FORMATION, STABILITY AND BASIN OF PHASE-LOCKING FOR KURAMOTO OSCILLATORS BIDIRECTIONALLY COUPLED IN A RING

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Abstract. We consider the dynamics of bidirectionally coupled identical Kuramoto oscillators in a ring, where each oscillator is influenced sinusoidally by two neighboring oscillator. Our purpose is to understand its dynamics in the following aspects: 1. identify all the phase-locked states (or equilibria) with stability or instability; 2. estimate the basins for stable phase-locked states; 3. identify the convergence rate towards phase-locked states. The crucial tool in this work is the celebrated theory of Łojasiewicz inequality.

1. Introduction

Collective behaviors in coupled nonlinear oscillators have attracted numerous attentions owing to its significance in both dynamical theory and various applications. Among coupled nonlinear systems, we concern the sinusoidally coupled oscillators which was pioneered by Kuramoto [12], and recently it has been a hot topic in many scientific disciplines such as neuroscience, nonlinear dynamics, statistical physics, engineering and network theory [1, 18]. In this work, we study dynamical behavior of a finite group of Kuramoto oscillators bidirectionally coupled in a ring by performing nonlinear stability analysis.

The classic Kuramoto model was set as a population of sinusoidally coupled oscillators with all-to-all coupling. A lot of studies have been done for this model, see [4, 5, 6, 7, 9] for example, and a nice property that is useful in analysis is the mean-filed property. For identical Kuramoto oscillators with all-to-all coupling, it is well known that the phase synchronization (in short, sync) is the only stable phase-locked state, which denotes the collapse of all phases into a single phase, see [17]. Hence, almost all initial configurations of phases converge to the phase sync asymptotically. It is reasonable to guess that different asymptotic patterns for Kuramoto oscillators can emerge depending on different network topologies. For example, the literature [17] studies stability properties of the Kuramoto model with identical oscillators by linear stability analysis and the authors presented a six-node example to point out that a stable non-sync equilibrium arises for oscillators bidirectionally coupled in a ring. Recently, Wiley, Strogatz, and Girvan [20] addressed the problem of "the size of the sync basin" for the Kuramoto model with k-neighbor coupling

$$\dot{\theta}_i = \omega + \sum_{j=i-k}^{i+k} \sin(\theta_j - \theta_i), \qquad i = 1, 2, \dots, N,$$

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by setting $\theta_{N+j} := \theta_j$ and investigated the size of the sync basin. They found that when k/N is greater than a critical value, then the phase sync is the only stable phase-locked state; as k/N passes below this critical value, other stable phase-locked states are born which takes the form of twisted waves (or splay-state). They also numerically investigate how the size of basin of phase sync or splay-state depends on the winding number of the state. In [11, 19], the stability of phase-locking was considered for identical Kuramoto oscillators unidirectionally coupled in a ring, which takes the form of a system on an asymmetric network:

$$\dot{\theta}_i = \omega + \sin(\theta_{i+1} - \theta_i), \qquad i = 1, 2, \ldots, N.$$

In particular, in [19], Rogge and Aeyels used an extended Gershgorin disc theorem to derive the linear stability/instability of phase locking when the phase difference of neighboring oscillators is in either $(-\frac{\pi}{2}, \frac{\pi}{2})$ or $(\frac{\pi}{2}, \frac{3\pi}{2})$; however, when phase differences equal to $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, the system is not hyperbolic and zero eigenvalues arise so that the linearization approach does not work well. In [11], Ha and Kang performed nonlinear stability analysis and presented nontrivial proper subsets of synchronization and splay-state basins with positive Lebesgue measure in N-phase space. In this paper, we will concern the identical Kuramoto oscillators bidirectionally coupled in a ring, which is given by the following system

$$\dot{\theta}_i = \sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i), \qquad i = 1, 2, \ldots, N. \tag{1.1}$$

The main tool is the theory of Łojasiewicz inequality of analytic potential, by which we can determine the stability/instability of all possible phase locked states of (1.1), including those with phase differences equal to $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The main contributions of this paper are as follows. First, we will identify the formation and stability/instability of all phase-locked states for system (1.1) (see Theorems 3.1 and 3.2). This also enables us to determine the exact number of (asymptotically) stable phase-locked states (see Remark 3.2). Second, for the stable phase-locked states, we present their basins with positive Lebesgue measure in N-phase space (see Theorem 4.2). Third, we calculate the Łojasiewicz exponent for the stable equilibriums of (1.1) and clarify that the convergence towards the stable phase-locked states is exponentially fast (see Theorems 3.3 and 4.2).

Organization of paper.- In Section 2, we give some preliminaries. In Section 3, we identify the formation and stability/instability of all phase-locked states. The Łojasiewicz exponent of stable phase-locked states is also presented. In Section 4, we prove the convergence of (1.1) and give an estimate on the basin of stable phase-locked state. Section 5 is devoted to be a brief summary of this paper.

