# ENERGY EQUALITY FOR WEAK SOLUTIONS TO THE 3D MAGNETOHYDRODYNAMIC EQUATIONS IN A BOUNDED DOMAIN

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ABSTRACT. In this paper, we study the energy equality for weak solutions to the 3D homogeneous incompressible magnetohydrodynamic equations with viscosity and magnetic diffusion in a bounded domain. Two types of regularity conditions are imposed on weak solutions to ensure the energy equality. For the first type, some global integrability condition for the velocity $\mathbf{u}$ is required, while for the magnetic field $\mathbf{b}$ and the magnetic pressure $\pi$, some suitable integrability conditions near the boundary are sufficient. In contrast with the first type, the second type claims that if some additional interior integrability is imposed on $\mathbf{b}$, then the regularity on $\mathbf{u}$ can be relaxed.

## 1. INTRODUCTION

Magnetohydrodynamics (MHD) describes the evolution of an electrically conducting fluid, such as plasma and liquid metal, which has attracted a lot of attention from researchers in mathematical physics in the past decades. In this paper, we consider the following three-dimensional homogeneous incompressible MHD equations with viscosity and magnetic diffusion

$$\begin{cases}\partial_t\mathbf{u}+\mathrm{div}(\mathbf{u}\otimes\mathbf{u})-\mathrm{div}(\mathbf{b}\otimes\mathbf{b})+\nabla\pi-\mu\triangle\mathbf{u}=0,\\ \partial_t\mathbf{b}+\mathrm{div}(\mathbf{b}\otimes\mathbf{u})-\mathrm{div}(\mathbf{u}\otimes\mathbf{b})-\nu\triangle\mathbf{b}=0,\\ \mathrm{div}\mathbf{u}=\mathrm{div}\mathbf{b}=0,\end{cases}\tag{1.1}$$

where $\mathbf{u}\in\mathbb{R}^3$ is the velocity field, $\mathbf{b}\in\mathbb{R}^3$ is the magnetic field, the positive constants $\mu$ and $\nu$ are the viscosity coefficient and magnetic diffusivity respectively, and $\pi$ is the magnetic pressure given by

$$\pi=P+\frac{1}{2}|\mathbf{b}|^2,$$

with $P$ being the pressure of the fluid.

We are concerned with the above system in an open bounded domain $\Omega\subset\mathbb{R}^3$ and consider the boundary conditions

$$\mathbf{u}=\mathbf{b}=0\quad\text{on }\ [0,T]\times\partial\Omega,\tag{1.2}$$

and the initial conditions

$$(\mathbf{u},\mathbf{b})(0,x)=(\mathbf{u}_0,\mathbf{b}_0)(x),\qquad x\in\Omega.\tag{1.3}$$

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Formally, one can easily check by integration by parts that for any $t \in [0,T]$, the following energy equality holds for smooth solutions to (1.1):

$$\frac{1}{2}\int_\Omega \left(|\mathbf{u}|^2+|\mathbf{b}|^2\right)(t)dx+\int_0^t\int_\Omega \left(\mu|\nabla \mathbf{u}|^2+\nu|\nabla \mathbf{b}|^2\right)dxds=\frac{1}{2}\int_\Omega\left(|\mathbf{u}_0|^2+|\mathbf{b}_0|^2\right)dx. \tag{1.4}$$

However, for weak solutions with less regularity, (1.4) may fail. It is natural to ask the following question: under what regularity conditions on the solutions as weak as possible, does the energy equality (1.4) still hold true?

In (1.1), if $\mathbf{b}=\mathbf{0}$ and $\mu=0$, then the system reduces to the homogeneous incompressible Euler equations. Regarding this case, Onsager [25] conjectured that for any weak solution belonging to the Hölder space $C^\alpha$, the energy is conserved provided that the Hölder exponent $\alpha>\frac{1}{3}$; and if $\alpha<\frac{1}{3}$, then there exists a weak solution which dissipates energy. This is the famous "Onsager's conjecture". The conservation part was partly obtained by Eyink [12] in 1994, it is proved that for any weak solution in a subspace $C^\alpha_*\subset C^\alpha$, with $\alpha>\frac{1}{3}$, the energy is conserved. Then, Constantin, E and Titi [8] proved and improved the conservation part of "Onsager's conjecture", to be specific, they proved that the energy conservation holds true provided that the weak solution belongs to the Besov space $B^\alpha_{3,\infty}$, with $C^\alpha\subset B^\alpha_{3,\infty}$ and $\alpha>\frac{1}{3}$. After that, Duchon and Robert [10] and Cheskidov, Constantin, Friedlander and Shvydkoy [7] improved the previous results and proved the energy conservation for the whole space or a periodic domain. Breakthroughs for domains with physical boundaries were achieved very recently by Bardos and Titi [2], Bardos, Titi and Wiedemann [3], Drivas and Nguyen [9] and Nguyen and Nguyen [22]. For the inhomogeneous incompressible case or the compressible case in a periodic domain, the energy conservation was proved by Akramov, Debiec, Skipper and Wiedemann [1], Chen and Yu [6] and Feireisl, Gwiazda, Świerczewska-Gwiazda and Wiedemann [14]. Recently, Nguyen, Nguyen and Tang [23] improved the results in [1, 6, 14] and they also obtained similar results for a general bounded domain.

In (1.1), if $\mathbf{b}=\mathbf{0}$ and $\mu>0$, then the system reduces to the homogeneous incompressible Navier-Stokes equations, for this case, Serrin [27] first proved the energy equality for weak solutions under the condition $\mathbf{u}\in L^p(0,T;L^q(\mathbb{T}^d))$ with $\frac{2}{p}+\frac{d}{q}\le 1$. In [28], Shinbrot improved Serrin's result by proposing a regularity condition independent of the dimension $d$, that is, $\mathbf{u}\in L^p(0,T;L^q(\mathbb{T}^d))$ with $\frac{1}{p}+\frac{1}{q}\le\frac{1}{2}$ and $q\ge 4$. See also [32] for an alternative proof given by Yu by a different method. Recently, Yu [34] extended Shinbrot's result to a general bounded domain, with an additional Besov regularity imposed on the velocity, which is essential to handle the boundary effect. There are also some results for the inhomogeneous incompressible case and the compressible case for different types of domains. See [5, 20, 24, 33] and the references therein.

For the MHD equations, which possess more complicated nonlinear terms, there are few results concerning the energy conservation or energy equality in the literature. For the ideal MHD equations, Caflisch, Klapper and Steele [4], Kang and Lee [17] and Yu [35] proved the energy conservation in the whole space or a periodic domain. Then, Wang and Zuo [31] dealt with the bounded domain case, with suitable Besov-type continuity imposed near the boundary in order to cope with the boundary effect. For the viscous and diffusive MHD equations, to our knowledge, the only revelent paper is [30] by Wang, Zhao, Chen and Zhang, where energy equality is proved for a periodic domain.

In this paper, we mainly consider the energy equality (1.4) of weak solutions to the homogeneous incompressible viscous and diffusive MHD equations (1.1) in a general bounded domain. To the best of our knowledge, this paper seems to be the first one dealing with the relation between the energy equality (1.4) and the regularity of the solutions to (1.1) when there is a boundary. Finally, we apply our method to the inhomogeneous incompressible and isentropic compressible cases.

Compared with the Euler or the Navier-Stokes equations, the MHD equations have stronger nonlinearity due to the couplings of the fluid velocity $\mathbf{u}$ and the magnetic field $\mathbf{b}$. In the light of this difficulty, two types of sufficient conditions are imposed on weak solutions to ensure the energy equality. Our first result requires some global integrability condition for the velocity $\mathbf{u}$ and suitable integrability conditions on the magnetic field $\mathbf{b}$ and magnetic pressure $\pi$ near the boundary to tackle the boundary effect, which in some sense means that $\mathbf{u}$ plays a dominant role in the energy equality. The other result claims that if additional interior integrability is imposed on the magnetic filed $\mathbf{b}$, then the regularities on the velocity filed $\mathbf{u}$ could be relaxed, which implies it is possible to "trade" regularity between $\mathbf{u}$ and $\mathbf{b}$.

Another difficulty is how to handle the boundary effect. Unlike the whole space or the periodic domain case, we additionally need to consider the boundary effect. In [31], Wang and Zuo studied the energy conservation for the ideal MHD equations in a bounded domain, where the Besov type continuity near the boundary for $\mathbf{u}$ and $\mathbf{b}$ is needed. However, for the viscous and diffusive MHD equations considered in this paper, such a requirement is not necessary. In fact, since $\mathbf{u}$, $\mathbf{b} \in L^2(0,T;H_0^1(\Omega))$, by the classical form of Hardy inequality (see Lemma 2.2 in Section 2), we are able to get a satisfactory estimate for $\mathbf{u}$ and $\mathbf{b}$ near the boundary, which is impossible for the ideal MHD equations.

Note that we consider the inhomogeneous incompressible and isentropic compressible cases in Sections 4 and 5, the proofs of which are similar. Compared with the homogeneous case, there is an additional nonlinear term $\partial_t(\rho\mathbf{u})$ in the inhomogeneous MHD equations. To tackle this term, stronger regularity on $\mathbf{u}$ is required (see Theorem 4.1 in Section 4). For the isentropic compressible case, the pressure $p$ is a function of the density $\rho$, thus to deal with the pressure, better regularity for the density is required (see Theorem 5.1 in Section 5).

### 1.1. **Main results.**

The weak solutions to the initial-boundary value problem (1.1)-(1.3) that we are interested in is as follows.

**Definition 1.1.** *We call* $(\mathbf{u},\mathbf{b},\pi)$ *a weak solution to the initial-boundary value problem* (1.1) *if*

*(i)* $\mathbf{u}$ *and* $\mathbf{b}$ *satisfy*

$$\mathbf{u},\ \mathbf{b} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)); \tag{1.5}$$

*(ii) the equation* (1.1) *holds in* $\mathcal{D}'((0,T)\times\Omega)$*, that is, for any* $\Phi\in C_c^\infty([0,T]\times\Omega)$*, we have*

$$\begin{aligned}\int_0^T\int_\Omega \partial_t\Phi\cdot\mathbf{u}dxdt+\int_0^T\int_\Omega\nabla\Phi:(\mathbf{u}\otimes\mathbf{u}-\mathbf{b}\otimes\mathbf{b})\,dxdt\\ +\int_0^T\int_\Omega \mathrm{div}\Phi\,\pi dxdt-\mu\int_0^T\int_\Omega\nabla\Phi:\nabla\mathbf{u}dxdt=0,\end{aligned}\tag{1.6}$$

$$\int_0^T\int_\Omega \partial_t\Phi\cdot\mathbf{b}dxdt+\int_0^T\int_\Omega\nabla\Phi:(\mathbf{b}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{b})\,dxdt-\nu\int_0^T\int_\Omega\nabla\Phi:\nabla\mathbf{b}dxdt=0,\tag{1.7}$$

*and for any* $\varsigma\in C_c^\infty(\Omega)$*, it holds that for any* $t\in[0,T]$

$$\int_\Omega\mathbf{u}\cdot\nabla\varsigma dx=\int_\Omega\mathbf{b}\cdot\nabla\varsigma dx=0;\tag{1.8}$$

*(iii) the following energy inequality holds for any* $t\in[0,T]$

$$\frac{1}{2}\int_\Omega\left(|\mathbf{u}|^2+|\mathbf{b}|^2\right)(t)dx+\int_0^t\int_\Omega\left(\mu|\nabla\mathbf{u}|^2+\nu|\nabla\mathbf{b}|^2\right)dxds\le\frac{1}{2}\int_\Omega\left(|\mathbf{u}_0|^2+|\mathbf{b}_0|^2\right)dx.\tag{1.9}$$

Two types of sufficient regularity conditions are imposed on the weak solutions of (1.1) to ensure the energy equality (1.4). To be specific, the first result is as follows.

