in \mathfrak{Q}_{i-n} \\ Q \cap \{x+\widetilde{E}_j^{n,v}\} \neq \emptyset}} \frac{\lambda}{\|a\|_\infty} |Q| \\
&\lesssim 2^{-n\gamma(d-1)} 2^{-jd} 2^{jd-n\gamma(d-1)} \frac{\lambda}{\|a\|_\infty} \\
&\lesssim \frac{\lambda}{\|a\|_\infty} 2^{-2n\gamma(d-1)},
\end{aligned} \tag{4.7}
$$

where in the third inequality above, we use $\int |b_Q(y)| dy \lesssim \lambda |Q| / \|a\|_\infty$ (see the property (iv) in Lemma 2.2) and in the fourth inequality we use fact that the cubes in $\mathcal{Q}$ are disjoint (see the property (iii) in Lemma 2.2). By (4.6) and (4.7), we obtain (4.5) and complete the proof of Lemma 3.2. □

5. Proof of Lemma 3.3

To prove Lemma 3.3, we have to deal with some oscillatory integrals which come from $(I - G_{n,v}) T_j^{n,v}$. We first introduce Mihlin multiplier theorem, which can be found in [8].

Lemma 5.1. *Let m be a complex-value bounded function on $\mathbb{R}^d \setminus \{0\}$ that satisfies*

$$
|\partial_\xi^\alpha m(\xi)| \le A |\xi|^{-|\alpha|}
$$

for all multi indices $|\alpha| \le [\frac{d}{2}] + 1$, then the operator T_m defined by

$$
\widehat{T_m f}(\xi) = m(\xi) \hat{f}(\xi)
$$

can be extended to a weak type (1,1) *bounded operator with bound* $C_d(A+\|m\|_\infty)$.

Before stating the proof of Lemma 3.3, let us give some notations. We first introduce the Littlewood-Paley decomposition. Let ψ be a radial $C^\infty(\mathbb{R}^d)$ function such that $\psi(\xi)=1$ for $|\xi|\le 1$, $\psi(\xi)=0$ for $|\xi|\ge 2$ and $0\le\psi(\xi)\le 1$ for all $\xi\in\mathbb{R}^d$. Define $\psi_k(\xi)=\psi(2^k\xi)$ and $\beta_k(\xi)=\psi_k(\xi)-\psi_{k+1}(\xi)$, then β_k is supported in $\{\xi: 2^{-k-1}\le|\xi|\le 2^{-k+1}\}$ and $\sum_k\beta_k(\xi)=1$ for $\xi\in\mathbb{R}^d\setminus\{0\}$. Define the convolution operators V_k and Λ_k with Fourier multipliers $\psi_k(\cdot)$ and β_k, respectively. That is,

$$\widehat{V_k f}(\xi)=\psi_k(\xi)\hat{f}(\xi)$$

and

$$\widehat{\Lambda_k f}(\xi)=\beta_k(\xi)\hat{f}(\xi).$$

Then by the construction of β_k and ψ, we have

$$I=\sum_{k\in\mathbb{Z}}\Lambda_k=V_m+\sum_{k<m}\Lambda_k \quad \text{for every } m\in\mathbb{Z}.$$

Set $A^{n,v}_{j,m}=V_mT^{n,v}_j$ and $D^{n,v}_{j,k}=(I-G_{n,v})\Lambda_kT^{n,v}_j$. Write

$$\begin{aligned}(I-G_{n,v})T^{n,v}_j&=(I-G_{n,v})V_mT^{n,v}_j+\sum_{k<m}(I-G_{n,v})\Lambda_kT^{n,v}_j\\&=:(I-G_{n,v})A^{n,v}_{j,m}+\sum_{k<m}D^{n,v}_{j,k},\end{aligned}$$

where $m=j-[n\varepsilon_0]$, $\varepsilon_0>0$ will be chosen later. To prove Lemma 3.3, we split the measure in Lemma 3.3 into two parts,

$$\begin{aligned}&\Big|\Big\{x\in\mathbb{R}^d:\Big|\sum_v\sum_j(I-G_{n,v})T^{n,v}_j\mathfrak{B}_{j-n}(x)\Big|>\lambda\Big\}\Big|\\&\le\Big|\Big\{x\in\mathbb{R}^d:\Big|\sum_v(I-G_{n,v})\big(\sum_jA^{n,v}_{j,m}\mathfrak{B}_{j-n}\big)(x)\Big|>\frac{\lambda}{2}\Big\}\Big|\\&\quad+\Big|\Big\{x\in\mathbb{R}^d:\Big|\sum_v\sum_j\sum_{k<m}D^{n,v}_{j,k}\mathfrak{B}_{j-n}(x)\Big|>\frac{\lambda}{2}\Big\}\Big|\\&=:I+II.\end{aligned}\tag{5.1}$$