2. Preliminaries

We consider $N (N \geq 3)$ coupled oscillators. The dynamics of the i-th phase θ_i is governed by the following system:

$$\begin{aligned} &\dot{\theta}_i = \sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i), \\ &\theta_i(0) = \theta_{i0}, \qquad i = 1, 2, \ldots, N. \end{aligned} \tag{2.1}$$

where we set $\theta_{N+1} := \theta_1$. It is easy to see that $\sum_{i=1}^{N} \dot{\theta}_i = 0$, so the phase-locked solutions of the system (2.1) are its equilibrium points. Let

$$f(\theta) = \sum_{i=1}^{N} \big[1 - \cos(\theta_{i+1} - \theta_i)\big], \tag{2.2}$$

then system (2.1) can be written as a gradient system

$$\dot{\theta} = -\nabla f(\theta). \tag{2.3}$$

Next, we present some results for gradient systems with analytic potential. Consider the following system

$$\dot{x}(t) = -\nabla f(x(t)). \tag{2.4}$$

A crucial tool in this study is the nice theory for gradient inequality which was first developed by Łojasiewicz [16]. In his earlier work he proved the following result.

Proposition 2.1. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a real analytic function.*

(1) For any $x_* \in \mathbb{R}^N$, there exist a neighborhood $\mathcal{N}(x_*)$ of x_* and some constants $c > 0$ and $r \in (0, \frac{1}{2}]$ such that

$$|f(x) - f(x_*)|^{1-r} \leq c\|\nabla f(x)\|, \ \forall\ x \in \mathcal{N}(x_*). \tag{2.5}$$

(2) Let $x(\cdot)$ be a solution of (2.4). If $\{x(t)\}_{t=0}^{\infty}$ is bounded, then there exists an equilibrium x_∞ such that $x(t) \to x_\infty$.

The inequality (2.5) is referred as the celebrated Łojasiewicz's inequality and the constant $r \in \big(0, \frac{1}{2}\big]$ is called the Łojasiewicz exponent of f at x_0. We note that for a non-equilibrium, i.e., $\nabla f(x) \neq 0$, the inequality (2.5) certainly holds. However, when an equilibrium is concerned, the above inequality reveals a fundamental relation between the potential and its gradient near the equilibrium. Owing to this inequality, he could prove the second assertion in Proposition 2.1. Thus, the Łojasiewicz inequality provides a powerful tool to analyze the convergence of a trajectory towards a single equilibrium. To catch some idea from Łojasiewicz inequality to convergence, we refer to [3] for the so-called finite-length argument.

Furthermore, the value of Łojasiewicz exponent gives some information on the convergence rate; more precisely, the convergence is at least algebraically slow if $r \in \big(0, \frac{1}{2}\big)$, and exponentially fast if $r = \frac{1}{2}$ (see [3]). Using similar argument as in [3] with Łojasiewicz inequality, the following result can be obtained.

Proposition 2.2. [3, 14] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a real analytic function and $x(\cdot)$ be a bounded trajectory of (2.4) which converges to an equilibrium x_∞. Let $r_* \in (0, \frac{1}{2}]$ be the Łojasiewicz exponent of f at x_∞. Then we have:*

(1) If $r_* = \frac{1}{2}$, then there exist constants $C, T, \lambda > 0$ such that

$$\|x(t) - x_\infty\| \leq Ce^{-\lambda t}, \quad t \geq T.$$

(2) If $r_* < \frac{1}{2}$, then there exist constants $C, T > 0$ such that

$$\|x(t) - x_\infty\| \leq Ct^{-\frac{r_*}{1-2r_*}}, \quad t \geq T.$$

Based on Łojasiewicz inequality, Absil and Kurdyka [2] gave a sufficient and necessary condition for the stability of equilibrium of gradient system. We restate it here and this will be the main tool to identify the stability/instability for each equilibrium of (2.1).

Lemma 2.1. [2] *Let f be real analytic in a neighbourhood of $z_* \in \mathbb{R}^n$. Then, z_* is a stable equilibrium of* (2.4) *if and only if z_* is a local minimum of f. Furthermore, it is asymptotically stable if and only if it is a strict local minimum.*

Before we close this section, we present an inequality which will be useful in this paper. Let us consider a symmetric and connected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, 2, \dots, N\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ are vertex and edge sets, respectively. For a given network or graph $\mathcal{G}$, we assume that Kuramoto oscillators are located at the nodes of the network, and they interact symmetrically through the interacting channels registered by the interaction topology $\mathcal{E}$. We say $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is connected if for any two nodes i and j, there exists a path $i \to k_1 \to k_2 \to \cdots \to k_s \to j$ with $(i, k_1), (k_1, k_2), (k_2, k_3), \dots, (k_s, j) \in \mathcal{E}$. The distance between i and j is the number of arcs in a shortest path connecting i and j. We now state a lemma from [10] as follows.