**Theorem 1.1.** *Let* $(\mathbf{u},\mathbf{b},\pi)$ *be a weak solution to the initial-boundary value problem* (1.1)-(1.3) *in the sense of Definition 1.1. Assume that*

$$\mathbf{u}_0,\ \mathbf{b}_0\in L^2(\Omega),\tag{1.10}$$

*and*

$$\mathbf{u}\in L^p(0,T;L^q(\Omega))\tag{1.11}$$

*with* $\frac{2}{p}+\frac{3}{q}\le 1$ *and* $q\ge 4$*. Also assume that there exists a* $\delta>0$ *such that*

$$\mathbf{b}\in L^4((0,T)\times\Omega_\delta),\quad \pi\in L^2((0,T)\times\Omega_\delta),\tag{1.12}$$

*where* $\Omega_\delta=\{x\in\Omega|dist(x,\partial\Omega)<\delta\}$*.*

*Then the energy equality* (1.4) *holds for all* $t\in[0,T]$*.*

**Remark 1.1.** *Zhou* [36] *proved a regularity result for the viscous and diffusive MHD equations in the whole space. More precisely, Zhou proved that if the initial data* $\mathbf{u}_0,\mathbf{b}_0\in H^s(\mathbb{R}^3)$*, with* $s\ge 3$*, then under the condition* (1.11)*, with* $\frac{2}{p}+\frac{3}{q}\le 1$ *and* $q>3$*, the solution is in fact smooth, from which the energy equality* (1.4) *can be easily obtained. However, under the conditions of Theorem 1.1, the smoothness of solutions can not be guaranteed. On the one hand, the initial data* $\mathbf{u}_0,\mathbf{b}_0$ *are only in* $L^2$*. On the other hand, we are concerned with domains with boundary.*

Our second theorem requires additional regularity on the magnetic field $\mathbf{b}$ in the interior domain, which allows to relax the regularity imposed on the velocity.

**Theorem 1.2.** *Let* $(\mathbf{u}, \mathbf{b}, \pi)$ *be a weak solution to the initial-boundary value problem* (1.1)-(1.3) *in the sense of Definition 1.1. Assume that*

$$\mathbf{u}_0,\ \mathbf{b}_0 \in L^2(\Omega), \tag{1.13}$$

$$\mathbf{u},\ \mathbf{b} \in L^4((0,T)\times\Omega), \tag{1.14}$$

*and*

$$\pi \in L^2((0,T)\times\Omega_\delta). \tag{1.15}$$

*Then the energy equality* (1.4) *holds for all* $t \in [0,T]$.

Our paper is organized as follows. In Section 2, we fix some notation and terminologies and recall several important lemmas to be used in the sequel. Section 3 will be devoted to the proofs of Theorem 1.1 and Theorem 1.2. In Section 4 and Section 5, we will apply our method to the inhomogeneous incompressible case and the isentropic compressible case respectively.

## 2. Preliminaries

To begin with, we introduce some notation and terminologies.

Denote

$$\Omega_\varepsilon := \{x \in \Omega \mid dist(x, \partial\Omega) < \varepsilon\}, \quad \Omega^\varepsilon := \Omega \setminus \overline{\Omega}_\varepsilon,$$

where $\overline{\Omega}_\varepsilon$ is the closure of $\Omega_\varepsilon$ in the usual Euclidean topology.

Let $l, h$ be two positive parameters with $0 < l < \frac{h}{16}$, and define $\eta_{h,l} : \mathbb{R} \to \mathbb{R}$ to be a smooth function with compact support satisfying

$$\eta_{h,l}(z) = \begin{cases} 0, & z \in (-\infty, h-l], \\ 1, & z \in [h, +\infty). \end{cases}$$

Define the cut-off function $\theta_{h,l} : \Omega \to \mathbb{R}$ as follows

$$\theta_{h,l}(x) := \eta_{h,l}(dist(x, \partial\Omega)). \tag{2.16}$$

It is easy to check that $supp\,\theta_{h,l} \subset \Omega^{h-l}$ and $supp\,\nabla\theta_{h,l} \subset \Omega_h$.

Then we define the mollification of a function $f \in L^1(\Omega)$ as follows:

$$f^l(x) := \frac{1}{l^3}\int_\Omega \phi(\frac{x-y}{l}) f(y)dy, \tag{2.17}$$

where $\phi$ is the standard mollifier in three dimensions.

The following commutator estimates will be essential in our proofs.

**Lemma 2.1.** (Lemma 2.3 in [21]) *Suppose* $f_1 \in L^{p_1}(0,T; W_0^{1,p_1}(\Omega))$, $f_2 \in L^{p_2}((0,T)\times\Omega)$, *with* $1 \le p, p_1, p_2 \le \infty$, *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then*

$$\|\partial_x(f_1 f_2)^l - \partial_x(f_1 f_2^l)\|_{L^p((0,T)\times\Omega)} \le C\|\partial_x f_1\|_{L^{p_1}((0,T)\times\Omega)}\|f_2\|_{L^{p_2}((0,T)\times\Omega)}, \tag{2.18}$$

*for some constant $C$ independent of $l$, $f_1$ and $f_2$, where $f^l$ is defined as in* (2.17). *In addition, if $p<\infty$, then*

$$\partial_x(f_1f_2)^l - \partial_x(f_1f_2^l) \to 0 \text{ in } L^p((0,T)\times\Omega), \text{ as } l\to 0^+. \tag{2.19}$$

Finally we introduce the following Hardy inequality that will be frequently used in the subsequent sections.

**Lemma 2.2.** ([18]) *Let $p\in[1,\infty)$ and $f\in W_0^{1,p}(\Omega)$. Then there exists a constant $C$ depending on $p$ and $\Omega$, such that*

$$\left\| \frac{f}{dist(\cdot,\partial\Omega)} \right\|_{L^p(\Omega)} \le C\|f\|_{W_0^{1,p}(\Omega)}. \tag{2.20}$$

## 3. Proof of our main results

### 3.1. Proof of Theorem 1.1.

*Proof.* Since $(\mathbf{u},\mathbf{b},\pi)$ is a weak solution to (1.1) in the sense of Definition 1.1, we see that for any test function $\Phi\in C_c^\infty([0,T]\times\Omega)$, the maps $t\to\int_\Omega \mathbf{u}(t,x)\cdot\Phi(t,x)dx$ and $t\to\int_\Omega \mathbf{b}(t,x)\cdot\Phi(t,x)dx$ are Lipschitz continuous. As mentioned in [17] (or [31, 35]), by the standard density argument, we can choose the test functions in (1.6) and (1.7) to be smooth and compactly supported in space and weakly Lipschitz continuous in time. Therefore, we can take $\Phi=\left(\psi_\tau\theta_{h,l}\mathbf{u}^l\right)^l$ in (1.6) and $\Phi=\left(\psi_\tau\theta_{h,l}\mathbf{b}^l\right)^l$ in (1.7), where $\tau$ is a fixed small positive constant and $\psi_\tau(t)\in C_c^1((\tau,T-\tau))$.

Taking $\Phi=\left(\psi_\tau\theta_{h,l}\mathbf{u}^l\right)^l$ in (1.6), we deduce that

$$\int_0^T\int_\Omega \partial_t\left(\psi_\tau\theta_{h,l}\mathbf{u}^l\right)\cdot\mathbf{u}^l dxdt = I_1+I_2+I_3+I_4, \tag{3.21}$$

where

$$\begin{aligned}
I_1 &:= \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \operatorname{div}\left(\mathbf{u}\otimes\mathbf{u}\right)^l dxdt,\\
I_2 &:= -\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \operatorname{div}\left(\mathbf{b}\otimes\mathbf{b}\right)^l dxdt,\\
I_3 &:= \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \nabla\pi^l dxdt,\\
I_4 &:= -\mu\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \triangle\mathbf{u}^l dxdt.
\end{aligned}$$

Similarly, taking $\Phi=\left(\psi_\tau\theta_{h,l}\mathbf{b}^l\right)^l$ in (1.7), we deduce that

$$\int_0^T\int_\Omega \partial_t\left(\psi_\tau\theta_{h,l}\mathbf{b}^l\right)\cdot\mathbf{b}^l dxdt = J_1+J_2+J_3, \tag{3.22}$$

where

$$
\begin{aligned}
J_1 &:= \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \operatorname{div}(\mathbf{b} \otimes \mathbf{u})^l \, dxdt, \\
J_2 &:= - \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \operatorname{div}(\mathbf{u} \otimes \mathbf{b})^l \, dxdt, \\
J_3 &:= -\nu \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \triangle \mathbf{b}^l dxdt.
\end{aligned}
$$

Our next step is to pass the limit $l \to 0^+$ first and subsequently, $h \to 0^+$ in (3.21) and (3.22) for fixed $\tau > 0$. To this end, we need a sequence of lemmas. For convenience, in the rest of this paper we shall use $C$ to denote a generic positive constant independent of $l$, $h$ and $\tau$, whose value may change from line to line.

**Lemma 3.1.** *For fixed $\tau > 0$, for the terms on left-hand side of* (3.21) *and* (3.22)*, we have*

$$
\lim_{h\to 0^+} \lim_{l \to 0^+} \int_0^T \int_\Omega \partial_t \left( \psi_\tau \theta_{h,l} \mathbf{u}^l \right) \cdot \mathbf{u}^l dxdt = \frac{1}{2} \int_0^T \int_\Omega \psi_\tau' |\mathbf{u}|^2 dxdt, \tag{3.23}
$$

$$
\lim_{h\to 0^+} \lim_{l \to 0^+} \int_0^T \int_\Omega \partial_t \left( \psi_\tau \theta_{h,l} \mathbf{b}^l \right) \cdot \mathbf{b}^l dxdt = \frac{1}{2} \int_0^T \int_\Omega \psi_\tau' |\mathbf{b}|^2 dxdt. \tag{3.24}
$$

*Proof.* For the term on the left-hand side of (3.21), we have

$$
\begin{aligned}
&\int_0^T \int_\Omega \partial_t \left( \psi_\tau \theta_{h,l} \mathbf{u}^l \right) \cdot \mathbf{u}^l dxdt \\
&= \int_0^T \int_\Omega \psi_\tau' \theta_{h,l} |\mathbf{u}^l|^2 dxdt + \frac{1}{2} \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \partial_t |\mathbf{u}^l|^2 dxdt \\
&= \frac{1}{2} \int_0^T \int_\Omega \psi_\tau' \theta_{h,l} |\mathbf{u}^l|^2 dxdt.
\end{aligned} \tag{3.25}
$$

Thus to prove (3.23), it is sufficient to prove

$$
\lim_{h\to 0^+} \lim_{l \to 0^+} \int_0^T \int_\Omega \psi_\tau' \theta_{h,l} |\mathbf{u}^l|^2 dxdt = \frac{1}{2} \int_0^T \int_\Omega \psi_\tau' |\mathbf{u}|^2 dxdt.
$$

In fact,

$$
\begin{aligned}
&\left| \int_0^T \int_\Omega \psi_\tau' \theta_{h,l} |\mathbf{u}^l|^2 dxdt - \int_0^T \int_\Omega \psi_\tau' |\mathbf{u}|^2 dxdt \right| \\
&= \left| \int_0^T \int_\Omega \psi_\tau' \left( \theta_{h,l} |\mathbf{u}^l|^2 - |\mathbf{u}|^2 \right) dxdt \right| \\
&\leq \int_0^T \int_\Omega \psi_\tau' \theta_{h,l} \left| |\mathbf{u}^l|^2 - |\mathbf{u}|^2 \right| dxdt + \int_0^T \int_{\Omega_h} \psi_\tau' |\theta_{h,l} - 1| |\mathbf{u}|^2 dxdt \\
&\leq C \int_0^T \int_\Omega \left| |\mathbf{u}^l|^2 - |\mathbf{u}|^2 \right| dxdt + C \int_0^T \int_{\Omega_h} |\mathbf{u}|^2 dxdt,
\end{aligned}
$$

where for fixed $h$, the first term on the last inequality convergence to 0, as $l \to 0$, and then, letting $h \to 0$, the second term vanishes.

The term on the left-hand side of (3.22) can be dealt with similarly, therefore we omit the details here. □

Below we pass the limit $l \to 0^+$ and $h \to 0^+$ successively to the terms $I_i$ $(i = 1,2,3,4)$ and $J_j$ $(j = 1,2,3)$.