5.1. **First step: basic estimates of I and II.**

For I, notice that $\mathcal{F}[(I-G_{n,v})f](\xi)=(1-\Phi(2^{n\gamma}\langle e^n_v,\xi/|\xi|\rangle))\cdot\hat{f}(\xi)$. It is easy to see that $(1-\Phi(2^{n\gamma}\langle e^n_v,\xi/|\xi|\rangle))$ is bounded and

$$|\partial^\alpha_\xi(1-\Phi(2^{n\gamma}\langle e^n_v,\xi/|\xi|\rangle))|\lesssim 2^{n\gamma([\frac{d}{2}]+1)}|\xi|^{-|\alpha|}$$

for all multi indices $|\alpha|\le[\frac{d}{2}]+1$. Then by Lemma 5.1, $I-G_{n,v}$ is of weak type (1,1) with bound $C2^{n\gamma([\frac{d}{2}]+1)}$.

Since $\operatorname{card}(\Theta_n) \approx 2^{n\gamma(d-1)}$, then there exists $C_{\gamma,d}$ such that $\sum_{e_v^n \in \Theta_n} C_d 2^{-n\gamma(d-1)} = \frac{1}{2}$. Therefore

$$\begin{aligned}
&\Big|\Big\{x \in \mathbb{R}^d : \Big|\sum_v (I - G_{n,v})\Big(\sum_j A_{j,m}^{n,v}\mathfrak{B}_{j-n}\Big)(x)\Big| > \frac{\lambda}{2}\Big\}\Big| \\
&= \Big|\Big\{x \in \mathbb{R}^d : \Big|\sum_v (I - G_{n,v})\Big(\sum_j A_{j,m}^{n,v}\mathfrak{B}_{j-n}\Big)(x)\Big| > \sum_v C_{\gamma,d}2^{-n\gamma(d-1)}\lambda\Big\}\Big| \\
&\le \sum_v \Big|\Big\{x \in \mathbb{R}^d : \Big|(I - G_{n,v})\Big(\sum_j A_{j,m}^{n,v}\mathfrak{B}_{j-n}\Big)(x)\Big| > C_{\gamma,d}2^{-n\gamma(d-1)}\lambda\Big\}\Big| \\
&\le \sum_j \sum_v \frac{1}{C_{\gamma,d}\lambda} 2^{n\gamma(d-1)+n\gamma([\frac{d}{2}]+1)} \|A_{j,m}^{n,v}\mathfrak{B}_{j-n}\|_1 \\
&\le \sum_j \sum_v \sum_{l(Q)=2^{j-n}} \frac{1}{C_{\gamma,d}\lambda} 2^{n\gamma(d-1)+n\gamma([\frac{d}{2}]+1)} \|A_{j,m}^{n,v} b_Q\|_1,
\end{aligned} \tag{5.2}$$

where the second inequality follows from (2.8) and in the third inequality we use $I - G_{n,v}$ is weak type (1,1) bounded and Minkowski's inequality.

For II, we use L^1 estimate directly

$$II \le \frac{2}{\lambda}\sum_v \sum_j \sum_{k<m} \|D_{j,k}^{n,v}\mathfrak{B}_{j-n}\|_1 \le \frac{2}{\lambda}\sum_v \sum_j \sum_{k<m} \sum_{l(Q)=2^{j-n}} \|D_{j,k}^{n,v} b_Q\|_1 \tag{5.3}$$