Lemma 2.2. [10] *Suppose that the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is connected and the set $\{\gamma_i : i = 1, 2, \dots, N\}$ has zero mean:*

$$\sum_{i=1}^{N} \gamma_i = 0.$$

Then, we have

$$\frac{2N}{1 + \mathrm{diam}(\mathcal{G})|\mathcal{E}^c|} \sum_{i=1}^{N} |\gamma_i|^2 \le \sum_{(i,j)\in\mathcal{E}} |\gamma_i - \gamma_j|^2 \le 2N \sum_{i=1}^{N} |\gamma_i|^2.$$

where $\mathrm{diam}(\mathcal{G})$ *denotes the diameter of a graph which is the shortest distance between any pair of nodes in $\mathcal{G}$.*

3. Formation and stability of phase locking

Let

$$\phi_i = (\theta_{i+1} - \theta_i) \mod 2\pi, \qquad i = 1, 2, \dots, N. \tag{3.1}$$

Without loss of generality, we may set $\phi_i \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for all $i \in \{1, 2, \dots, N\}$. By (2.1), the variable $(\phi_1, \phi_2, \dots, \phi_N)$ satisfies the following equations:

$$\begin{aligned} &\dot{\phi}_i = \sin\phi_{i+1} - 2\sin\phi_i + \sin\phi_{i-1}, \\ &\phi_i(0) = \phi_{i0} := \theta_{i+1,0} - \theta_{i0}, \qquad i = 1, 2, \dots, N. \end{aligned} \tag{3.2}$$

Next we identify the equilibriums of (3.2) which also gives the formation of phase-locked states for system (2.1).

Theorem 3.1. *Every equilibrium $\phi = (\phi_1, \phi_2, \dots, \phi_N)$ to system* (3.2) *corresponds to a permutation of the vector*

$$(\underbrace{\alpha, \dots, \alpha}_{m}, \underbrace{\pi - \alpha, \dots, \pi - \alpha}_{N-m}), \tag{3.3}$$

where $\alpha \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$ and $m \in \{0, 1, \dots, N\}$ satisfy $m\alpha + (N - m)(\pi - \alpha) = 2\pi k$ for some $k \in \mathbb{Z}$.

Proof. Let $\phi = (\phi_1, \phi_2, \dots, \phi_N)$ be an equilibrium of (3.2), that is,

$$\sin\phi_{i+1} - 2\sin\phi_i + \sin\phi_{i-1} = 0, \qquad i = 1, 2, \dots, N.$$

Let $\phi_1 = \alpha, \phi_2 = \beta$, then we have

$$\begin{cases} \sin\phi_3 = 2\sin\beta - \sin\alpha \\ \sin\phi_4 = 3\sin\beta - 2\sin\alpha \\ \sin\phi_5 = 4\sin\beta - 3\sin\alpha \\ \cdots \\ \sin\phi_N = (N-1)\sin\beta - (N-2)\sin\alpha \\ \sin\phi_1 = N\sin\beta - (N-1)\sin\alpha \\ \sin\phi_2 = (N+1)\sin\beta - N\sin\alpha. \end{cases} \tag{3.4}$$

Using $\sin\phi_2 = \sin\beta, \sin\phi_1 = \sin\alpha$, we can get $\sin\beta = (N+1)\sin\beta - N\sin\alpha$. Thus, $\sin\alpha = \sin\beta$. We plug this relation to (3.4) to find that

$$\sin\phi_1 = \sin\phi_2 = \cdots = \sin\phi_N.$$

In view of the setting (3.1) and $\phi_i \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the equilibrium of equation (3.2) is a permutation of the following configuration

$$(\underbrace{\alpha, \dots, \alpha}_{m}, \underbrace{\pi-\alpha, \dots, \pi-\alpha}_{N-m}), \tag{3.5}$$

where $m \in \{0, 1, \dots, N\}$. As $(\theta_2 - \theta_1) + (\theta_3 - \theta_2) + \cdots + (\theta_N - \theta_{N-1}) + (\theta_{N+1} - \theta_N) = 0$, we have

$$m\alpha + (N-m)(\pi - \alpha) = 2\pi k$$

for some $k \in \mathbb{Z}$ with $-\frac{N\pi}{2} < 2\pi l \le \frac{3N\pi}{2}$. □

Next we prove that the equilibrium of the form

$$\phi_\alpha = (\underbrace{\alpha, \alpha, \dots, \alpha}_{N}), \qquad \text{with } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \; N\alpha = 2\pi k, \; k \in \mathbb{Z}. \tag{3.6}$$

is the only stable equilibrium for (3.2) or equivalently (2.1).