**Lemma 3.2.** *For fixed $\tau > 0$, for the term $I_1$, we have*

$$\lim_{h\to 0^+}\lim_{l\to 0^+} I_1 = \lim_{h\to 0^+}\lim_{l\to 0^+}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u})^l\,dxdt = 0, \tag{3.26}$$

*provided $q \geq 4$ and $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$.*

*Proof.* We first decompose the term $I_1$ into two parts

$$\begin{aligned} I_1 =& \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u})^l\,dxdt \\ =& \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \Big(\operatorname{div}(\mathbf{u}\otimes\mathbf{u})^l - \operatorname{div}\big(\mathbf{u}^l\otimes\mathbf{u}\big)\Big)\,dxdt + \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \operatorname{div}\big(\mathbf{u}^l\otimes\mathbf{u}\big)\,dxdt \\ :=& I_{11}+I_{12}. \end{aligned} \tag{3.27}$$

For $I_{12}$, by the divergence-free property of $\mathbf{u}$, we obtain

$$\begin{aligned} I_{12} =& \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \operatorname{div}\Big(\mathbf{u}^l\otimes\mathbf{u}\Big)\,dxdt \\ =& \int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\mathbf{u}^l|^2\operatorname{div}\mathbf{u}dxdt + \frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}\cdot\nabla|\mathbf{u}^l|^2dxdt \\ =& \frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\mathbf{u}^l|^2\operatorname{div}\mathbf{u}dxdt - \frac{1}{2}\int_0^T\int_\Omega \psi_\tau\nabla\theta_{h,l}\cdot\mathbf{u}|\mathbf{u}^l|^2dxdt \\ =& -\frac{1}{2}\int_0^T\int_{\Omega_h} \psi_\tau\nabla\theta_{h,l}\cdot\mathbf{u}|\mathbf{u}^l|^2dxdt \\ \leq& C\||\nabla\theta_{h,l}||\mathbf{u}|\|_{L^2((0,T)\times\Omega_h)}\|\mathbf{u}\|^2_{L^4((0,T)\times\Omega_h)} \\ \leq& C\left\|\frac{\mathbf{u}}{dist(x,\partial\Omega)}\right\|_{L^2((0,T)\times\Omega_h)}\|\mathbf{u}\|^2_{L^4((0,T)\times\Omega_h)}. \end{aligned} \tag{3.28}$$

Taking into account (1.5) and (1.11) and using the interpolation inequality, we have

$$\|\mathbf{u}\|_{L^4((0,T)\times\Omega)} \leq \|\mathbf{u}\|^{\frac{q-4}{2(q-2)}}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}\|^{\frac{q}{2(q-2)}}_{L^{\frac{2q}{q-2}}(0,T;L^q(\Omega))} \leq C, \tag{3.29}$$

provided $0 \leq \frac{q}{2(q-2)} \leq 1$ and $\frac{2q}{q-2} \leq p$, or equivalently, $q \geq 4$ and $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$. Thus, we obtain

$$\lim_{h\to 0^+}\lim_{l\to 0^+} I_{12} = 0, \tag{3.30}$$

provided $q \geq 4$ and $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$.

For the term $I_{11}$, we claim that

$$\lim_{l\to 0^+} I_{11} = 0, \tag{3.31}$$

provided $q \geq 4$ and $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. In fact, by Hölder's inequality and (2.18), we have

$$
\begin{aligned}
I_{11} &= \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{u}^l \cdot \Big( \operatorname{div}(\mathbf{u}\otimes\mathbf{u})^l - \operatorname{div}\Big(\mathbf{u}^l \otimes \mathbf{u}\Big)\Big)\, dxdt \\
&\leq C \int_\tau^{T-\tau} \|\mathbf{u}\|_{L^q(\Omega)} \|\operatorname{div}(\mathbf{u}\otimes\mathbf{u})^l - \operatorname{div}\Big(\mathbf{u}^l\otimes\mathbf{u}\Big)\|_{L^{\frac{q}{q-1}}(\Omega)} dt \\
&\leq C \|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \|\nabla \mathbf{u}\|_{L^2((0,T)\times\Omega)} \|\mathbf{u}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}.
\end{aligned} \tag{3.32}
$$

Again by (1.5), (1.11) and the interpolation inequality, we have

$$
\|\mathbf{u}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))} \leq \|\mathbf{u}\|^{\frac{q-4}{q-2}}_{L^\infty(0,T;L^2(\Omega))} \|\mathbf{u}\|^{\frac{2}{q-2}}_{L^{\frac{4p}{(p-2)(q-2)}}(0,T;L^q(\Omega))} \leq C, \tag{3.33}
$$

provided $0 \leq \frac{2}{q-2} \leq 1$ and $\frac{4p}{(p-2)(q-2)} \leq p$, which is equivalent to $q \geq 4$ and $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. This combined with Lemma 2.1 and (3.32) gives (3.31).

Thus, substituting (3.30) and (3.31) into (3.27), we get the desired result. □

**Lemma 3.3.** *For fixed $\tau > 0$, for the term $J_1$, we have*

$$
\lim_{h\to0^+}\lim_{l\to0^+} J_1 = \lim_{h\to0^+}\lim_{l\to0^+} \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \operatorname{div}(\mathbf{b}\otimes\mathbf{u})^l\, dxdt = 0, \tag{3.34}
$$

*provided $\frac{2}{p} + \frac{3}{q} \leq 1$.*

*Proof.* We first decompose the term $J_1$ into three parts

(3.35)

$$
\begin{aligned}
J_1 =& \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \operatorname{div}(\mathbf{b}\otimes\mathbf{u})^l\, dxdt \\
=& \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \Big(\operatorname{div}(\mathbf{b}\otimes\mathbf{u})^l - \operatorname{div}\big(\mathbf{b}\otimes\mathbf{u}^l\big)\Big)\, dxdt \\
&+ \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \big(\operatorname{div}\big(\mathbf{b}\otimes\mathbf{u}^l\big) - \operatorname{div}\big(\mathbf{b}^l\otimes\mathbf{u}\big)\big)\, dxdt + \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \operatorname{div}\big(\mathbf{b}^l\otimes\mathbf{u}\big)\, dxdt \\
:=& J_{11} + J_{12} + J_{13}.
\end{aligned}
$$

For the term $J_{13}$, by the divergence-free property of $\mathbf{u}$, we have

$$
\begin{aligned}
J_{13} &= \int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{b}^l \cdot \operatorname{div}\Big(\mathbf{b}^l\otimes\mathbf{u}\Big)\, dxdt \\
&= \int_0^T \int_\Omega \psi_\tau \theta_{h,l} |\mathbf{b}^l|^2 \operatorname{div}\mathbf{u} dxdt + \frac{1}{2}\int_0^T \int_\Omega \psi_\tau \theta_{h,l} \mathbf{u}\cdot\nabla|\mathbf{b}^l|^2 dxdt \\
&= \frac{1}{2}\int_0^T \int_\Omega \psi_\tau \theta_{h,l} |\mathbf{b}^l|^2 \operatorname{div}\mathbf{u} dxdt - \frac{1}{2}\int_0^T \int_\Omega \psi_\tau (\mathbf{u}\cdot\nabla\theta_{h,l})|\mathbf{b}^l|^2 dxdt \\
&= -\frac{1}{2}\int_0^T \int_{\Omega_h} \psi_\tau (\mathbf{u}\cdot\nabla\theta_{h,l})|\mathbf{b}^l|^2 dxdt \\
&\leq C \||\mathbf{u}||\nabla\theta_{h,l}|\|_{L^2((0,T)\times\Omega_h)} \|\mathbf{b}\|^2_{L^4((0,T)\times\Omega_h)} \\
&\leq C \left\| \frac{\mathbf{u}}{dist(x,\partial\Omega)} \right\|_{L^2((0,T)\times\Omega_h)} \|\mathbf{b}\|^2_{L^4((0,T)\times\Omega_h)} \\
&\quad \to 0, \quad \text{as } h\to0^+.
\end{aligned} \tag{3.36}
$$

For the term $J_{11}$, we claim that

$$\lim_{l\to 0^+} J_{11}=0, \tag{3.37}$$

provided $\frac{2}{p}+\frac{3}{q}\le 1$. Indeed, similarly as the term $I_{11}$, by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
J_{11}=&\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\left(\mathrm{div}\,(\mathbf{b}\otimes\mathbf{u})^l-\mathrm{div}\left(\mathbf{b}\otimes\mathbf{u}^l\right)\right)dxdt\\
\le& C\int_\tau^{T-\tau}\|\mathbf{b}\|_{L^{\frac{2q}{q-2}}(\Omega)}\|\mathrm{div}\,(\mathbf{b}\otimes\mathbf{u})^l-\mathrm{div}\left(\mathbf{b}\otimes\mathbf{u}^l\right)\|_{L^{\frac{2q}{q+2}}(\Omega)}dt\\
\le& C\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\|\nabla\mathbf{b}\|_{L^2((0,T)\times\Omega)}\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}.
\end{aligned}\tag{3.38}$$

By virtue of (1.5) and the interpolation inequality, we have

$$\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\le\|\mathbf{b}\|^{1-\frac{3}{q}}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{b}\|^{\frac{3}{q}}_{L^{\frac{6p}{q(p-2)}}(0,T;L^6(\Omega))}\le C, \tag{3.39}$$

provided $0\le\frac{3}{q}\le 1$ and $\frac{6p}{q(p-2)}\le 2$, which is equivalent to $\frac{2}{p}+\frac{3}{q}\le 1$. Thus, thanks to Lemma 2.1, (3.38) and (3.39), we can obtain (3.37).

For the term $J_{12}$, by (3.39) and the divergence-free property of $\mathbf{u}$ and $\mathbf{u}^l$, we have

$$\begin{aligned}
J_{12}=&\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\left(\mathrm{div}\left(\mathbf{b}\otimes\mathbf{u}^l\right)-\mathrm{div}\left(\mathbf{b}^l\otimes\mathbf{u}\right)\right)dxdt\\
=&\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\left(\mathbf{b}\mathrm{div}\mathbf{u}^l+(\mathbf{u}^l\cdot\nabla)\mathbf{b}-\mathbf{b}^l\mathrm{div}\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{b}^l\right)dxdt\\
=&\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\left[((\mathbf{u}^l-\mathbf{u})\cdot\nabla)\mathbf{b}+(\mathbf{u}\cdot\nabla)(\mathbf{b}-\mathbf{b}^l)\right]dxdt\\
\le& C\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\|\mathbf{u}^l-\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla\mathbf{b}\|_{L^2((0,T)\times\Omega)}\\
&+C\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla(\mathbf{b}-\mathbf{b}^l)\|_{L^2((0,T)\times\Omega)}\\
&\to 0,\ \text{as } l\to 0^+,
\end{aligned}\tag{3.40}$$

provided $\frac{2}{p}+\frac{3}{q}\le 1$.

Substituting (3.36), (3.37) and (3.40) into (3.35), we obtain (3.34). □

**Lemma 3.4.** *For fixed $\tau>0$, for the terms $I_2$ and $J_2$, we have*

$$\begin{aligned}
&\lim_{h\to 0^+}\lim_{l\to 0^+}(I_2+J_2)\\
&=-\lim_{h\to 0^+}\lim_{l\to 0^+}\int_0^T\int_\Omega\psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\mathrm{div}\,(\mathbf{b}\otimes\mathbf{b})^l\,dxdt\\
&\quad-\lim_{h\to 0^+}\lim_{l\to 0^+}\int_0^T\int_\Omega\psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\mathrm{div}\,(\mathbf{u}\otimes\mathbf{b})^l\,dxdt\\
&=0,
\end{aligned}\tag{3.41}$$

*provided $q\ge 4$ and $\frac{2}{p}+\frac{3}{q}\le 1$.*

*Proof.* We first decompose the terms $I_2$ and $J_2$ into two parts respectively

(3.42)
$$\begin{aligned}I_2 =&-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot \mathrm{div}\,(\mathbf{b}\otimes\mathbf{b})^l\,dxdt\\ =&-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\Big(\mathrm{div}\,(\mathbf{b}\otimes\mathbf{b})^l-\mathrm{div}\,\big(\mathbf{b}^l\otimes\mathbf{b}\big)\Big)\,dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\mathrm{div}\,\big(\mathbf{b}^l\otimes\mathbf{b}\big)\,dxdt\\ :=&I_{21}+I_{22},\end{aligned}$$

and

(3.43)
$$\begin{aligned}J_2 =&-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot \mathrm{div}\,(\mathbf{u}\otimes\mathbf{b})^l\,dxdt\\ =&-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\Big(\mathrm{div}\,(\mathbf{u}\otimes\mathbf{b})^l-\mathrm{div}\,\big(\mathbf{u}^l\otimes\mathbf{b}\big)\Big)\,dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\mathrm{div}\,\big(\mathbf{u}^l\otimes\mathbf{b}\big)\,dxdt\\ :=&J_{21}+J_{22}.\end{aligned}$$

Owing to (3.29) and the divergence-free property of $\mathbf{b}$, for the second terms in (3.42) and (3.43), we have

$$\begin{aligned}&I_{22}+J_{22}\\ &=-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\mathrm{div}\,\big(\mathbf{b}^l\otimes\mathbf{b}\big)\,dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\mathrm{div}\,\big(\mathbf{u}^l\otimes\mathbf{b}\big)\,dxdt\\ &=-2\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\mathbf{b}^l\mathrm{div}\mathbf{b}dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}\cdot\nabla(\mathbf{u}^l\cdot\mathbf{b}^l)dxdt\\ &=\int_0^T\int_\Omega \psi_\tau\mathbf{b}\cdot\nabla\theta_{h,l}\mathbf{u}^l\cdot\mathbf{b}^ldxdt+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}(\mathrm{div}\mathbf{b})\mathbf{u}^l\cdot\mathbf{b}^ldxdt\\ &=\int_0^T\int_{\Omega_h} \psi_\tau\mathbf{b}\cdot\nabla\theta_{h,l}\mathbf{u}^l\cdot\mathbf{b}^ldxdt\\ &\le C\||\mathbf{b}||\nabla\theta_{h,l}|\|_{L^2((0,T)\times\Omega_h)}\|\mathbf{u}\|_{L^4((0,T)\times\Omega_h)}\|\mathbf{b}\|_{L^4((0,T)\times\Omega_h)}\\ &\le C\left\|\frac{\mathbf{b}}{dist(x,\partial\Omega)}\right\|_{L^2((0,T)\times\Omega_h)}\|\mathbf{u}\|_{L^4((0,T)\times\Omega_h)}\|\mathbf{b}\|_{L^4((0,T)\times\Omega_h)}\\ &\to 0,\ \text{as } h\to 0^+,\end{aligned}\tag{3.44}$$

provided $q\ge 4$ and $\frac{1}{p}+\frac{1}{q}\le\frac{1}{2}$.

