On sign changes of Fourier coefficients of modular forms

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1. Introduction

Let f be a non-zero cusp form of weight k for a congruence subgroup of the full modular group $\Gamma_1 := SL_2(\mathbf{Z})$ with real Fourier coefficients $a(n)(n \geq 1)$. Then it is well-known [1] that the sequence $(a(n))_{n\geq 1}$ has infinitely many sign changes, i.e. there are infinitely many n with $a(n) > 0$ and infinitely many n with $a(n) < 0$.

On the other hand, let

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n$$

be the "normalized" Eisenstein series of even integral weight $k \geq 4$ for Γ_1. Here $q = e^{2\pi i z}$ for z in the complex upper half-plane $\mathcal{H}$. Furthermore, B_k is the k-th Bernoulli number and

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

Then by definition all Fourier coefficients of G_k with positive index are positive.

However, if one admits levels $N > 1$ then there exist linear combinations of Eisenstein series whose Fourier coefficients change signs infinitely often, for example the function

$$G_k(pz) - G_k(\ell z)$$

where p and ℓ are different primes, clearly has this property. Sign changes of Eisenstein series that are "newforms" were studied quite generally in [2].

The question arises if one can explicitly decompose spaces of modular forms according to a "sign change property" of their Fourier coefficients. In this paper, using certain linear combinations of Eisenstein series we will give a simple answer to this question if the level N is squarefree and the weight is at least 4. A similar result can be derived also in weight 2, the precise formulation however is slightly different due to the slightly different structure of the space of Eisenstein series in weight 2. We leave the exact formulation and a detailed work-out to the reader.

At the end of the paper we shall give a simple application to the growth of the number of representations of a positive integer n by a positive definite integral quadratic form of discriminant a square and of squarefree level.

2. Statement of result

For N a positive integer we denote by $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \,|\, N|c\}$ the Hecke congruence subgroup of level N. For k an even positive integer we let $M_k(N)$ be the space of modular forms of weight k for $\Gamma_0(N)$ and denote by $S_k(N)$ the subspace of cusp forms.

If N is squarefree and $k \geq 4$, then it is well-known that

$$M_k(N) = \mathcal{E}_k(N) \oplus S_k(N)$$

where the space of Eisenstein series is given by

$$\mathcal{E}_k(N) = \oplus_{t|N} \mathbf{C} G_k | V_t.$$

Here for any function $g : \mathcal{H} \to \mathbf{C}$ we have put

$$(g|V_t)(z) := g(tz) \qquad (z \in \mathcal{H}).$$

For $t|N, t > 1$ we set

$$G_{k,t} := G_k - t^{2k} G_k | V_t.$$

Clearly the functions $G_{k,t}(t|N, t > 1)$ are linearly independent over $\mathbf{C}$. We denote their span by

$$\tilde{\mathcal{E}}_k(N) := \oplus_{t|N, t>1} \mathbf{C} G_{k,t}. \tag{1}$$

Recall that for every $m \in \mathbf{N}$ with $(m, N) = 1$ there is a Hecke operator $T(m)$ on $M_k(N)$ leaving $S_k(N)$ stable. It acts on Fourier coefficients by

$$\Big(\sum_{n\geq 0} a(n) q^n\Big) | T(m) = \sum_{n \geq 0} \Big(\sum_{d|(m,n)} d^{k-1} a(\frac{mn}{d^2}) \Big) q^n.$$

The operators $T(m)((m, N) = 1)$ and $V_t(t|N)$ commute.

Theorem. *Suppose that N is squarefree and $k \geq 4$. Then there exists a codimension one subspace $\tilde{\mathcal{M}}_k(N) \subset M_k(N)$ stable under all Hecke operators $T(m)((m, N) = 1)$ such that*

$$M_k(N) = \mathbf{C} G_k \oplus \tilde{\mathcal{M}}_k(N) \tag{2}$$

and such that if $f \in \tilde{\mathcal{M}}_k(N) \setminus \{0\}$ has real Fourier coefficients $a(n)(n \geq 0)$, then the sequence $(a(n))_{n\geq 1}$ has infinitely many sign changes. Explicitly, one can take

$$\tilde{\mathcal{M}}_k(N) := \tilde{\mathcal{E}}_k(N) \oplus S_k(N) \tag{3}$$

where $\tilde{\mathcal{E}}_k(N)$ is defined by (1).

Remarks. i) As expected, a codimension one subspace $\tilde{\mathcal{M}}_k(N) \subset M_k(N)$ satisfying (2) and having "the sign change property" as described above of course is not uniquely determined. For example, in the definition of $\tilde{\mathcal{E}}_k(N)$ one can modify the functions $G_{k,t}$ by replacing the power t^{2k} by t^A where A is any constant larger than $k-1$. Then the resulting subspace still has the required properties as easily follows from the arguments given in the next section.

ii) It is natural to ask if a codimension one subspace as in i) also is invaraint under the action of the Fricke involution W_N or more generally the action of the Atkin-Lehner involutions $W_t(t|N)$ (defined in the usual way). Here the answer seems to be negative in general. For example, in the simplest case where $N = p$ is a prime, it is easy to see that the image of $G_k - p^A G_k|V_p$ under W_p up to a non-zero scalar equals $G_k - p^{k-A} G_k|V_p$, and the latter function never changes sign under our assumption $A > k-1$.

iii) Let us recall again that $M_k(1) = \mathbf{C}G_k \oplus S_k(1)$ and so in particular any $f \in M_k(1)$ up to the addition of a constant multiple of the Eisenstein series G_k on Γ_1 is either zero or has "the sign change property". Equation (2) then says that exactly the same assertion holds if $N = 1$ is replaced by an arbitrary squarefree level $N > 1$.