Theorem 3.2. *The equilibrium in* (3.6) *is the only stable equilibrium of* (3.2)*. Moreover, The equilibrium in* (3.6) *is asymptotically stable.*

Proof. First, we use Lemma 2.1 to show that the equilibrium of this type is the only stable equilibrium for (2.1). Let

$$f(\theta) = \sum_{i=1}^{N} \left[1 - \cos(\theta_{i+1} - \theta_i)\right], \tag{3.7}$$

then system (2.1) can be written as a gradient system

$$\dot\theta = -\nabla f(\theta). \tag{3.8}$$

Let $\theta^* = (\theta_1^*, \theta_2^*, \dots, \theta_N^*)$ be an equilibrium of system (2.1). By Theorem 3.1 it corresponds to an equilibrium $\phi^* = (\phi_1^*, \phi_2^*, \dots, \phi_N^*)$ of (3.2) which is given by a permutation of

$$(\underbrace{\alpha, \dots, \alpha}_{m}, \underbrace{\pi-\alpha, \dots, \pi-\alpha}_{N-m}). \tag{3.9}$$

We denote the index set $\mathcal{G}_1 = \{i : \phi_i^* = \alpha\}$ and $\mathcal{G}_2 = \{i : \phi_i^* = \pi - \alpha\}$ so that $\mathcal{G}_1 \cup \mathcal{G}_2 = \{1, 2, \ldots, N\}$. Let $\mathcal{N}(\theta^*)$ be a neighborhood of θ^*. For convenience we set

$$x_i = \theta_i - \theta_i^*, \qquad \gamma_i = x_{i+1} - x_i, \quad i = 1, 2, \ldots, N.$$

For any $\theta \in \mathcal{N}(\theta^*)$, we have $\sum_{i=1}^N \gamma_i = 0$ and

$$\begin{aligned}
&\cos(\theta_{i+1} - \theta_i) - \cos(\theta_{i+1}^* - \theta_i^*) \\
&= \cos(\theta_{i+1} - \theta_i - \theta_{i+1}^* + \theta_i^* + \theta_{i+1}^* - \theta_i^*) - \cos(\theta_{i+1}^* - \theta_i^*) \\
&= \cos(x_{i+1} - x_i)\cos(\theta_{i+1}^* - \theta_i^*) - \sin(x_{i+1} - x_i)\sin(\theta_{i+1}^* - \theta_i^*) - \cos(\theta_{i+1}^* - \theta_i^*).
\end{aligned}$$

Then we have

$$\begin{aligned}
&f(\theta^*) - f(\theta) \\
&= \sum_{i=1}^N \cos(\theta_{i+1} - \theta_i) - \sum_{i=1}^N \cos(\theta_{i+1}^* - \theta_i^*) \\
&= \sum_{i=1}^N \left[\cos(\theta_{i+1} - \theta_i) - \cos(\theta_{i+1}^* - \theta_i^*)\right] \\
&= \sum_{i=1}^N \left[\cos(x_{i+1} - x_i)\cos(\theta_{i+1}^* - \theta_i^*) - \sin(x_{i+1} - x_i)\sin(\theta_{i+1}^* - \theta_i^*) - \cos(\theta_{i+1}^* - \theta_i^*)\right] \\
&= \sum_{i \in \mathcal{G}_1} \left[\cos(x_{i+1} - x_i)\cos\alpha - \sin(x_{i+1} - x_i)\sin\alpha - \cos\alpha\right] \\
&\quad + \sum_{i \in \mathcal{G}_2} \left[(1 - \cos(x_{i+1} - x_i))\cos\alpha - \sin(x_{i+1} - x_i)\sin\alpha\right] \\
&= \cos\alpha \sum_{i \in \mathcal{G}_1} (\cos(x_{i+1} - x_i) - 1) - \sin\alpha \sum_{i \in \mathcal{G}_1} \sin(x_{i+1} - x_i) \\
&\quad + \cos\alpha \sum_{i \in \mathcal{G}_2} (1 - \cos(x_{i+1} - x_i)) - \sin\alpha \sum_{i \in \mathcal{G}_2} \sin(x_{i+1} - x_i) \\
&= \cos\alpha \left[\sum_{i \in \mathcal{G}_2} (1 - \cos(x_{i+1} - x_i)) - \sum_{i \in \mathcal{G}_1} (1 - \cos(x_{i+1} - x_i))\right] - \sin\alpha \sum_{i \in \mathcal{G}_1} \sin(x_{i+1} - x_i) \\
&= \frac{\cos\alpha}{2} \left[\sum_{i \in \mathcal{G}_2} \left((x_{i+1} - x_i)^2 + o((x_{i+1} - x_i)^4)\right) - \sum_{i \in \mathcal{G}_1} \left((x_{i+1} - x_i)^2 + o((x_{i+1} - x_i)^4)\right)\right] \\
&\quad - \sin\alpha \sum_{i=1}^N \left[(x_{i+1} - x_i) + o\left((x_{i+1} - x_i)^3\right)\right] \\
&= \frac{\cos\alpha}{2} \left[\sum_{i \in \mathcal{G}_2} \left(\gamma_i^2 + o\left(\gamma_i^4\right)\right) - \sum_{i \in \mathcal{G}_1} \left(\gamma_i^2 + o\left(\gamma_i^4\right)\right)\right] - \sin\alpha \left(\sum_{i=1}^N \gamma_i\right) + \sum_{i=1}^N o\left(\gamma_i^3\right) \\
&= \frac{\cos\alpha}{2} \left[\sum_{i \in \mathcal{G}_2} \left(\gamma_i^2 + o\left(\gamma_i^4\right)\right) - \sum_{i \in \mathcal{G}_1} \left(\gamma_i^2 + o\left(\gamma_i^4\right)\right)\right] - \sin\alpha \sum_{i=1}^N o\left(\gamma_i^3\right).
\end{aligned}$$

We now consider several cases depending on α and the configurations of $\mathcal{G}_1$ and $\mathcal{G}_2$.