For $I_{21}$, we have

$$\begin{aligned}I_{21}=&-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\Big(\mathrm{div}\,(\mathbf{b}\otimes\mathbf{b})^l-\mathrm{div}\,\Big(\mathbf{b}^l\otimes\mathbf{b}\Big)\Big)\,dxdt\\ \le& C\int_\tau^{T-\tau}\|\mathbf{u}\|_{L^q(\Omega)}\|\mathrm{div}(\mathbf{b}\otimes\mathbf{b})^l-\mathrm{div}(\mathbf{b}^l\otimes\mathbf{b})\|_{L^{\frac{q}{q-1}}(\Omega)}dt\\ \le& C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla\mathbf{b}\|_{L^2((0,T)\times\Omega)}\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))},\end{aligned}\tag{3.45}$$

which combined with Lemma 2.1 and (3.39), yields

$$\lim_{l\to0^+}I_{21}=0,\tag{3.46}$$

provided $\frac{2}{p}+\frac{3}{q}\le 1$.

Similarly, for $J_{21}$, we have

$$
\begin{aligned}
J_{21} =& -\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\Big(\mathrm{div}\,(\mathbf{u}\otimes\mathbf{b})^l-\mathrm{div}\,\big(\mathbf{u}^l\otimes\mathbf{b}\big)\Big)\,dxdt\\
\leq& C\int_\tau^{T-\tau}\|\mathbf{b}\|_{L^{\frac{2q}{q-2}}(\Omega)}\|\mathrm{div}(\mathbf{u}\otimes\mathbf{b})^l-\mathrm{div}(\mathbf{u}^l\otimes\mathbf{b})\|_{L^{\frac{2q}{q+2}}(\Omega)}dt\\
\leq& C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla\mathbf{b}\|_{L^2((0,T)\times\Omega)}\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\\
&\to 0,\ \text{as } l\to 0^+,
\end{aligned}
\tag{3.47}
$$

provided $\frac{2}{p}+\frac{3}{q}\leq 1$.

Substituting (3.44), (3.46) and (3.47) into (3.42) and (3.43), we obtain our desired result. □

**Lemma 3.5.** *For fixed $\tau>0$, the pressure term satisfies*

$$
\lim_{h\to 0^+}\lim_{l\to 0^+} I_3=\lim_{h\to 0^+}\lim_{l\to 0^+}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\nabla\pi^l dxdt=0. \tag{3.48}
$$

*Proof.* Integrating by parts gives

$$
\begin{aligned}
I_3=&\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\nabla\pi^l dxdt\\
=&-\int_0^T\int_\Omega \psi_\tau\nabla\theta_{h,l}\cdot\mathbf{u}^l\pi^l dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathrm{div}\mathbf{u}^l\pi^l dxdt\\
=&-\int_0^T\int_{\Omega_h}\psi_\tau\nabla\theta_{h,l}\cdot\mathbf{u}^l\pi^l dxdt\\
\leq& C\||\mathbf{u}^l||\nabla\theta_{h,l}|\|_{L^2((0,T)\times\Omega_h)}\|\pi\|_{L^2((0,T)\times\Omega_h)}\\
\leq& C\left\|\frac{\mathbf{u}^l}{dist(x,\partial\Omega)}\right\|_{L^2((0,T)\times\Omega_h)}\|\pi\|_{L^2((0,T)\times\Omega_h)}\\
&\to 0,\ \text{as } h\to 0^+,
\end{aligned}
\tag{3.49}
$$

where we used the divergence-free property of $\mathbf{u}^l$ in the third equality. □

**Lemma 3.6.** *For fixed $\tau>0$, the viscous term satisfies*

$$
\lim_{h\to 0^+}\lim_{l\to 0^+} I_4=-\lim_{h\to 0^+}\lim_{l\to 0^+}\mu\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\triangle\mathbf{u}^l dxdt=\mu\int_0^T\int_\Omega\psi_\tau|\nabla\mathbf{u}|^2dxdt. \tag{3.50}
$$

*Proof.* First, we decompose the viscous term into two parts

$$
\begin{aligned}
I_4=&-\mu\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\triangle\mathbf{u}^l dxdt\\
=&\mu\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\nabla\mathbf{u}^l|^2dxdt+\mu\int_0^T\int_{\Omega_h}\psi_\tau(\nabla\theta_{h,l}\cdot\nabla)\mathbf{u}^l\cdot\mathbf{u}^l dxdt\\
:=&I_{41}+I_{42}.
\end{aligned}
\tag{3.51}
$$

For $I_{42}$, we have

$$
\begin{aligned}
I_{42} =&\mu \int_0^T \int_{\Omega_h} \psi_\tau (\nabla\theta_{h,l}\cdot\nabla)\mathbf{u}^l\cdot\mathbf{u}^l dxdt \\
\leq& C\||\nabla\theta_{h,l}||\mathbf{u}^l|\|_{L^2((0,T)\times\Omega_h)}\|\nabla\mathbf{u}\|_{L^2((0,T)\times\Omega_h)} \\
\leq& C\left\|\frac{\mathbf{u}^l}{dist(x,\partial\Omega)}\right\|_{L^2((0,T)\times\Omega_h)}\|\nabla\mathbf{u}\|_{L^2((0,T)\times\Omega_h)} \\
&\to 0, \text{ as } h\to 0^+.
\end{aligned}
\tag{3.52}
$$

For $I_{41}$, we have

$$
\lim_{h\to0^+}\lim_{l\to0^+} I_{41} = \lim_{h\to0^+}\lim_{l\to0^+} \mu\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\nabla\mathbf{u}^l|^2dxdt = \mu\int_0^T\int_\Omega\psi_\tau|\nabla\mathbf{u}|^2dxdt. \tag{3.53}
$$

In fact,

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\nabla\mathbf{u}^l|^2dxdt - \int_0^T\int_\Omega\psi_\tau|\nabla\mathbf{u}|^2dxdt\right| \\
&= \left|\int_0^T\int_\Omega\psi_\tau\left(\theta_{h,l}|\nabla\mathbf{u}^l|^2-|\nabla\mathbf{u}|^2\right)dxdt\right| \\
&\leq \int_0^T\int_\Omega\psi_\tau\theta_{h,l}\left||\nabla\mathbf{u}^l|^2-|\nabla\mathbf{u}|^2\right|dxdt + \int_0^T\int_{\Omega_h}\psi_\tau|\theta_{h,l}-1||\nabla\mathbf{u}|^2dxdt \\
&\leq C\int_0^T\int_\Omega\left||\nabla\mathbf{u}^l|^2-|\nabla\mathbf{u}|^2\right|dxdt + C\int_0^T\int_{\Omega_h}|\nabla\mathbf{u}|^2dxdt,
\end{aligned}
$$

where for fixed $h$, the first term on the last inequality tends to 0, as $l\to 0$, and then, letting $h\to 0$, the second term convergence to 0.

Substituting (3.52) and (3.53) into (3.51), we obtain (3.50). □

**Lemma 3.7.** *For fixed $\tau>0$, the diffusive term satisfies*

$$
\lim_{h\to0^+}\lim_{l\to0^+} J_3 = -\lim_{h\to0^+}\lim_{l\to0^+}\nu\int_0^T\int_{\Omega^l}\psi_\tau\theta_{h,l}\mathbf{b}^l\cdot\triangle\mathbf{b}^ldxdt = \nu\int_0^T\int_\Omega\psi_\tau|\nabla\mathbf{b}|^2dxdt. \tag{3.54}
$$

*Proof.* The proof of Lemma 3.7 is similar to that of Lemma 3.6, thus we omit it. □

Combining Lemma 3.1-3.7 together, we obtain

$$
-\frac{1}{2}\int_0^T\int_\Omega\psi_\tau'\left(|\mathbf{u}|^2+|\mathbf{b}|^2\right)dxdt + \int_0^T\int_\Omega\psi_\tau\left(\mu|\nabla\mathbf{u}|^2+\nu|\nabla\mathbf{b}|^2\right)dxdt = 0. \tag{3.55}
$$

Denote

$$
E(t) := \frac{1}{2}\int_\Omega\left(|\mathbf{u}|^2+|\mathbf{b}|^2\right)(t)dx, \tag{3.56}
$$

and

$$
D(t) := \int_0^t\int_\Omega\left(\mu|\nabla\mathbf{u}|^2+\nu|\nabla\mathbf{b}|^2\right)dxds. \tag{3.57}
$$

Then (3.55) implies

$$
(E+D)' = 0 \text{ in } \mathcal{D}'((0,T)), \tag{3.58}
$$

which means that the energy equality (1.4) holds for a.e. $t \in [0,T]$. To complete the proof, we only need to show that (1.4) actually holds for all $t \in [0,T]$. To this end, we proceed as follows.

First, by the Arzela-Ascoli theorem (cf. Corollary 2.1 in [13]), we have

$$\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;W^{-1,\frac{3}{2}}(\Omega)) \hookrightarrow C([0,T];L^2_{weak}(\Omega)), \tag{3.59}$$

and

$$\mathbf{b} \in L^\infty(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;W^{-1,\frac{3}{2}}(\Omega)) \hookrightarrow C([0,T];L^2_{weak}(\Omega)). \tag{3.60}$$

Then by the energy inequality (1.9), (3.59) and (3.60), we have

$$\begin{aligned} 0 &\le \overline{\lim_{t\to 0^+}} \int_\Omega \left(|\mathbf{u}-\mathbf{u}_0|^2 + |\mathbf{b}-\mathbf{b}_0|^2\right) dx \\ &= 2\overline{\lim_{t\to 0^+}} \left(\int_\Omega \frac{1}{2}(|\mathbf{u}|^2+|\mathbf{b}|^2)dx - \int_\Omega \frac{1}{2}(|\mathbf{u}_0|^2+|\mathbf{b}_0|^2)dx\right) \\ &\quad + 2\overline{\lim_{t\to 0^+}} \left(\int_\Omega \mathbf{u}_0(\mathbf{u}_0-\mathbf{u})dx + \int_\Omega \mathbf{b}_0(\mathbf{b}_0-\mathbf{b})dx\right) \\ &\le 0, \end{aligned} \tag{3.61}$$

which implies

$$\lim_{t\to 0^+} \|\mathbf{u}(t,x)-\mathbf{u}(0,x)\|_{L^2(\Omega)} = \lim_{t\to 0^+} \|\mathbf{b}(t,x)-\mathbf{b}(0,x)\|_{L^2(\Omega)} = 0. \tag{3.62}$$