Now the problem is reduced to estimate $\|A_{j,m}^{n,v} b_Q\|_1$ and $\|D_{j,k}^{n,v} b_Q\|_1$. Recall in (3.5), the kernel of operator $T_j^{n,v}$ is

$$K_{j,y}^{n,v}(x) = \Gamma_v^n(x-y)K_j^n(x-y)m_{x,y}^n a.$$

Now we see $K_{j,y}^{n,v}(x)$ as a function of x for a fixed $y \in Q$. Thus, by Fubini's theorem,

$$A_{j,m}^{n,v} b_Q(x) = \int_Q V_m K_{j,y}^{n,v}(x) \cdot b_Q(y)dy =: \int_Q A_m(x,y) b_Q(y)dy$$

and

$$D_{j,k}^{n,v} b_Q(x) = \int_Q (I - G_{n,v})\Lambda_k K_{j,y}^{n,v}(x) \cdot b_Q(y)dy =: \int_Q D_k(x,y) b_Q(y)dy.$$

5.2. L^1 **estimate of** D_k**.**

Lemma 5.2. *For a fixed $y \in Q$, there exists $N > 0$, such that*

$$\|D_k(\cdot,y)\|_1 \le C2^{\tau n}2^{-n\gamma(d-1)}2^{(-j+k)+n\gamma(1+2N)}\|a\|_\infty, \tag{5.4}$$

where C is independent of y.

Proof. Denote $h_{k,n,v}(\xi) = (1 - \Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle))\beta_k(\xi)$, then

$$D_k(x,y) = (I - G_{n,v})\Lambda_k K_{j,y}^{n,v}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} h_{k,n,v}(\xi) \int_{\mathbb{R}^d} e^{-i\xi\cdot\omega} K_j^{n,v}(\omega - y) m_{\omega,y}^n a \cdot \mathrm{d}\omega \mathrm{d}\xi.$$

Next we make a polar transform $\omega - y = r\theta$. By Fubini's theorem, the integral above can be written as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}} \Gamma_v^n(\theta)\Big\{\int_{\mathbb{R}^d}\int_0^\infty e^{i\langle x-y-r\theta,\xi\rangle} h_{k,n,v}(\xi) K_j^n(r\theta) r^{d-1} m_{y+r\theta,y}^n a \cdot \mathrm{d}r\mathrm{d}\xi\Big\}\mathrm{d}\theta. \tag{5.5}$$

Consider the support of $K_j^n(x)$ in (3.1), we see $2^{j-2} \le r \le 2^{j+2}$. So integrate by parts with r first. Then the integral involving r can be rewritten as

$$\int_0^\infty e^{i\langle -r\theta,\xi\rangle}(i\langle\theta,\xi\rangle)^{-1}\partial_r[K_j^n(r\theta)r^{d-1}m^n_{y+r\theta,y}a]\mathrm{d}r.$$

Since $\theta \in \operatorname{supp} \Gamma_v^n$, then $|\theta - e_v^n| \le 2^{-n\gamma}$. By the support of Φ, we see $|\langle e_v^n, \xi/|\xi|\rangle| \ge 2^{1-nr}$. Thus,

$$|\langle\theta,\xi/|\xi|\rangle| \ge |\langle e_v^n,\xi/|\xi|\rangle| - |\langle e_v^n-\theta,\xi/|\xi|\rangle| \ge 2^{-n\gamma}. \tag{5.6}$$

Note that ξ is supported in $\{2^{-k-1} \le |\xi| \le 2^{-k+1}\}$, so we can integrate by parts with ξ. Therefore the integral in (5.5) can be rewritten as

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Gamma_v^n(\theta)\int_{\mathbb{R}^d}e^{i\langle x-y-r\theta,\xi\rangle}\int_0^\infty \partial_r\Big(K_j^n(r\theta)r^{d-1}m^n_{y+r\theta,y}a\Big)\times\\ \frac{(I-2^{-2k}\Delta_\xi)^N}{(1+2^{-2k}|x-y-r\theta|^2)^N}\Big(h_{k,n,v}(\xi)(i\langle\theta,\xi\rangle)^{-1}\Big)\mathrm{d}r\mathrm{d}\xi\mathrm{d}\theta.\end{aligned} \tag{5.7}$$