(1) If $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathcal{G}_2 = \emptyset$, we can get that $f(\theta^*) \le f(\theta)$, i.e., θ^* is a local minimum of $f(\theta)$ in $\mathcal{N}(\theta^*)$. By Lemma 2.1, it is stable.

(2) If $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathcal{G}_1 = \emptyset$, then $f(\theta^*) \ge f(\theta)$ and $f(\theta^*)$ is a local maximum of $f(\theta)$ in $\mathcal{N}(\theta^*)$. Hence, it is unstable.

(3) If $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, this is equivalent to $\pi - \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. By cases (1) and (2), we see that the equilibrium is stable if $\mathcal{G}_1 = \emptyset$ and it is unstable if $\mathcal{G}_2 = \emptyset$.

(4) If $\alpha \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$ and both of $\mathcal{G}_1$ and $\mathcal{G}_2$ are not empty, we will show that θ^* is not a local minimum of f. Since $N \ge 3$ and both of $\mathcal{G}_1$ and $\mathcal{G}_2$ are not empty, we find at least one of $\mathcal{G}_1$ and $\mathcal{G}_2$ contains two nodes. without loss of generality we assume $1 \in \mathcal{G}_1$ and $2, 3 \in \mathcal{G}_2$. Let $\theta^1_\epsilon \in \mathcal{N}(\theta^*)$ be corresponding to

$$\phi^1_\epsilon = (\phi^*_1 + 2\epsilon, \phi^*_2 - \epsilon, \phi^*_3 - \epsilon, \phi^*_4, \phi^*_5, \dots, \phi^*_N), \quad \text{with} \quad 0 < |\epsilon| \ll 1.$$

This means $\gamma_1 = 2\epsilon$, $\gamma_2 = \gamma_3 = -\epsilon$ and $\gamma_4 = \cdots = \gamma_N = 0$. On the other hand, we let $\theta^2_\epsilon \in \mathcal{N}(\theta^*)$ be corresponding to

$$\phi^2_\epsilon = (\phi^*_1, \phi^*_2 + \epsilon, \phi^*_3 - \epsilon, \phi^*_4, \phi^*_5, \dots, \phi^*_N), \quad \text{with} \quad 0 < |\epsilon| \ll 1,$$

which means $\gamma_1 = 0$, $\gamma_2 = -\gamma_3 = \epsilon$ and $\gamma_4 = \cdots = \gamma_N = 0$. Note that these two choices make $f(\theta^*) - f(\theta^1)$ and $f(\theta^*) - f(\theta^2)$ produce different signs. So θ^* cannot be a local minimum of f and this equilibria is unstable.

(5) If $\alpha = \frac{\pi}{2}$, that is, θ^* is corresponding to $\phi^* = \left(\frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right)$. Let $\theta_\epsilon \in \mathcal{N}(\theta^*)$ be corresponding to

$$\phi_\epsilon = \left(\frac{\pi}{2} + 2\epsilon, \frac{\pi}{2} - \epsilon, \frac{\pi}{2} - \epsilon, \frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right), \quad \text{with} \quad 0 < |\epsilon| \ll 1.$$

We use (3.7) to find that

$$\begin{aligned} f(\theta_\epsilon) - f(\theta^*) &= \sum_{i=1}^N \cos\phi^*_i - \sum_{i=1}^N \cos\phi_{\epsilon,i} = -\cos\left(\frac{\pi}{2} + 2\epsilon\right) - 2\cos\left(\frac{\pi}{2} - \epsilon\right) \\ &= \sin 2\epsilon - 2\sin\epsilon = 2\sin\epsilon(\cos\epsilon - 1). \end{aligned}$$

The sign of $f(\theta_\epsilon) - f(\theta^*)$ depends on the sign of ϵ. This means that θ^* is not a minimum of f in $\mathcal{N}(\theta^*)$. Therefore, it is unstable.

(6) If $\alpha = \frac{3\pi}{2}$, we note that $\pi - \alpha = -\frac{\pi}{2} \sim \frac{3\pi}{2}$, so θ^* is corresponding to $\phi^* = \left(\frac{3\pi}{2}, \frac{3\pi}{2}, \dots, \frac{3\pi}{2}\right)$. In this case, we consider $\theta_\epsilon \in \mathcal{N}(\theta^*)$ corresponding to

$$\phi_\epsilon = \left(\frac{3\pi}{2} + 2\epsilon, \frac{3\pi}{2} - \epsilon, \frac{3\pi}{2} - \epsilon, \frac{3\pi}{2}, \frac{3\pi}{2}, \dots, \frac{3\pi}{2}\right), \quad \text{with} \quad 0 < |\epsilon| \ll 1.$$

Similar to Case (5), we find the equilibrium is unstable.