Similarly, we can deduce the right continuity of $\mathbf{u}$ and $\mathbf{b}$ in $L^2(\Omega)$ for any $t_0 \ge 0$, that is,

$$\lim_{t\to t_0^+} \|\mathbf{u}(t,x)-\mathbf{u}(t_0,x)\|_{L^2(\Omega)} = \lim_{t\to t_0^+} \|\mathbf{b}(t,x)-\mathbf{b}(t_0,x)\|_{L^2(\Omega)} = 0. \tag{3.63}$$

Now, as in [5], for $t_0 > 0$, we take some positive $\tau$ and $\alpha$ satisfying $\tau + \alpha < t_0$, and define

$$\psi_\tau(t) := \begin{cases} 0, & 0 \le t \le \tau, \\ \frac{t-\tau}{\alpha}, & \tau \le t \le \tau+\alpha, \\ 1, & \tau+\alpha \le t \le t_0, \\ \frac{t_0-t}{\alpha}, & t_0 \le t \le t_0+\alpha, \\ 0, & t \ge t_0+\alpha. \end{cases} \tag{3.64}$$

Substituting (3.64) into (3.55), we deduce

$$\begin{aligned} &\frac{1}{2\alpha}\int_\tau^{\tau+\alpha}\int_\Omega (|\mathbf{u}|^2+|\mathbf{b}|^2)dxdt - \frac{1}{2\alpha}\int_{t_0}^{t_0+\alpha}\int_\Omega (|\mathbf{u}|^2+|\mathbf{b}|^2)dx \\ &= \int_\tau^{t_0+\alpha}\int_\Omega \psi_\tau\left(\mu|\nabla\mathbf{u}|^2+\nu|\nabla\mathbf{b}|^2\right)dxdt = D(t_0+\alpha) - D(\tau). \end{aligned} \tag{3.65}$$

Passing to the limit $\alpha \to 0^+$, using the right continuity of $\mathbf{u}$ and $\mathbf{b}$ in $L^2$ and the continuity of $D(t)$, we obtain

$$E(\tau) - E(t_0) = D(t_0) - D(\tau). \tag{3.66}$$

Then, letting $\tau \to 0^+$, by (3.62), we have

$$(E+D)(t_0) = (E+D)(0), \tag{3.67}$$

which is exactly (1.4). □

### 3.2. **Proof of Theorem 1.2.**

*Proof.* Following the proof of Theorem 1.1, we have

$$\int_0^T \int_\Omega \partial_t \left(\psi_\tau \theta_{h,l} \mathbf{u}^l\right) \cdot \mathbf{u}^l dxdt = I_1 + I_2 + I_3 + I_4, \tag{3.68}$$

and

$$\int_0^T \int_\Omega \partial_t \left(\psi_\tau \theta_{h,l} \mathbf{b}^l\right) \cdot \mathbf{b}^l dxdt = J_1 + J_2 + J_3, \tag{3.69}$$

with $I_i$ $(i = 1, 2, 3, 4)$ and $J_j$ $(j = 1, 2, 3)$ being as previous subsection.

In accordance with Lemma 3.1, Lemma 3.5-Lemma 3.7, to prove Theorem 1.2, we need to show

$$\begin{aligned} &\lim_{h\to 0^+} \lim_{l\to 0^+} I_1 = 0, \\ &\lim_{h\to 0^+} \lim_{l\to 0^+} J_1 = 0, \\ &\lim_{h\to 0^+} \lim_{l\to 0^+} (I_2 + J_2) = 0. \end{aligned} \tag{3.70}$$

In fact, by the proof of Lemma 3.2-Lemma 3.4 in Section 3, we can check that the three limits in (3.70) still holds under the condition (1.14).

□

## 4. The inhomogeneous incompressible case

Now we apply our method to the inhomogeneous incompressible MHD system, which reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(\mathbf{b} \otimes \mathbf{b}) + \nabla \pi - \mu \triangle \mathbf{u} = 0, \\ \partial_t \mathbf{b} + \operatorname{div}(\mathbf{b} \otimes \mathbf{u}) - \operatorname{div}(\mathbf{u} \otimes \mathbf{b}) - \nu \triangle \mathbf{b} = 0, \qquad \text{in } (0,T) \times \Omega, \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{b} = 0, \end{cases} \tag{4.71}$$

together with the initial conditions

$$(\rho, \rho\mathbf{u}, \mathbf{b})(0, x) = (\rho_0, \rho_0 \mathbf{u}_0, \mathbf{b}_0)(x), \qquad x \in \Omega, \tag{4.72}$$

and the boundary conditions

$$\mathbf{u} = \mathbf{b} = 0 \quad \text{on } [0,T] \times \partial\Omega. \tag{4.73}$$

Compared with the homogeneous case, we require stronger regularity condition for velocity $\mathbf{u}$ to tackle the new nonlinear term $\partial_t(\rho\mathbf{u})$, which will be stated in Theorem 4.1. First, we give the definition of weak solutions to (4.71)-(4.73).

**Definition 4.1.** *We call* $(\rho, \mathbf{u}, \mathbf{b}, \pi)$ *a weak solution to the initial-boundary value problem* (4.71)-(4.73) *if* *(i)* $\rho \geq 0$ *and* $\mathbf{u}$ *satisfy*

$$\mathbf{u} \in L^2(0,T;H_0^1(\Omega)),\ \sqrt{\rho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega)), \tag{4.74}$$

*and* $\mathbf{b}$ *satisfies*

$$\mathbf{b} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)); \tag{4.75}$$

*(ii) the equation* (4.71) *holds in* $\mathcal{D}'((0,T)\times\Omega)$*;*

*(iii)* $\rho(t,\cdot) \to \rho_0$, $\rho\mathbf{u}(t,\cdot) \to \mathbf{m}_0$ *in* $\mathcal{D}'(\Omega)$ *as* $t \to 0^+$*, that is, for any* $\chi \in C_c^\infty(\Omega)$*, it holds*

$$\lim_{t\to 0^+} \int_\Omega \rho(t,x)\chi(x)dx = \int_\Omega \rho_0(x)\chi(x)dx,$$

$$\lim_{t\to 0^+} \int_\Omega (\rho\mathbf{u})(t,x)\chi(x)dx = \int_\Omega \mathbf{m}_0(x)\chi(x)dx;$$

*(iv) the following energy inequality holds for any* $t \in [0,T]$

$$\frac{1}{2}\int_\Omega (\rho|\mathbf{u}|^2 + |\mathbf{b}|^2)(t)dx + \int_0^t \int_\Omega \left(\mu|\nabla\mathbf{u}|^2 + \nu|\nabla\mathbf{b}|^2\right) dxds \leq \frac{1}{2}\int_\Omega (\rho_0|\mathbf{u}_0|^2 + |\mathbf{b}_0|^2)dx. \tag{4.76}$$

Now we state our main result about the inhomogeneous case.

**Theorem 4.1.** *Let* $(\rho, \mathbf{u}, \mathbf{b}, \pi)$ *be a weak solution to the initial-boundary value problem* (4.71)-(4.73) *in the sense of Definition 4.1. Assume that*

$$\rho_0 \in L^\infty(\Omega), \quad \mathbf{u}_0 \in L^4(\Omega), \quad \mathbf{b}_0 \in L^2(\Omega), \tag{4.77}$$

$$\nabla\sqrt{\rho} \in L^\infty(0,T;L^2(\Omega)), \tag{4.78}$$

*and*

$$\mathbf{u} \in L^p(0,T;L^q(\Omega)) \tag{4.79}$$

*with* $p \geq 4$ *and* $q \geq 6$*. Also assume that there exists a* $\delta > 0$ *such that*

$$\mathbf{b} \in L^4((0,T)\times\Omega_\delta), \quad \pi \in L^2((0,T)\times\Omega_\delta). \tag{4.80}$$

*Then the following energy equality*

$$\frac{1}{2}\int_\Omega (\rho|\mathbf{u}|^2 + |\mathbf{b}|^2)(t)dx + \int_0^t \int_\Omega \left(\mu|\nabla\mathbf{u}|^2 + \nu|\nabla\mathbf{b}|^2\right) dxds = \frac{1}{2}\int_\Omega (\rho_0|\mathbf{u}_0|^2 + |\mathbf{b}_0|^2)dx. \tag{4.81}$$

*holds for all* $t \in [0,T]$*.*

**Remark 4.1.** *If the density is away from zero, the condition* (4.79) *can be replaced by the following weaker one:*

$$\mathbf{u} \in L^p(0,T;L^q(\Omega)),$$

*with* $\frac{2}{p} + \frac{3}{q} \leq 1$ *and* $q \geq 6$*, which will be proved in Remark 4.3 below.*

**Remark 4.2.** *With regard to the condition* (4.78)*, as proved in* [6]*, it can be replaced by imposing additional time regularity on* $\mathbf{u}$*, and if the density* $\rho$ *is bounded from below by a positive constant, the condition* (4.78) *can be omitted, just as Nguyen, Nguyen and Tang did in* [24]*.*

Before proving Theorem 4.1, we first give the definition of mollification in time and space and introduce an important lemma.

We define the mollification of a function $f \in L^1(\Omega)$ in time and space as follows:

$$f^{,l}(t,x) := \frac{1}{l^4}\int_0^t \int_\Omega \varphi(\frac{t-s}{l}, \frac{x-y}{l}) f(s,y)dyds, \tag{4.82}$$

where $\varphi$ is the standard mollifier in four dimensions.

Corresponding to (4.82), as Lemma 2.1, we introduce the following commutator estimates.

**Lemma 4.1.** ([13, 19]) *Suppose $g_1 \in W^{1,p_1}((0,T)\times\Omega)$, $g_2 \in L^{p_2}((0,T)\times\Omega)$, with $1 \leq p, p_1, p_2 \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\|\partial(g_1g_2)^{,l} - \partial(g_1g_2^{,l})\|_{L^p((0,T)\times\Omega)} \leq C\|\partial g_1\|_{L^{p_1}((0,T)\times\Omega)}\|g_2\|_{L^{p_2}((0,T)\times\Omega)}, \tag{4.83}$$

*for some constant $C$ independent of $l$, $g_1$ and $g_2$, where $\partial = \partial_t$ or $\partial = \partial_x$, and $f^{,l}$ is defined as in* (4.82)*. In addition, if $p < \infty$, then*

$$\partial(g_1g_2)^{,l} - \partial(g_1g_2^{,l}) \to 0 \text{ in } L^p((0,T)\times\Omega), \text{ as } l \to 0^+. \tag{4.84}$$

Now we begin our proof.

*Proof.* Choosing $\left(\psi_\tau\theta_{h,l}\mathbf{u}^{,l}\right)^{,l}$ as a test function of $(4.71)_2$, we have

$$\int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt + K_1 + K_2 + K_3 + K_4 = 0, \tag{4.85}$$

where

$$\begin{aligned}
K_1 &:= \int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})^{,l}dxdt,\\
K_2 &:= -\int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\operatorname{div}(\mathbf{b}\otimes\mathbf{b})^{,l}dxdt,\\
K_3 &:= \int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla\pi^{,l}dxdt,\\
K_4 &:= -\mu\int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\triangle\mathbf{u}^{,l}dxdt.
\end{aligned}$$

Similarly, choosing $\left(\psi_\tau\theta_{h,l}\mathbf{b}^{,l}\right)^{,l}$ as a test function of $(4.71)_3$, we have

$$\int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\partial_t\mathbf{b}^{,l}dxdt + L_1 + L_2 + L_3 = 0, \tag{4.86}$$

where

$$\begin{aligned}
L_1 &:= \int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\operatorname{div}\left(\mathbf{b}\otimes\mathbf{u}\right)^{,l}dxdt,\\
L_2 &:= -\int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\operatorname{div}\left(\mathbf{u}\otimes\mathbf{b}\right)^{,l}dxdt,\\
L_3 &:= -\nu\int_0^T \int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\triangle\mathbf{b}^{,l}dxdt.
\end{aligned}$$

Next we will focus our attention on the terms $\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt$ and $K_1$, since the other terms can be handled as in Section 3.