In the following, we give an exploit estimate of the term in (5.7). By the definition of $K_j^n(x)$ and $m^n_{y+r\theta,y}a$,

$$\begin{aligned}|\partial_x^\alpha K_j^n(x)| &= 2^{-(j-l_\tau(n))|\alpha|}\Big|\int(\partial_x^\alpha\eta)_{j-l_\tau(n)}(x-z)K_j(z)dz\Big|\\ &\le 2^{-(j-l_\tau(n))|\alpha|}\|K_j(\cdot,y)\|_\infty\|\partial_x^\alpha\eta\|_1\\ &\lesssim 2^{-(j-l_\tau(n))|\alpha|-jd},\end{aligned} \tag{5.8}$$

where the third inequality follows from (3.1) and

$$|\partial_r(m^n_{y+r\theta,y}a)| = \Big|\partial_r\Big(\frac{1}{r}\int_0^r\rho_n(\frac{s}{r})a(y+s\theta)ds\Big)\Big| \lesssim \frac{1}{r}\|a\|_\infty. \tag{5.9}$$

By using product rule, (5.8) and (5.9), and note that $2^{j-2} \le r \le 2^{j+2}$,

$$\Big|\partial_r\Big(K_j^n(r\theta)r^{d-1}m_{y+r\theta,y}a\Big)\Big| \lesssim 2^{l_\tau(n)-2j}\|a\|_\infty. \tag{5.10}$$

Now we turn to give an estimate of $(I-2^{-2k}\Delta_\xi)^N[\langle\theta,\xi\rangle^{-1}h_{k,n,v}(\xi)]$. By (5.6), we get

$$|(-i\langle\theta,\xi\rangle)^{-1}\cdot h_{k,n,v}(\xi)| \lesssim |\langle\theta,\xi\rangle|^{-1} \lesssim 2^{n\gamma+k}.$$

Now using product rule,

$$|\partial_{\xi_i}h_{k,n,v}(\xi)| = \big|-\partial_{\xi_i}[\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle)]\cdot\beta_k(\xi)+\partial_{\xi_i}\beta_k(\xi)\cdot(1-\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle))\big| \lesssim 2^{n\gamma+k}.$$

Therefore by induction, we have $|\partial_\xi^\alpha h_{k,n,v}(\xi)| \lesssim 2^{(n\gamma+k)|\alpha|}$ for any multi-indices $\alpha \in \mathbb{Z}^n_+$. By using product rule again and (5.6), we have

$$\begin{aligned}&\big|\partial^2_{\xi_i}(\langle\theta,\xi\rangle)^{-1}h_{k,n,v}(\xi))\big|\\ &= \big|\langle\theta,\xi\rangle^{-3}\cdot 2\theta_i^2\cdot h_{k,n,v} - 2\langle\theta,\xi\rangle^{-2}\cdot\theta_i\partial_{\xi_i}h_{k,n,v}(\xi)+\langle\theta,\xi\rangle^{-1}\partial^2_{\xi_i}h_{k,n,v}(\xi)\big|\\ &\lesssim 2^{3(n\gamma+k)}.\end{aligned}$$

Hence

$$2^{-2k}\big|\Delta_\xi[(\langle\theta,\xi\rangle)^{-1}h_{k,n,v}(\xi)]\big|\lesssim 2^{(n\gamma+k)+2n\gamma}.$$

Proceeding by induction, we obtain

$$\big|(I-2^{-2k}\Delta_\xi)^N[\langle\theta,\xi\rangle^{-1}h_{k,n,v}(\xi)]\big|\lesssim 2^{(n\gamma+k)+2n\gamma N}. \tag{5.11}$$

Now we choose $N=[d/2]+1$. Since we need to get the L^1 estimate of (5.5), by the support of $h_{k,n,v}$,

$$\int_{\mathrm{supp}(h_{k,n,v})}\int\Big(1+2^{-2k}|x-y-r\theta|^2\Big)^{-N}dxd\xi\le C.$$

Integrating with r, we get a bound 2^j. Then integrating with θ, we get a bound $2^{-n\gamma(d-1)}$. Combining (5.10), (5.11) and above estimates, (5.4) is bounded by

$$2^{\tau n}2^{-n\gamma(d-1)}2^{(-j+k)+n\gamma(1+2N)}\|a\|_\infty.$$

Hence we complete the proof of Lemma 5.2 with $N=[\frac{d}{2}]+1$. □

5.3. L^1 estimate of $A^{n,v}_{j,m}$.

By using Fubini's theorem, we can write $A_m(x,y)$ as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i\langle x-\omega,\xi\rangle}\psi_m(\xi)K^{n,v}_j(\omega-y)\cdot m^n_{\omega,y}a\,d\omega d\xi.$$