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2. Proof

We define $\tilde{M}_k(N)$ by (3). Property (2) and the invaraince under Hecke operators are then clear. So we only need to show the sign change property of any non-zero $f \in \tilde{M}_k(N)$. If $N = 1$ then $\tilde{\mathcal{E}}_k(N) = \{0\}$ and the assertion follows from [1]. We will therefore suppose that $N > 1$.

We start with a general remark. Let $f = \sum_{n \geq 0} a(n) q^n \in \tilde{\mathcal{E}}_k(N)$. The n-th Fourier coefficient ($n \geq 1$) of $G_{k,t}$ by definition equals

$$\sigma_{k-1}(n) - t^{2k} \sigma_{k-1}(\frac{n}{t})$$

where we have used the convention $\sigma_{k-1}(x) = 0$ if x is not a positive integer. Write $n = n_0 n_1$ with $n_0 | N^\infty$ (i.e. n_0 has only prime factors dividing N) and $(n_0, n_1) = 1$. Then we see that

$$a(n) = \sigma_{k-1}(n_1) a(n_0),$$

by the multiplicativity of the divisor function. In particular it holds that $a(n) \geq 0$ for all $n \geq 1$ if and only $a(n_0) \geq 0$ for all $n_0 \geq 1$ with $n_0 | N^\infty$. The above (and similar) reasonings will tacitly be used several times in similar situations below.

Suppose that $f \in \tilde{\mathcal{M}}_k(N)$ has real Fourier coefficients and write $f = g + h$ with $g \in \tilde{\mathcal{E}}_k(N)$ and $h \in S_k(N)$. Since complex conjugation acts on $M_k(N)$ through its action on Fourier coefficients we see that h has real Fourier coefficients and the coefficients of g w.r.t. the basis given in (1) are real.

We claim that if the coefficients $b(n)(n \geq 1)$ of g have at least one sign change, then the Fourier coefficients of f must have infinitely many sign changes. Indeed, suppose that $b(n_1) > 0$ and $b(n_2) < 0$ for some $n_1, n_2 \geq 1$ and denote by $c(n)(n \geq 1)$ the Fourier coefficients of h. Since h is a cusp form, we have $c(n) = \mathcal{O}_h(n^{k/2})$ by the classical Hecke estimate. Choose a prime p not dividing $n_1 n_2 N$. It then follows that

$$b(p^r n_1) = \sigma_{k-1}(p^r) b(n_1), b(p^r n_2) = \sigma_{k-1}(p^r) b(n_2) \qquad (r \geq 1).$$

From this we deduce that

$$b(p^r n_1) + c(p^r n_1) > 0, b(p^r n_2) + c(p^r n_2) < 0$$

whenever r is large, and our claim is proved.

Let $f \in \tilde{M}_k(N), f \neq 0$ with real Fourier coefficients $a(n)$. We will now prove the sign change property for f. Since the Fourier coefficients of non-zero cusp forms change signs infinitely often (loc. cit.), by the above proved claim we may assume that $f \in \tilde{\mathcal{E}}_k(N)$ and then only have to prove that the sequence $(a(n))_{n \geq 1}$ has at least one sign change. The arguments will be elementary, though a bit involved.

If $N = p$ is a prime, then f is a constant multiple of $G_{k,p}$, and the first Fourier coefficient of the latter equals 1, while its p-th coefficient equals $\sigma_{k-1}(p) - p^{2k} = 1 + p^{k-1} - p^{2k} < 0$.

By induction, now suppose that $N = p_1 M$ with p_1 the smallest prime divisor of N and $M > 1$. We will argue by contradiction and for this may suppose that $a(n) \geq 0$ for all $n \geq 1$.

It will be convenient to change our basis $G_{k,t}(t|N, t > 1)$ slightly. Namely, for each $t|N, t > 1$ we set

$$E_{k,t}(z) := G_k(\frac{t}{p_*} z) - p_*^{2k} G_k(tz)$$

where p_* is the largest prime factor of t. Then for $t = p$ a prime we have

$$E_{k,p}(z) = G_{k,p}(z) \tag{4}$$

and for t having at least two prime factors we have

$$(t/p_*)^{2k} E_{k,t}(z) = G_{k,t}(z) - G_{k,t/p_*}(z). \tag{5}$$

From (4) and (5) we easily find by induction that the functions $E_{k,t}(t|N, t > 1)$ indeed are a basis of $\tilde{\mathcal{E}}_k(N)$.