**Lemma 4.2.** *For fixed $\tau>0$, we have*

$$\begin{aligned}
&\lim_{h\to0^+}\lim_{l\to0^+}\left[\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt+K_1\right]\\
&=\lim_{h\to0^+}\lim_{l\to0^+}\left[\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})^{,l}dxdt\right]\\
&=-\frac12\int_0^T\int_\Omega \psi_\tau'\rho|\mathbf{u}|^2dxdt,
\end{aligned}\tag{4.87}$$

*provided $p\ge4$ and $q\ge6$.*

*Proof.* We first decompose the terms $\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt$ and $K_1$ into two parts respectively

$$\begin{aligned}
&\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt\\
&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\Big(\partial_t(\rho\mathbf{u})^{,l}-\partial_t(\rho\mathbf{u}^{,l})\Big)\,dxdt+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u}^{,l})dxdt\\
&:=M+N,
\end{aligned}\tag{4.88}$$

and

$$\begin{aligned}
K_1&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})^{,l}dxdt\\
&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\Big(\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})^{,l}-\operatorname{div}(\mathbf{u}^{,l}\otimes\rho\mathbf{u})\Big)\,dxdt\\
&\quad+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\operatorname{div}(\mathbf{u}^{,l}\otimes\rho\mathbf{u})dxdt\\
&:=K_{11}+K_{12}.
\end{aligned}\tag{4.89}$$

For the terms $N$ and $K_{12}$, integrating by parts and using the continuity equation $(4.71)_1$, we have

$$\begin{aligned}
N+K_{12}&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u}^{,l})dxdt+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\operatorname{div}(\mathbf{u}^{,l}\otimes\rho\mathbf{u})dxdt\\
&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\mathbf{u}^{,l}|^2\partial_t\rho dxdt+\frac12\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\rho\partial_t|\mathbf{u}^{,l}|^2dxdt\\
&\quad+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\mathbf{u}^{,l}|^2\operatorname{div}(\rho\mathbf{u})dxdt+\frac12\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\rho\mathbf{u}\cdot\nabla|\mathbf{u}^{,l}|^2dxdt\\
&=\frac12\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\mathbf{u}^{,l}|^2\partial_t\rho dxdt-\frac12\int_0^T\int_\Omega \psi_\tau'\theta_{h,l}\rho|\mathbf{u}^{,l}|^2dxdt\\
&\quad+\frac12\int_0^T\int_\Omega \psi_\tau\theta_{h,l}|\mathbf{u}^{,l}|^2\operatorname{div}(\rho\mathbf{u})dxdt-\frac12\int_0^T\int_\Omega \psi_\tau\mathbf{u}\cdot\nabla\theta_{h,l}\rho|\mathbf{u}^{,l}|^2dxdt\\
&=-\frac12\int_0^T\int_\Omega \psi_\tau'\theta_{h,l}\rho|\mathbf{u}^{,l}|^2dxdt-\frac12\int_0^T\int_{\Omega_h} \psi_\tau\mathbf{u}\cdot\nabla\theta_{h,l}\rho|\mathbf{u}^{,l}|^2dxdt,
\end{aligned}\tag{4.90}$$

where thanks to Lemma 2.2, the second term can be controlled by

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\int_{\Omega_h}\psi_\tau\mathbf{u}\cdot\nabla\theta_{h,l}\rho|\mathbf{u}^{,l}|^2dxdt\\
&\le C\||\mathbf{u}||\nabla\theta_{h,l}|\|_{L^2((0,T)\times\Omega_h)}\|\rho\|_{L^\infty((0,T)\times\Omega_h)}\|\mathbf{u}\|^2_{L^4((0,T)\times\Omega_h)}\\
&\le C\left\|\frac{\mathbf{u}}{dist(x,\partial\Omega)}\right\|_{L^2((0,T)\times\Omega_h)}\|\rho\|_{L^\infty((0,T)\times\Omega_h)}\|\mathbf{u}\|^2_{L^4((0,T)\times\Omega_h)}\to 0,\ \text{as } h\to 0^+.
\end{aligned}\tag{4.91}
$$

Thus, we have

$$
\lim_{h\to 0^+}\lim_{l\to 0^+}(N+K_{12})=-\frac{1}{2}\int_0^T\int_\Omega\psi'_\tau\rho|\mathbf{u}|^2dxdt.\tag{4.92}
$$

For the terms $M$ and $K_{11}$, by Hölder's inequality and Lemma 4.1, we have

$$
\begin{aligned}
&M+K_{11}\\
&=\int_0^T\int_\Omega\psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\big(\partial_t(\rho\mathbf{u})^l-\partial_t(\rho\mathbf{u}^l)+\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})^l-\mathrm{div}(\mathbf{u}^l\otimes\rho\mathbf{u})\big)\,dxdt\\
&\le C\int_\tau^{T-\tau}\|\mathbf{u}\|_{L^q(\Omega)}\Big(\|\partial_t(\rho\mathbf{u})^l-\partial_t(\rho\mathbf{u}^l)\|_{L^{\frac{q}{q-1}}(\Omega)}+\|\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})^l-\mathrm{div}(\mathbf{u}^l\otimes\rho\mathbf{u})\|_{L^{\frac{q}{q-1}}(\Omega)}\Big)\,dt\\
&\le C\|\mathbf{u}\|^2_{L^p(0,T;L^q(\Omega))}\Big(\|\partial_t\rho\|_{L^{\frac{p}{p-2}}(0,T;L^{\frac{q}{q-2}}(\Omega))}+\|\mathrm{div}(\rho\mathbf{u})\|_{L^{\frac{p}{p-2}}(0,T;L^{\frac{q}{q-2}}(\Omega))}\Big).
\end{aligned}\tag{4.93}
$$

Then, by the continuity equation $(4.71)_1$, the divergence-free property of $\mathbf{u}$, (4.78) and (4.79), we have

$$
\partial_t\rho=-\mathrm{div}(\rho\mathbf{u})=-2\nabla\sqrt{\rho}\cdot\sqrt{\rho}\mathbf{u}\in L^p(0,T;L^{\frac{2q}{q+2}}(\Omega)).\tag{4.94}
$$

Thus, by Lemma 4.1, we have

$$
\lim_{l\to 0^+}(M+K_{11})=0,\tag{4.95}
$$

provided $\frac{p}{p-2}\le p$ and $\frac{q}{q-2}\le\frac{2q}{q+2}$, or equivalently, $p\ge 3$ and $q\ge 6$.

Substituting (4.92) and (4.95) into (4.88) and (4.89), we get our desired result. □

**Remark 4.3.** *As mentioned in Remark 4.1, if the density is away from zero, the condition* (4.79) *can be replaced by the following weaker condition:*

$$
\mathbf{u}\in L^p(0,T;L^q(\Omega)),
$$

*with* $\frac{2}{p}+\frac{3}{q}\le 1$ *and* $q\ge 6$.

*In fact, if the density is away from zero, by* (4.74)*, we have*

$$
\mathbf{u}\in L^\infty(0,T;L^2(\Omega)),\tag{4.96}
$$

*which implies for fixed* $\tau>0$

$$
\lim_{h\to 0^+}\lim_{l\to 0^+}\left[\int_0^T\int_\Omega\psi_\tau\theta_{h,l}\mathbf{u}^l\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt+K_1\right]=-\frac{1}{2}\int_0^T\int_\Omega\psi'_\tau\rho|\mathbf{u}|^2dxdt,\tag{4.97}
$$

*provided* $q\ge 6$ *and* $\frac{1}{p}+\frac{1}{q}\le\frac{1}{2}$.

*On the one hand, by* (4.79) *and* (4.96)*, we have*

$$
\|\mathbf{u}\|_{L^4((0,T)\times\Omega)}\le C,
$$

*provided* $q \geq 4$ *and* $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, *which implies* (4.92).

*On the other hand, for* $M$ *and* $K_{11}$, *we have*

$$\begin{aligned}&M + K_{11}\\ &\leq C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\mathbf{u}\|_{L^{\frac{p}{p-2}}(0,T;L^{\frac{2q}{q-4}}(\Omega))}\\ &\quad\left(\|\partial_t\rho\|_{L^p(0,T;L^{\frac{2q}{q+2}}(\Omega))} + \|\mathrm{div}(\rho\mathbf{u})\|_{L^p(0,T;L^{\frac{2q}{q+2}}(\Omega))}\right).\end{aligned}$$

*By* (4.96) *and interpolation inequality, we have*

$$\|\mathbf{u}\|_{L^{\frac{p}{p-2}}(0,T;L^{\frac{2q}{q-4}}(\Omega))} \leq C\|\mathbf{u}\|^{\frac{q-6}{q-2}}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}\|^{\frac{4}{q-2}}_{L^{\frac{4p}{(p-2)(q-2)}}(0,T;L^q(\Omega))} \leq C,$$

*provided* $0 \leq \frac{4}{q-2} \leq 1$ *and* $\frac{4p}{(p-2)(q-2)} \leq p$, *which is equivalent to* $q \geq 6$ *and* $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. *Thus, we can obtain* (4.95) *provided* $q \geq 6$ *and* $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$.

*Taking into account* (4.92) *and* (4.95), *we can obtain* (4.97). *Therefore, combining* (4.97) *and Lemma 3.3-Lemma 3.7, we can obtain our desired result.*

Now we return to our proof. By Lemma 3.3-Lemma 3.7 and Lemma 4.2, letting $l \to 0^+$ and $h \to 0^+$ successively, we have

$$-\frac{1}{2}\int_0^T\int_\Omega \psi_\tau'\left(\rho|\mathbf{u}|^2 + |\mathbf{b}|^2\right)dxdt + \int_0^T\int_\Omega \psi_\tau(\mu|\nabla\mathbf{u}|^2 + \nu|\nabla\mathbf{b}|^2)dxdt = 0. \tag{4.98}$$

Denote

$$E_1(t) := \frac{1}{2}\int_\Omega(\rho|\mathbf{u}|^2 + |\mathbf{b}|^2)dx, \tag{4.99}$$

and

$$D_1(t) := \int_0^t\int_\Omega(\mu|\nabla\mathbf{u}|^2 + \nu|\nabla\mathbf{b}|^2)dxds. \tag{4.100}$$

Then (4.98) implies

$$(E_1 + D_1)' = 0 \text{ in } \mathcal{D}'((0,T)). \tag{4.101}$$

As the homogeneous case, in order to prove the energy equality (4.81), it is sufficient to prove the right continuity of $\sqrt{\rho}\mathbf{u}$ and $\mathbf{b}$ in $L^2$.

First, by the continuity equation $(4.71)_1$, we have

$$\partial_t\sqrt{\rho} = -\mathrm{div}(\sqrt{\rho}\mathbf{u}), \tag{4.102}$$

which, with help of (4.74), implies

$$\partial_t\sqrt{\rho} \in L^\infty(0,T;H^{-1}(\Omega)). \tag{4.103}$$

Thanks to the Aubin-Lions lemma (cf. [29]), we have

$$\sqrt{\rho} \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega)) \hookrightarrow C([0,T];L^2(\Omega)). \tag{4.104}$$

Meanwhile, by the Arzela-Ascoli theorem (cf. Corollary 2.1 in [13]), we have

$$\rho\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;W^{-1,\frac{3}{2}}(\Omega)) \hookrightarrow C([0,T];L^2_{weak}(\Omega)), \tag{4.105}$$

and

$$\mathbf{b} \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;W^{-1,\frac{3}{2}}(\Omega)) \hookrightarrow C([0,T];L^2_{weak}(\Omega)). \tag{4.106}$$

Thus, by the energy inequality (4.76) and (4.104)-(4.106), we have

$$
\begin{aligned}
0 &\le \overline{\lim_{t\to 0^+}} \int_\Omega \left(|\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2 + |\mathbf{b}-\mathbf{b}_0|^2\right) dx \\
&= 2\overline{\lim_{t\to 0^+}} \left(\int_\Omega \frac{1}{2}(\rho|\mathbf{u}|^2 + |\mathbf{b}|^2)dx - \int_\Omega \frac{1}{2}(\rho_0|\mathbf{u}_0|^2 + |\mathbf{b}_0|^2)dx\right) \\
&\quad + 2\overline{\lim_{t\to 0^+}} \left(\int_\Omega \mathbf{u}_0(\rho_0\mathbf{u}_0 - \sqrt{\rho_0}\sqrt{\rho}\mathbf{u})dx + \int_\Omega \mathbf{b}_0(\mathbf{b}_0 - \mathbf{b})dx\right) \\
&\le 2\overline{\lim_{t\to 0^+}} \int_\Omega \mathbf{u}_0(\rho_0\mathbf{u}_0 - \sqrt{\rho_0}\sqrt{\rho}\mathbf{u})dx \\
&= 2\overline{\lim_{t\to 0^+}} \int_\Omega \left(\mathbf{u}_0(\rho_0\mathbf{u}_0 - \rho\mathbf{u}) + \mathbf{u}_0\sqrt{\rho}\mathbf{u}(\sqrt{\rho} - \sqrt{\rho_0})\right) dx \\
&= 0,
\end{aligned} \tag{4.107}
$$

provided $q \ge 4$ and $\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}$, where we used $\mathbf{u} \in L^4((0,T)\times\Omega)$ and $\mathbf{u}_0 \in L^4(\Omega)$ in the last equality, which implies

$$
\lim_{t\to 0^+} \|(\sqrt{\rho}\mathbf{u})(t,x) - (\sqrt{\rho_0}\mathbf{u}_0)(x)\|_{L^2(\Omega)} = \lim_{t\to 0^+} \|\mathbf{b}(t,x) - \mathbf{b}_0(x)\|_{L^2(\Omega)} = 0. \tag{4.108}
$$