Integrating by part $N=[d/2]+1$ times with ξ in the above integral and using Fubini's theorem again, the above integral is equal to

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_0^1\rho_n(s)\Big\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}&e^{i\langle x-\omega,\xi\rangle}K^{n,v}_j(\omega-y)a(s\omega+(1-s)y)\\&\times\frac{(I-2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)}{\big(1+2^{-2m}|x-\omega|^2\big)^N}d\xi d\omega\Big\}ds.\end{aligned}$$

By making a transform $\omega+\frac{1-s}{s}y=z$, the above integral is equal to

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_0^1\rho_n(s)\int_{\mathbb{R}^d}a(sz)\int_{\mathbb{R}^d}&e^{i\langle x-z+\frac{1-s}{s}y,\xi\rangle}K^{n,v}_j(z-\frac{y}{s})\\&\times\frac{(I-2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)}{\big(1+2^{-2m}|x-z+\frac{1-s}{s}y|^2\big)^N}d\xi dzds.\end{aligned}$$

Using the cancellation of b_Q (see the property (iv) in Lemma 2.2), we get

$$A^{n,v}_{j,m}b_Q(x)=\int_Q\big(A_m(x,y)-A_m(x,y_0)\big)b_Q(y)dy,$$

where y_0 is the center of Q. Split $A_m(x,y)-A_m(x,y_0)$ into three parts:

$$A_m(x,y)-A_m(x,y_0)=:F_{m,1}(x,y)+F_{m,2}(x,y)+F_{m,3}(x,y),$$

where

$$F_{m,1}(x,y)=\frac{1}{(2\pi)^d}\int_0^1\rho_n(s)\int_{\mathbb{R}^d}a(sz)\int_{\mathbb{R}^d}\big(e^{i\langle\frac{1-s}{s}y,\xi\rangle}-e^{i\langle\frac{1-s}{s}y_0,\xi\rangle}\big)e^{i\langle x-z,\xi\rangle}$$
$$\times K_j^{n,v}(z-\frac{y}{s})\frac{(I-2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)}{\big(1+2^{-2m}|x-z+\frac{1-s}{s}y|^2\big)^N}d\xi dzds,$$

$$F_{m,2}(x,y)=\frac{1}{(2\pi)^d}\int_0^1\rho_n(s)\int_{\mathbb{R}^d}a(sz)\int_{\mathbb{R}^d}e^{i\langle x-z+\frac{1-s}{s}y_0,\xi\rangle}\frac{(I-2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)}{\big(1+2^{-2m}|x-z+\frac{1-s}{s}y|^2\big)^N}$$
$$\times\big(K_j^{n,v}(z-\frac{y}{s})-K_j^{n,v}(z-\frac{y_0}{s})\big)d\xi dzds$$

and

$$F_{m,3}(x,y)=\frac{1}{(2\pi)^d}\int_0^1\rho_n(s)\int_{\mathbb{R}^d}a(sz)\int_{\mathbb{R}^d}e^{i\langle x-z+\frac{1-s}{s}y_0,\xi\rangle}\Big\{(I-2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)\Big\}K_j^{n,v}(z-\frac{y_0}{s})$$
$$\times\Big(\frac{1}{\big(1+2^{-2m}|x-z-\frac{1-s}{s}y|^2\big)^N}-\frac{1}{\big(1+2^{-2m}|x-z-\frac{1-s}{s}y_0|^2\big)^N}\Big)d\xi dzds.$$

Hence

$$\|A_{j,m}^{n,v}b_Q\|_1\le\sup_{y\in Q}(\|F_{m,1}(\cdot,y)\|_1+\|F_{m,2}(\cdot,y)\|_1+\|F_{m,3}(\cdot,y)\|_1)\|b_Q\|_1. \tag{5.12}$$

For $F_{m,1}(x,y)$ and $F_{m,3}(x,y)$, we have the following similar estimates.

Lemma 5.3. *For a fixed $y\in Q$,*

$$\|F_{m,1}(\cdot,y)\|_1\le Cn2^{-n\gamma(d-1)+j-n-m}\|a\|_\infty,$$

where C is independent of y.