We now summarize the above cases (1)-(6) to conclude that the only stable equilibrium is the state $\phi = (\alpha, \alpha, \dots, \alpha)$ with $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Furthermore, we can easily see from Case (1) that θ^* is a strict local minimum. Therefore, by Lemma 2.1 we conclude that it is asymptotically stable.

□

Remark 3.1. *A classic method for stability/instability analysis is based on the linearization and eigenvalues. In* [19]*, Rogge and Aeyels applied Gershgorin disc theorem to perform the linear stability analysis for phase locking with phase differences in* $(-\frac{\pi}{2}, \frac{\pi}{2})$ *or* $(\frac{\pi}{2}, \frac{3\pi}{2})$*, when the oscillators are unidirectionally coupled in a ring. However, this approach based* *on linearization does not work when the phase difference is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ which produces zero eigenvalue (system (3.2) is not hyperbolic). In contrast, our analysis in Theorem 3.2 does not rely on any information on the algebraic spectrum of linearization, and we can determine the stability/instability for all possible phase locked states.*

Remark 3.2. *In [8], the authors considered the number of different, linearly stable phase-locked solutions for Kuramoto oscillators on a single-circle graph (unidirectionally coupled in a ring) and gave an upper bound $2[\frac{N}{4}]+1$. Here, $[\cdot]$ denotes the integer part of a given positive number. For Kuramoto model (2.1), by Theorem 3.2 we can see that the number of stable phase-locked states is exactly*

$$2\left[\frac{N-1}{4}\right]+1.$$

Actually, the number of the stable non-phase-sync equilibriums of (2.1) is exactly the number of integer k's with

$$2k\pi = N\alpha, \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \alpha \neq 0,$$

which is given by $2\left[\frac{N-1}{4}\right]$. Therefore, the total number of stable phase-locked states is given by $2\left[\frac{N-1}{4}\right]+1$.

Remark 3.3. *In the proof of Theorem 3.2, we can find that for the stable equilibrium given by $\phi_\alpha = (\alpha, \alpha, \dots, \alpha)$ with $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\theta \in \mathcal{N}(\theta^*)$, the following relation holds:*

$$|f(\theta^*) - f(\theta)| = \left| -\frac{\cos\alpha}{2} \sum_{i=1}^N \left(\gamma_i^2 + o(\gamma_i^4)\right) - \sin\alpha \sum_{i=1}^N o\left(\gamma_i^3\right) \right| \le C \sum_{i=1}^N \gamma_i^2,$$

where C is a positive constant.

Next,we will clarify that the Łojasiewicz exponent of the equilibrium $(\alpha, \alpha, \dots, \alpha)$ with $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is exactly $\frac{1}{2}$.

Theorem 3.3. *The Łojasiewicz exponent of f at equilibrium θ^* corresponding to $\phi_\alpha = (\alpha, \alpha, \dots, \alpha)$ with $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $\frac{1}{2}$.*

Proof. For $\theta \in \mathcal{N}(\theta^*)$, we have

$$\cos\frac{\theta_{i+1} - \theta_i + \theta^*_{i+1} - \theta^*_i}{2} \approx \cos\alpha > 0, \qquad \cos\frac{\theta_{i-1} - \theta_i + \theta^*_{i-1} - \theta^*_i}{2} \approx \cos\alpha > 0.$$

We now estimate the gradient of f at $\theta \in \mathcal{N}(\theta^*)$ as follows:

$$\begin{aligned}
&\|\nabla f(\theta)\|_2^2 \\
&= \sum_{i=1}^N \left[\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)\right]^2 \\
&= \sum_{i=1}^N \left[\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i) - \sin(\theta^*_{i+1} - \theta^*_i) - \sin(\theta^*_{i-1} - \theta^*_i)\right]^2 \\
&= \sum_{i=1}^N \Big[2\cos\frac{\theta_{i+1} - \theta_i + \theta^*_{i+1} - \theta^*_i}{2} \sin\frac{\theta_{i+1} - \theta_i - \theta^*_{i+1} + \theta^*_i}{2} \\
&\qquad + 2\cos\frac{\theta_{i-1} - \theta_i + \theta^*_{i-1} - \theta^*_i}{2} \sin\frac{\theta_{i-1} - \theta_i - \theta^*_{i-1} + \theta^*_i}{2}\Big]^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \left[2\cos\frac{\theta_{i+1}-\theta_i+\theta^*_{i+1}-\theta^*_i}{2} \sin\frac{x_{i+1}-x_i}{2} + 2\cos\frac{\theta_{i-1}-\theta_i+\theta^*_{i-1}-\theta^*_i}{2} \sin\frac{x_{i-1}-x_i}{2} \right]^2 \\
&= \sum_{i=1}^{N} \left[2\cos\frac{\theta_{i+1}-\theta_i+\theta^*_{i+1}-\theta^*_i}{2} \sin\frac{\gamma_i}{2} - 2\cos\frac{\theta_{i-1}-\theta_i+\theta^*_{i-1}-\theta^*_i}{2} \sin\frac{\gamma_{i-1}}{2} \right]^2 \\
&= \sum_{i=1}^{N} \left[2\cos\frac{\theta_{i+1}-\theta_i+\theta^*_{i+1}-\theta^*_i}{2} \left(\frac{\gamma_i}{2} + o(\gamma_i^3)\right) - 2\cos\frac{\theta_{i-1}-\theta_i+\theta^*_{i-1}-\theta^*_i}{2} \left(\frac{\gamma_{i-1}}{2} + o(\gamma_{i-1}^3)\right) \right]^2 \\
&\geq C_1 \sum_{i=1}^{N} \left[\gamma_i - \gamma_{i-1} + o(\gamma_i^3) - o(\gamma_{i-1}^3)\right]^2 \\
&\geq C_2 \sum_{i=1}^{N} \gamma_i^2 .
\end{aligned}
$$