Similarly, we can deduce the right continuity of $\sqrt{\rho}\mathbf{u}$ and $\mathbf{b}$ in $L^2$ for any $t_0 \ge 0$, that is,

$$
\lim_{t\to t_0^+} \|(\sqrt{\rho}\mathbf{u})(t,x) - (\sqrt{\rho}\mathbf{u})(t_0,x)\|_{L^2(\Omega)} = \lim_{t\to t_0^+} \|\mathbf{b}(t,x) - \mathbf{b}(t_0,x)\|_{L^2(\Omega)} = 0. \tag{4.109}
$$

The rest of this proof is similar to that of Theorem 1.1, thus we omit it here. □

## 5. The isentropic compressible case

Now we apply our method to the isentropic compressible MHD system, which reads

$$
\begin{cases}
\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \\
\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mathrm{div}(\mathbf{b}\otimes\mathbf{b}) + \nabla\pi = \mathrm{div}\mathbb{S}, \\
\partial_t\mathbf{b} + \mathrm{div}(\mathbf{b}\otimes\mathbf{u}) - \mathrm{div}(\mathbf{u}\otimes\mathbf{b}) - \nu\triangle\mathbf{b} = 0, \qquad \text{in } (0,T)\times\Omega, \\
\mathrm{div}\mathbf{b} = 0,
\end{cases} \tag{5.110}
$$

together with the initial conditions

$$
(\rho, \rho\mathbf{u}, \mathbf{b})(0,x) = (\rho_0, \rho_0\mathbf{u}_0, \mathbf{b}_0)(x), \qquad x \in \Omega, \tag{5.111}
$$

and the boundary conditions

$$
\mathbf{u} = \mathbf{b} = 0 \quad \text{on } [0,T]\times\partial\Omega. \tag{5.112}
$$

Here $P$ is given by $P := P(\rho) = a\rho^\gamma$, with $a$ being a positive constant and the adiabatic exponent $\gamma$ satisfying $\gamma > 1$, and the viscous stress tensor $\mathbb{S}$ is given by

$$
\mathbb{S} = \mu(\nabla\mathbf{u} + \nabla^{\mathbf{t}}\mathbf{u}) + \lambda(\mathrm{div}\mathbf{u})\mathbb{I},
$$

with $\mathbb{I}$ being the unit matrix, $\mu$ and $\lambda$ being the shear and bulk viscosity coefficients respectively, satisfying

$$
\mu > 0,\ \lambda + \frac{2}{3}\mu \ge 0. \tag{5.113}
$$

We first give the definition of weak solutions to (5.110)-(5.112) as follows.

**Definition 5.1.** *We call* $(\rho, \mathbf{u}, \mathbf{b})$ *a weak solution to the initial-boundary value problem* (5.110)-(5.112) *if*

*(i)* $\rho \geq 0$ *and* $\mathbf{u}$ *satisfy*

$$\rho \in L^\infty(0,T; L^\gamma(\Omega)), \ \mathbf{u} \in L^2(0,T; H_0^1(\Omega)), \ \sqrt{\rho}\mathbf{u} \in L^\infty(0,T; L^2(\Omega)), \tag{5.114}$$

*and* $\mathbf{b}$ *satisfies*

$$\mathbf{b} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)); \tag{5.115}$$

*(ii) the equation* (5.110) *holds in* $\mathcal{D}'((0,T) \times \Omega)$*;*

*(iii)* $\rho(t,\cdot) \to \rho_0$, $\rho\mathbf{u}(t,\cdot) \to \mathbf{m}_0$ *in* $\mathcal{D}'(\Omega)$ *as* $t \to 0^+$*;*

*(iv) the following energy inequality holds for any* $t \in [0,T]$

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{b}|^2 + \frac{a\rho^\gamma}{\gamma-1}\right)(t)dx \\ &\qquad + \int_0^t \int_\Omega \left(\mu|\nabla\mathbf{u}|^2 + (\mu+\lambda)(\mathrm{div}\mathbf{u})^2 + \nu|\nabla\mathbf{b}|^2\right) dxds \\ &\leq \int_\Omega \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{1}{2}|\mathbf{b}_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1}\right) dx. \end{aligned} \tag{5.116}$$

Our main result related to system (5.110) is as follows.

**Theorem 5.1.** *Let* $(\rho, \mathbf{u}, \mathbf{b})$ *be a weak solution to the initial-boundary value problem* (5.110)-(5.112) *in the sense of Definition 5.1. Assume that*

$$\sqrt{\rho_0}\mathbf{u}_0 \in L^2(\Omega), \ \rho_0^\gamma \in L^1(\Omega), \ \mathbf{u}_0 \in L^4(\Omega), \ \mathbf{b}_0 \in L^2(\Omega), \tag{5.117}$$

$$\rho \in L^\infty((0,T) \times \Omega), \ \nabla\sqrt{\rho} \in L^\infty(0,T; L^2(\Omega)), \tag{5.118}$$

*and*

$$\mathbf{u} \in L^p(0,T; L^q(\Omega)) \tag{5.119}$$

*with* $p \geq 4$ *and* $q \geq 6$*. Also assume that there exists a* $\delta > 0$ *such that*

$$\mathbf{b} \in L^4((0,T) \times \Omega_\delta). \tag{5.120}$$

*Then the following energy equality*

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{b}|^2 + \frac{a\rho^\gamma}{\gamma-1}\right)(t)dx \\ &\qquad + \int_0^t \int_\Omega \left(\mu|\nabla\mathbf{u}|^2 + (\mu+\lambda)(\mathrm{div}\mathbf{u})^2 + \nu|\nabla\mathbf{b}|^2\right) dxds \\ &= \int_\Omega \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{1}{2}|\mathbf{b}_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1}\right) dx. \end{aligned} \tag{5.121}$$

*holds for all* $t \in [0,T]$.

**Remark 5.1.** *Note that our result is consistent with the result* [5] *for the compressible Navier-Stokes equations in a bounded domain. In addition, in the absence of vacuum, as proved in* [24]*, the condition* $\nabla\sqrt{\rho} \in L^\infty(0,T; L^2(\Omega))$ *in* (5.118) *can be relaxed to*

$$\sup_{t\in(0,T)} \sup_{|h|<\varepsilon} |h|^{-\frac{1}{2}} \|\rho(\cdot+h,t) - \rho(\cdot,t)\|_{L^{\frac{12}{5}}(\Omega^\delta)} < \infty, \text{ for each } 0 < \delta < 1. \tag{5.122}$$

*Proof.* As in Section 4, choosing $\left(\psi_\tau\theta_{h,l}\mathbf{u}^{,l}\right)^{,l}$ as a test function of $(5.110)_2$, we obtain

$$\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\partial_t(\rho\mathbf{u})^{,l}dxdt+M_1+M_2+M_3+M_4+M_5+M_6=0, \tag{5.123}$$

where

$$\begin{aligned}
M_1 &:= \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})^{,l}dxdt,\\
M_2 &:= -\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\mathrm{div}(\mathbf{b}\otimes\mathbf{b})^{,l}dxdt,\\
M_3 &:= a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla(\rho^\gamma)^{,l}dxdt,\\
M_4 &:= \frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla(|\mathbf{b}|^2)^{,l}dxdt,\\
M_5 &:= -\mu\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\triangle\mathbf{u}^{,l}dxdt,\\
M_6 &:= -(\lambda+\mu)\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla\mathrm{div}\mathbf{u}^{,l}dxdt.
\end{aligned}$$

Choosing $\left(\psi_\tau\theta_{h,l}\mathbf{b}^{,l}\right)^{,l}$ as the test function of $(5.110)_3$, we have

$$\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\partial_t\mathbf{b}^{,l}dxdt+N_1+N_2+N_3=0, \tag{5.124}$$

where

$$\begin{aligned}
N_1 &:= \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\mathrm{div}\left(\mathbf{b}\otimes\mathbf{u}\right)^{,l}dxdt,\\
N_2 &:= -\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\mathrm{div}\left(\mathbf{u}\otimes\mathbf{b}\right)^{,l}dxdt,\\
N_3 &:= -\nu\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\triangle\mathbf{b}^{,l}dxdt.
\end{aligned}$$

For the isentropic compressible case, we mainly focus our attention on the pressure term $M_3$ and the nonlinear terms $M_4$ and $N_1$, since the other terms can be handled as in Section 3 and Section 4.

**Lemma 5.1.** *For fixed $\tau>0$, for the pressure term $M_3$, we have*

$$\lim_{h\to0^+}\lim_{l\to0^+}M_3=a\lim_{h\to0^+}\lim_{l\to0^+}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla(\rho^\gamma)^{,l}dxdt=-\frac{a}{\gamma-1}\int_0^T\int_\Omega \psi_\tau'\rho^\gamma dxdt. \tag{5.125}$$

*Proof.* We first decompose the pressure term into three parts

$$
\begin{aligned}
M_3 =& a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla(\rho^\gamma)^{,l}dxdt\\
=& a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\left(\nabla(\rho^\gamma)^{,l}-\nabla\rho^\gamma\right)dxdt + a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}(\mathbf{u}^{,l}-\mathbf{u})\cdot\nabla\rho^\gamma dxdt\\
&+ a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}\cdot\nabla\rho^\gamma dxdt\\
:=& M_{31}+M_{32}+M_{33}.
\end{aligned}
\tag{5.126}
$$

For $M_{33}$, we have

$$
\begin{aligned}
M_{33} =& a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}\cdot\nabla\rho^\gamma dxdt = \frac{a\gamma}{\gamma-1}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\rho\mathbf{u}\cdot\nabla\rho^{\gamma-1}dxdt\\
=& -\frac{a\gamma}{\gamma-1}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathrm{div}(\rho\mathbf{u})\rho^{\gamma-1}dxdt - \frac{a\gamma}{\gamma-1}\int_0^T\int_\Omega \psi_\tau\mathbf{u}\cdot\nabla\theta_{h,l}\rho^\gamma dxdt\\
=& \frac{a\gamma}{\gamma-1}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\rho^{\gamma-1}\partial_t\rho dxdt - \frac{a\gamma}{\gamma-1}\int_0^T\int_\Omega \psi_\tau\mathbf{u}\cdot\nabla\theta_{h,l}\rho^\gamma dxdt\\
=& -\frac{a}{\gamma-1}\int_0^T\int_\Omega \psi_\tau'\theta_{h,l}\rho^\gamma dxdt - \frac{a\gamma}{\gamma-1}\int_0^T\int_{\Omega_h} \psi_\tau\mathbf{u}\cdot\nabla\theta_{h,l}\rho^\gamma dxdt,
\end{aligned}
\tag{5.127}
$$

where the second term can be controlled by

$$
\begin{aligned}
&-\frac{a\gamma}{\gamma-1}\int_0^T\int_{\Omega_h} \psi_\tau\mathbf{u}\cdot\nabla\theta_{h,l}\rho^\gamma dxdt\\
&\le C\||\mathbf{u}||\nabla\theta_{h,l}|\|_{L^2((0,T)\times\Omega_h)}\|\rho^\gamma\|_{L^2((0,T)\times\Omega_h)}\\
&\le C\left\|\frac{\mathbf{u}}{dist(x,\partial\Omega)}\right\|_{L^2((0,T)\times\Omega_h)}\|\rho^\gamma\|_{L^2((0,T)\times\Omega_h)}\\
&\to 0,\ \text{as } h\to 0^+.
\end{aligned}
\tag{5.128}
$$

Thus, we have

$$
\lim_{h\to 0^+}\lim_{l\to 0^+} M_{33} = -\frac{a}{\gamma-1}\int_0^T\int_\Omega \psi_\tau'\rho^\gamma dxdt.
\tag{5.129}
$$