Proof. We use the same method in proving Lemma 5.2 but don't apply integrating by parts. Note that $y\in Q$ and y_0 is the center of Q, then $|y-y_0|\lesssim 2^{j-n}$. Therefore we see

$$\Big|e^{i\langle\frac{1-s}{s}y,\xi\rangle}-e^{i\langle\frac{1-s}{s}y_0,\xi\rangle}\Big|\lesssim\frac{1-s}{s}2^{j-n-m}.$$

It is easy to see that

$$|(I-2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)|\le C.$$

Since we need to get the L^1 estimate of $F_{m,1}(\cdot,y)$, by the support of $\psi_m(\xi)$, we obtain

$$\int_{|\xi|\le 2^{1-m}}\int\Big(1+2^{-2m}|x-z+\frac{1-s}{s}y|^2\Big)^{-N}dxd\xi\le C.$$

The function $a(sz)$ is bounded by $\|a\|_\infty$. Note that

$$\|K_j^{n,v}\|_1\lesssim 2^{-n\gamma(d-1)},$$
$$\Big|\int_0^1\rho_n(s)\frac{1-s}{s}ds\Big|\lesssim n. \tag{5.13}$$

Combining these, we can get the required estimate for $F_{m,1}(\cdot,y)$. ∎

Lemma 5.4. *For a fixed $y \in Q$,*

$$\|F_{m,3}(\cdot, y)\|_1 \leq Cn2^{-n\gamma(d-1)+j-n-m}\|a\|_\infty,$$

where C is independent of y.

Proof. For $F_{m,3}(\cdot, y)$, we can deal with it in the same way as $F_{m,1}(\cdot, y)$ once we have the following observation

$$\begin{aligned}\Big|\Psi(s,y) - \Psi(s,y_0)\Big| &= \Big|\int_0^1 \big\langle y - y_0, \nabla\Psi(s, ty + (1-t)y_0)\big\rangle dt\Big| \\ &\lesssim \frac{1-s}{s}|y-y_0|2^{-m}\int_0^1 \frac{N2^{-m}|x - z + \frac{1-s}{s}(ty+(1-t)y_0)|}{(1+2^{-2m}|x-z+\frac{1-s}{s}(ty+(1-t)y_0)|^2)^{N+1}}dt\end{aligned}$$

where $\Psi(s,y) = \Big(1 + 2^{-2m}|x - z + \frac{1-s}{s}y|^2\Big)^{-N}$. It is easy to see

$$|(I - 2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)| \leq C.$$

Since we need to get the L^1 estimate of $F_{m,3}(\cdot, y)$, by the support of $\psi_m(\xi)$, we obtain

$$\int_{|\xi|\leq 2^{1-m}}\int \frac{N2^{-m}|x-z+\frac{1-s}{s}(ty+(1-t)y_0)|}{(1+2^{-2m}|x-z+\frac{1-s}{s}(ty+(1-t)y_0)|^2)^{N+1}}dxd\xi \leq C.$$

Since $y \in Q$ and y_0 is the center of Q, we have $|y - y_0| \lesssim 2^{j-n}$. The function $a(sz)$ is bounded by $\|a\|_\infty$. Combining (5.13) and the above estimates, we can get the required estimate for $F_{m,3}(\cdot, y)$. □

Lemma 5.5. *For a fixed $y \in Q$, we get*

$$\|F_{m,2}(\cdot, y)\|_1 \leq Cn(2^{\tau n} + 2^{\gamma n})2^{-n\gamma(d-1)-n}\|a\|_\infty,$$

where C is independent of y.