Here, C_1 and C_2 denote some positive constants, and we used Lemma 2.2 for the last inequality. Then we combine the above relation and Remark 3.3 to find that

$$
|f(\theta) - f(\theta^*)|^{\frac{1}{2}} \leq \bar{C}\|\nabla f(\theta)\|, \qquad \theta \in \mathcal{N}(\theta^*),
$$

for some constant $\bar{C} > 0$. □

Remark 3.4. *Theorem 3.3 gives the first result for Łojasiewicz exponent of Kuramoto model when the phases of equilibrium are distributed a arc which is larger than a quarter of circle. Actually, in* [14, 15] *it was proved that the exponent is* $\frac{1}{2}$ *when the phases are distributed inside a quarter of circle. On the other hand, an example in* [14] *also shows that the exponent can be less that* $\frac{1}{2}$ *when they are distributed on the boundary of the quarter of circle, which implies the algebraic convergence shown in* [11]*. In this paper, we show that for Kuramoto model* (2.1) *the exponent is still* $\frac{1}{2}$ *which enables us to conclude the exponential rate by Proposition 2.2 once we can show the convergence.*

4. Convergence and a basin for stable phase locking

In this section, we will concern the global dynamics of system (2.1). We will show that any trajectory of (2.1) must converge and we present a non-trivial subset of the basin for the stable phase-locked state in (3.6).

4.1. Convergence.

A general form of the following theorem was presented in [13] for gradient system with analytic and periodic potential. For readers' convenience, we state it for system (2.1) and give a proof here.

Theorem 4.1. *For any initial data, the solution of system* (2.1) *converges to some equilibrium.*

Proof. First of all, system (2.1) is equivalent to the gradient form (2.3), where the potential f is real analytic and $2\pi-$periodic, i.e., $f(\theta+2\pi K) = f(\theta)$ for any $K \in \mathbb{Z}^n$. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^n$ be any solution of (2.1) or (2.3). By (2.3) we see that $f(\theta(t))$ is bounded and non-increasing in time, then it must converge and without loss of any generality let's say $\lim_{t\to\infty} f(\theta(t)) = 0$. We now choose $\hat{\theta}(t) \in [0, 2\pi)^n$ such that

$$
\hat{\theta}(t) = \theta(t) \mod 2\pi.
$$

Since $\hat{\theta}(\cdot)$ is bounded, there exists a sequence $\{t_k\}$ with $t_k \to \infty$ such that $\hat{\theta}(t_k) \to \hat{\theta}^*$ as $k \to \infty$ for some $\hat{\theta}^* \in [0, 2\pi]^n$. We denote the index sets

$$J_1 = \{j \, : \, \hat{\theta}^*_j = 0\}, \qquad J_2 = \{j \, : \, \hat{\theta}^*_j = 2\pi\}.$$

Obviously, we have

$$f(\theta(t_k)) = f(\hat{\theta}(t_k)) \to f(\hat{\theta}^*) = 0.$$

For this $\hat{\theta}^* \in [0, 2\pi]^n \subset \mathbb{R}^n$, by Proposition 2.1 (1) there exist $\sigma > 0$, $C > 0$ and $r \in \left(0, \frac{1}{2}\right]$ such that

$$|f(\theta) - f(\hat{\theta}^*)|^{1-r} \le C\|\nabla f(\theta)\|, \quad \forall\, \theta \in B_\sigma(\hat{\theta}^*), \tag{4.1}$$

where $B_\sigma(\hat{\theta}^*)$ denotes the ball in $\mathbb{R}^n$ centered at $\hat{\theta}^*$ with radius σ. We can choose σ sufficiently small with $\sigma < \min_{i\in\{1,2,\ldots,n\}\setminus(J_1\cup J_2)}\{\hat{\theta}^*_i, 2\pi - \hat{\theta}^*_i\}$. This implies that

$$\left.\begin{array}{l} \theta = (\theta_1, \theta_2, \ldots, \theta_n) \in B_\sigma(\hat{\theta}^*) \\ i \in \{1, 2, \ldots, n\} \setminus (J_1 \cup J_2) \end{array}\right\} \Rightarrow \theta_i \in (0, 2\pi). \tag{4.2}$$