For $M_{31}$, we have

$$
\begin{aligned}
M_{31} =& a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\left(\nabla(\rho^\gamma)^{,l}-\nabla\rho^\gamma\right)dxdt\\
\le& C\|\mathbf{u}\|_{L^2((0,T)\times\Omega)}\|\nabla(\rho^\gamma)^{,l}-\nabla\rho^\gamma\|_{L^2((0,T)\times\Omega)}\\
&\to 0,\ \text{as } l\to 0^+,
\end{aligned}
\tag{5.130}
$$

and for $M_{32}$, we have

$$
\begin{aligned}
M_{32} =& a\int_0^T\int_\Omega \psi_\tau\theta_{h,l}(\mathbf{u}^{,l}-\mathbf{u})\cdot\nabla\rho^\gamma dxdt\\
\le& C\|\mathbf{u}^{,l}-\mathbf{u}\|_{L^2((0,T)\times\Omega)}\|\nabla\rho^\gamma\|_{L^2((0,T)\times\Omega)}\\
&\to 0,\ \text{as } l\to 0^+.
\end{aligned}
\tag{5.131}
$$

Thus, substituting (5.129)-(5.131) into (5.126), we obtain (5.125). □

**Lemma 5.2.** *For fixed $\tau > 0$, for the nonlinear terms $M_4$ and $N_1$, we have*

$$\begin{aligned}
&\lim_{h\to 0^+}\lim_{l\to 0^+}(M_4+N_1)\\
&\quad=\lim_{h\to 0^+}\lim_{l\to 0^+}\left(\frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla(|\mathbf{b}|^2)^{,l}dxdt+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot \mathrm{div}\,(\mathbf{b}\otimes\mathbf{u})^{,l}\,dxdt\right)=0,
\end{aligned}\tag{5.132}$$

*provided $\frac{2}{p}+\frac{3}{q}\leq 1$.*

*Proof.* We first decompose the terms $M_4$ and $N_1$ into two parts respectively

$$\begin{aligned}
M_4&=\frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla\left(|\mathbf{b}|^2\right)^{,l}dxdt\\
&=\frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla\left(\left(|\mathbf{b}|^2\right)^{,l}-|\mathbf{b}|^2\right)dxdt+\frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla|\mathbf{b}|^2dxdt\\
&:=M_{41}+M_{42},
\end{aligned}\tag{5.133}$$

and

$$\begin{aligned}
N_1&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\mathrm{div}\,(\mathbf{b}\otimes\mathbf{u})^{,l}\,dxdt\\
&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\left(\mathrm{div}\,(\mathbf{b}\otimes\mathbf{u})^{,l}-\mathrm{div}(\mathbf{b}\otimes\mathbf{u}^{,l})\right)dxdt+\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\mathrm{div}(\mathbf{b}\otimes\mathbf{u}^{,l})dxdt\\
&:=N_{11}+N_{12}.
\end{aligned}\tag{5.134}$$

For $N_{12}$, we have

$$\begin{aligned}
N_{12}&=\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\mathrm{div}(\mathbf{b}\otimes\mathbf{u}^{,l})dxdt\\
&=-\int_0^T\int_\Omega \psi_\tau\mathbf{u}^{,l}\cdot\nabla\theta_{h,l}\mathbf{b}^{,l}\cdot\mathbf{b}dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}(\mathbf{u}^{,l}\cdot\nabla)\mathbf{b}^{,l}\cdot\mathbf{b}dxdt\\
&=-\int_0^T\int_{\Omega_h} \psi_\tau\mathbf{u}^{,l}\cdot\nabla\theta_{h,l}\mathbf{b}^{,l}\cdot\mathbf{b}dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}(\mathbf{u}^{,l}\cdot\nabla)\left(\mathbf{b}^{,l}-\mathbf{b}\right)\cdot\mathbf{b}dxdt\\
&\quad-\frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla|\mathbf{b}|^2dxdt,
\end{aligned}$$

which, combined with (3.39) and (5.133), yields

$$\begin{aligned}
M_{42}+N_{12}=&-\int_0^T\int_{\Omega_h} \psi_\tau\mathbf{u}^{,l}\cdot\nabla\theta_{h,l}\mathbf{b}^{,l}\cdot\mathbf{b}dxdt-\int_0^T\int_\Omega \psi_\tau\theta_{h,l}(\mathbf{u}^{,l}\cdot\nabla)\left(\mathbf{b}^{,l}-\mathbf{b}\right)\cdot\mathbf{b}dxdt\\
\leq& C\||\mathbf{u}^{,l}||\nabla\theta_{h,l}|\|_{L^2((0,T)\times\Omega_h)}\|\mathbf{b}\|^2_{L^4((0,T)\times\Omega_h)}\\
&+C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla(\mathbf{b}^{,l}-\mathbf{b})\|_{L^2((0,T)\times\Omega)}\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\\
\leq& C\left\|\frac{\mathbf{u}^{,l}}{dist(x,\partial\Omega)}\right\|_{L^2((0,T)\times\Omega_h)}\|\mathbf{b}\|^2_{L^4((0,T)\times\Omega_h)}\\
&+C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla(\mathbf{b}^{,l}-\mathbf{b})\|_{L^2((0,T)\times\Omega)}\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\\
&\to 0,\ \text{as } h\to 0^+,
\end{aligned}\tag{5.135}$$

provided $\frac{2}{p}+\frac{3}{q}\leq 1$.

For $M_{41}$, we have

$$\begin{aligned}
M_{41} =& \frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\nabla\left(\left(|\mathbf{b}|^2\right)^{,l}-|\mathbf{b}|^2\right)dxdt\\
=&\frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\left(\nabla\left(|\mathbf{b}|^2\right)^{,l}-\nabla(\mathbf{b}^{,l}\cdot\mathbf{b})\right)dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{u}^{,l}\cdot\left(\nabla\left(\mathbf{b}^{,l}\cdot\mathbf{b}\right)-\nabla|\mathbf{b}|^2\right)dxdt\\
\le& C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla\mathbf{b}\|_{L^2((0,T)\times\Omega)}\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\\
&+C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla\mathbf{b}\|_{L^2((0,T)\times\Omega)}\|\mathbf{b}^{,l}-\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\\
&+C\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\|\nabla(\mathbf{b}^{,l}-\mathbf{b})\|_{L^2((0,T)\times\Omega)}\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))},
\end{aligned}$$

which, combined with Lemma 4.1 and (3.39), gives

$$\lim_{l\to 0^+} M_{41}=0, \tag{5.136}$$

provided $\frac{2}{p}+\frac{3}{q}\le 1$.

Similarly, for $N_{11}$, we have

$$\begin{aligned}
N_{11} =& \int_0^T\int_\Omega \psi_\tau\theta_{h,l}\mathbf{b}^{,l}\cdot\left(\operatorname{div}\left(\mathbf{b}\otimes\mathbf{u}\right)^{,l}-\operatorname{div}(\mathbf{b}\otimes\mathbf{u}^{,l})\right)dxdt\\
\le& C\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\|\operatorname{div}(\mathbf{b}\otimes\mathbf{u})^{,l}-\operatorname{div}(\mathbf{b}\otimes\mathbf{u}^{,l})\|_{L^{\frac{2p}{p+2}}(0,T;L^{\frac{2q}{q+2}}(\Omega))}\\
\le& C\|\mathbf{b}\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\Omega))}\|\nabla\mathbf{b}\|_{L^2((0,T)\times\Omega)}\|\mathbf{u}\|_{L^p(0,T;L^q(\Omega))},
\end{aligned}$$

which implies

$$\lim_{l\to 0^+} N_{11}=0, \tag{5.137}$$

provided $\frac{2}{p}+\frac{3}{q}\le 1$.

Substituting (5.135)-(5.137) into (5.133) and (5.134), we obtain (5.132). □

Therefore, letting $l\to 0^+$ and $h\to 0^+$ successively in (5.123) and (5.124), combining Lemma 3.4, Lemma 3.6, Lemma 3.7, Lemma 4.2, Lemma 5.1 and Lemma 5.2 together, we have

$$\begin{aligned}
&-\int_0^T\int_\Omega \psi_\tau'\left(\frac{1}{2}\rho|\mathbf{u}|^2++\frac{a\rho^\gamma}{\gamma-1}+\frac{1}{2}|\mathbf{b}|^2\right)dxdt\\
&+\int_0^T\int_\Omega \psi_\tau\left(\mu|\nabla\mathbf{u}|^2+(\mu+\lambda)(\operatorname{div}\mathbf{u})^2+\nu|\nabla\mathbf{b}|^2\right)dxdt=0.
\end{aligned}\tag{5.138}$$

Denote

$$E_2(t):=\int_\Omega\left(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{a\rho^\gamma}{\gamma-1}+\frac{1}{2}|\mathbf{b}|^2\right)(t)dx, \tag{5.139}$$

and

$$D_2(t):=\int_0^t\int_\Omega\left(\mu|\nabla\mathbf{u}|^2+(\mu+\lambda)(\operatorname{div}\mathbf{u})^2+\nu|\nabla\mathbf{b}|^2\right)dxds. \tag{5.140}$$

Then (5.138) implies

$$(E_2+D_2)'=0 \text{ in } \mathcal{D}'((0,T)). \tag{5.141}$$

Now, as in Section 3 and 4, to prove the energy equality (5.121), it is enough to prove the right continuity of $\sqrt{\rho}\mathbf{u}$, $\mathbf{b}$ in $L^2$ and $\rho^\gamma$ in $L^1$.

First, by the continuity equation $(5.110)_1$, we obtain

$$\partial_t(\sqrt{\rho}) = -\mathrm{div}(\sqrt{\rho}\mathbf{u}) + \frac{1}{2}\sqrt{\rho}\mathrm{div}\mathbf{u}, \tag{5.142}$$

which, combined with (5.114) and (5.118), implies

$$\sqrt{\rho} \in L^\infty(0,T;H^1(\Omega)), \quad \partial_t(\sqrt{\rho}) \in L^2(0,T;H^{-1}(\Omega)). \tag{5.143}$$

Thanks to the Aubin-Lions Lemma (cf. [29]), we obtain

$$\sqrt{\rho} \in C([0,T];L^2(\Omega)). \tag{5.144}$$

For $\rho^\gamma$, we can use the continuity equation $(5.110)_1$ again to obtain

$$\partial_t(\rho^\gamma) = -\gamma\rho^\gamma\mathrm{div}\mathbf{u} - 2\gamma\rho^{\gamma-\frac{1}{2}}\nabla\sqrt{\rho}\cdot\mathbf{u}, \tag{5.145}$$

which, with the help of (5.114) and (5.118), yields

$$\rho^\gamma \in L^\infty(0,T;H^1(\Omega)), \quad \partial_t(\rho^\gamma) \in L^2(0,T;L^{\frac{3}{2}}(\Omega)). \tag{5.146}$$

Applying the Aubin-Lions Lemma (cf. [29]), we get

$$\rho^\gamma \in C([0,T];L^2(\Omega)). \tag{5.147}$$

Therefore, combining (4.105), (4.106), (5.144), (5.147) and the energy inequality (5.116), we deduce

$$\begin{aligned}
0 &\le \overline{\lim_{t\to 0^+}} \int_\Omega \left(|\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2 + |\mathbf{b}-\mathbf{b}_0|^2\right) dx \\
&= 2\overline{\lim_{t\to 0^+}} \left[\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{a\rho^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{b}|^2\right) dx - \int_\Omega \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{b}_0|^2\right) dx\right] \\
&\quad + 2\overline{\lim_{t\to 0^+}} \left(\int_\Omega [\mathbf{u}_0(\rho_0\mathbf{u}_0 - \rho\mathbf{u}) + \mathbf{u}_0\sqrt{\rho}\mathbf{u}(\sqrt{\rho}-\sqrt{\rho_0})]dx + \int_\Omega \mathbf{b}_0(\mathbf{b}_0-\mathbf{b})dx\right) \\
&\quad + 2\overline{\lim_{t\to 0^+}} \int_\Omega \left(\frac{a\rho_0^\gamma}{\gamma-1} - \frac{a\rho^\gamma}{\gamma-1}\right) dx \\
&\le 0,
\end{aligned} \tag{5.148}$$

where we used $\mathbf{u} \in L^4((0,T)\times\Omega)$, $\mathbf{u}_0 \in L^4(\Omega)$.

The rest of the proof is similar to that in the Section 4 and 5, thus we omit it here. □

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