Proof. By the mean value formula, we can write $K_j^{n,v}(z - \frac{y}{s}) - K_j^{n,v}(z - \frac{y_0}{s})$ as

$$\int_0^1 \Big\langle \frac{y-y_0}{s}, \nabla K_j^{n,v}(z - \frac{ty+(1-t)y_0}{s})\Big\rangle dt.$$

Since $y \in Q$ and y_0 is the center of Q, we have $|y - y_0| \lesssim 2^{j-n}$. It is easy to check

$$|(I - 2^{-2m}\Delta_\xi)^N(\psi_m)(\xi)| \leq C.$$

Since we need to get the L^1 estimate of $F_{m,2}(\cdot, y)$, by the support of $\psi_m(\xi)$, we obtain

$$\int_{|\xi|\leq 2^{1-m}}\int \Big(1 + 2^{-2m}|x - z + \frac{1-s}{s}y|^2\Big)^{-N} dxd\xi \leq C.$$

The function $a(sz)$ is bounded by $\|a\|_\infty$. Notice that by (3.2) and (3.4), we see

$$\|\nabla K_j^{n,v}\|_1 \lesssim \Big(2^{l_\tau(n)-j} + 2^{n\gamma-j}\Big)2^{-n\gamma(d-1)}.$$

Combining with these estimates, the L^1 norm of $F_{m,2}(\cdot, y)$ is bounded by

$$|y - y_0| \cdot \int_0^1 \rho_n(s)\frac{ds}{s} \cdot \|\nabla K_j^{n,v}\|_1\|a\|_\infty \lesssim n\Big(2^{\tau n} + 2^{n\gamma}\Big)2^{-n\gamma(d-1)-n}\|a\|_\infty,$$

which is the required bound. □

5.4. **Proof of Lemma 3.3.**

Let us come back to the proof of Lemma 3.3, it is sufficient to consider I and II in (5.1). By (5.2), (5.3) and (5.12), we have

$$\begin{aligned} I+II &\le \frac{2}{\lambda}\sum_j\sum_v\sum_{l(Q)=2^{j-n}}\Big[C_{\gamma,d}^{-1}2^{n\gamma(d-1)+n\gamma([\frac{d}{2}]+1)}\|A_{j,m}^{n,v}b_Q\|_1+\sum_{k<m}\|D_{j,k}^{n,v}b_Q\|_1\Big] \\ &\le \frac{2}{\lambda}\sum_j\sum_v\sum_{l(Q)=2^{j-n}}\sup_{y\in Q}\Big[C_{\gamma,d}^{-1}2^{n\gamma(d-1)+n\gamma([\frac{d}{2}]+1)}\Big(\|F_{m,1}(\cdot,y)\|_1 \\ &\quad +\|F_{m,2}(\cdot,y)\|_1+\|F_{m,3}(\cdot,y)\|_1\Big)+\sum_{k<m}\|D_k(\cdot,y)\|_1\Big]\|b_Q\|_1. \end{aligned}$$

Notice $m=j-[n\varepsilon_0]$ and $\mathrm{card}(\Theta_n)\lesssim 2^{n\gamma(d-1)}$. Now applying Lemma 5.2 with $N=[\frac{d}{2}]+1$, then Lemma 5.3, Lemma 5.4, Lemma 5.5 and the fact $[n\varepsilon_0]\le n\varepsilon_0<[n\varepsilon_0]+1$ imply

$$I+II\lesssim\frac{1}{\lambda}\sum_j\sum_{l(Q)=2^{j-n}}\|b_Q\|_1\|a\|_\infty\big[n(2^{s_1n}+2^{s_2n}+2^{s_3n})+2^{s_4n}\big],$$

where

$$\begin{aligned} s_1&=\gamma(d-1)+\gamma\big([\frac{d}{2}]+1\big)-1+\varepsilon_0,\\ s_2&=\gamma(d-1)+\gamma\big([\frac{d}{2}]+1\big)-1+\tau,\\ s_3&=\gamma(d-1)+\gamma\big([\frac{d}{2}]+1\big)-1+\gamma,\\ s_4&=-\varepsilon_0+\gamma+2\big([\frac{d}{2}]+1\big)\gamma+\tau. \end{aligned}$$

Now we choose $0<\gamma\ll\varepsilon_0\ll1$ and $0<\tau\ll\varepsilon_0$ such that

$$\max\{s_1,s_2,s_3,s_4\}<0.$$

Set $\varepsilon=-\max\{s_1,s_2,s_3,s_4\}$. Then by the property (iv) in Lemma 2.2,

$$I+II\lesssim\frac{\|a\|_\infty}{\lambda}n2^{-n\varepsilon}\sum_Q\|b_Q\|_1\lesssim n2^{-n\varepsilon}\sum_Q|Q|.$$

Hence we finish the proof of Lemma 3.3, thus we prove Theorem 1.1. □

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