Let us write

$$f = \sum_{t|N, t>1} c_t E_{k,t} \qquad (c_t \in \mathbf{R}).$$

Then we can split up

$$f = f_1 + f_2|V_{p_1} + c_{p_1} E_{k,p_1}, \tag{6}$$

where

$$f_1 := \sum_{t|N, t>1, t\not\equiv 0 \pmod{p_1}} c_t E_{k,t}$$

and

$$f_2 := \sum_{t|N, t>1, p_1|t, p_1<t} c_t E_{k,t/p_1}.$$

Regarding f_2, we note that if $t|N$ and $p_1|t, p_1 < t$, then p_1 must divide t/p_*. Indeed, p_1 is the smallest prime divisor of N (and hence of t).

We note that f_1 and f_2 are in $\tilde{\mathcal{E}}_k(M)$. We write

$$f_1 = \sum_{n\geq 0} a_1(n)q^n,\ f_2 = \sum_{n\geq 0} a_2(n)q^n.$$

Comparing the n_0-th and the $p_1 n_0$-th Fourier coefficients in (6), respectively, where $n_0|M^\infty$, we find

$$0 \leq a(n_0) = a_1(n_0) + c_{p_1}\sigma_{k-1}(n_0) \tag{7}$$

and

$$\begin{aligned} 0 \leq a(p_1 n_0) &= a_1(p_1 n_0) + a_2(n_0) + c_{p_1}\big(\sigma_{k-1}(p_1 n_0) - p_1^{2k}\sigma_{k-1}(n_0)\big) \\ &= \sigma_{k-1}(p_1)a_1(n_0) + a_2(n_0) + c_{p_1}\sigma_{k-1}(n_0)\big(\sigma_{k-1}(p_1) - p_1^{2k}\big). \end{aligned} \tag{8}$$

In an intermediate step using (7) we will now show that $c_{p_1} \geq 0$. If $f_1 = 0$, then taking $n_0 = 1$ in (7) our assertion is obvious. Suppose that $f_1 \neq 0$. Since f_1 has level M, by induction there exists $m \geq 1$ with $a_1(m) < 0$. Writing $m = m_0 m_1$ with $m_0|M^\infty$ and $(m_0, m_1) = 1$ we see that $a_1(m_0) < 0$ and we also have that p_1 does not divide m_0. Taking now $n_0 = m_0$ in (7) we again obtain $c_{p_1} \geq 0$.

We will now finish the proof. Since $\sigma_{k-1}(p_1) - p_1^{2k} < 0$ and $c_{p_1} \geq 0$ we find from (8) that

$$\sigma_{k-1}(p_1)a_1(n_0) + a_2(n_0) \geq 0$$

for all n_0 with $n_0|M^\infty$. It follows that

$$\sigma_{k-1}(p_1)a_1(n) + a_2(n) \geq 0$$

for all $n \geq 1$, in other words the modular form

$$F_1 := \sigma_{k-1}(p_1)f_1 + f_2$$

of level M has non-negative n-th Fourier coefficients for all $n \geq 1$. By the induction hypothesis we deduce therefore that

$$F_1 = 0. \tag{9}$$

On the other hand, we can multiply (7) with $p_1^{2k} - \sigma_{k-1}(p_1) > 0$ and then add up (7) and (8) to obtain

$$p_1^{2k} a_1(n_0) + a_2(n_0) \geq 0,$$

hence

$$p_1^{2k} a_1(n) + a_2(n) \geq 0$$

for all $n \geq 1$. Thus

$$F_2 := p_1^{2k} f_1 + f_2 = 0. \tag{10}$$

Now (9) and (10) imply that

$$f_1 = f_2 = 0,$$

hence by (6)

$$f = c_{p_1} G_{k,p_1}.$$

This gives a contradiction as already pointed out at the beginning. Therefore the proof of the Theorem is finished.

4. An application

Let $Q(x)(x \in \mathbf{R}^{2k})$ be a positive definite integral quadratic form of rank $2k$ with $k \geq 4$, discriminant a square and squarefree level N. Thus if we write

$$Q(x) = \frac{1}{2} x' A x$$

with A an even integral matrix of rank $2k$, then $\det A$ is a perfect square and N is the least positive integer such that NA^{-1} again is even integral.

We let

$$\begin{aligned} \theta_Q(z) &= \sum_{x \in \mathbf{Z}^{2k}} e^{2\pi i Q(x) z} \qquad (z \in \mathcal{H}) \\ &= 1 + \sum_{n \geq 1} r_Q(n) q^n \end{aligned}$$

be the associated theta series, where $r_Q(n)$ is the number of representations of n by Q. Then $\theta_Q \in M_k(N)$ as is well-known. Since $r_Q(n) \geq 0$ for all $n \geq 1$ we obtain immediately from the Theorem

Corollary. Suppose that $N > 1$. Then there exists a positive constant c_Q depending only on Q such that there exist infinitely many $n \geq 1$ with $\frac{r_Q(n)}{\sigma_{k-1}(n)} < c_Q$ and infinitely many $n \geq 1$ with $\frac{r_Q(n)}{\sigma_{k-1}(n)} > c_Q$.

Remark. It is quite possible that the Corollary can also be proved directly in a different way, still using modular forms.

References

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