According to the settings of $\hat{\theta}$ and $\hat{\theta}^*$, we can choose $t_N \in \{t_k\}_{k=1}^\infty$ such that

$$\|\hat{\theta}(t_N) - \hat{\theta}^*\| < \frac{\sigma}{3} \quad \text{and} \quad f^r(\hat{\theta}(t_N)) < \frac{r\sigma}{3C}. \tag{4.3}$$

Since $\theta(t_N) = \hat{\theta}(t_N) \mod 2\pi$, we can denote $\theta(t_N) = 2\pi K_0 + \hat{\theta}(t_N)$ where $K_0 \in \mathbb{Z}^n$. We then let $\theta^* = 2\pi K_0 + \hat{\theta}^*$, so by (4.3) we have $\|\theta(t_N) - \theta^*\| < \frac{\sigma}{3}$. Let

$$T = \inf\{t > t_N \, : \, \|\theta(s) - \theta^*\| < \sigma, \forall\, s \in (t_N, t)\}.$$

We claim that $T = \infty$. Otherwise, we have $T < \infty$ and

$$\|\theta(t) - \theta^*\| < \sigma, \quad \forall t \in [t_N, T), \quad \text{and} \quad \|\theta(T) - \theta^*\| = \sigma. \tag{4.4}$$

For $t \in [t_N, T)$, we define $\tilde{\theta}(t) = (\tilde{\theta}_1(t), \tilde{\theta}_2(t), \ldots, \tilde{\theta}_n(t)) \in [0, 2\pi]^n$ as follows:

$$\tilde{\theta}_i(t) = \begin{cases} \hat{\theta}_i(t), & i \in \{1, 2, \ldots, n\} \setminus (J_1 \cup J_2), \\ \hat{\theta}_i(t), & i \in J_1 \quad \text{and} \quad \theta_i(t) - \theta^*_i \ge 0, \\ \hat{\theta}_i(t) - 2\pi, & i \in J_1 \quad \text{and} \quad \theta_i(t) - \theta^*_i < 0, \\ \hat{\theta}_i(t) + 2\pi, & i \in J_2 \quad \text{and} \quad \theta_i(t) - \theta^*_i \ge 0, \\ \hat{\theta}_i(t), & i \in J_2 \quad \text{and} \quad \theta_i(t) - \theta^*_i < 0. \end{cases}$$

Then with (4.2) we can see that

$$\tilde{\theta}(t) = \hat{\theta}(t) \mod 2\pi, \quad \text{and} \quad \|\tilde{\theta}(t) - \hat{\theta}^*\| < \sigma, \quad \forall\, t \in [t_N, T). \tag{4.5}$$

Now for $t \in (t_N, T)$, we combine (2.3), (4.1) and (4.5) to derive

$$\begin{aligned} -\frac{d}{dt} f^r(\theta(t)) &= -r f^{r-1}(\theta(t)) \frac{d}{dt} f(\theta(t)) \\ &= r f^{r-1}(\tilde{\theta}(t)) \|\nabla f(\tilde{\theta}(t))\|^2 \\ &\ge \frac{r}{C} \|\nabla f(\tilde{\theta}(t))\| \\ &= \frac{r}{C} \|\nabla f(\theta(t))\|, \end{aligned}$$

where we used the 2π-periodicity of f and ∇f, and the relation $\theta(t) = \hat{\theta}(t) = \tilde{\theta}(t) \mod 2\pi$, to see that $f(\theta(t)) = f(\tilde{\theta}(t))$ and $\nabla f(\theta(t)) = \nabla f(\tilde{\theta}(t))$. By integrating the above relation we obtain

$$f^r(\theta(t_N)) - f^r(\theta(t)) \geq \frac{r}{C}\int_{t_N}^{t} \|\nabla f(\theta(s))ds\|, \quad \forall\, t \in (t_N, T).$$

This, together with (4.3), implies that

$$\int_{t_N}^{T} \|\nabla f(\theta(s))\|ds < \frac{\sigma}{3}.$$

Therefore, we have

$$\begin{aligned}\|\theta(T) - \theta^*\| &\leq \|\theta(T) - \theta(t_N)\| + \|\theta(t_N) - \theta^*\| \\ &\leq \int_{t_N}^{T} \|\dot{\theta}(s)\|ds + \|\theta(t_N) - \theta^*\| \\ &< \frac{2}{3}\sigma.\end{aligned}$$

This contradicts (4.4). So we conclude that $T = \infty$, that is, $\|\theta(t) - \theta^*\| < \sigma$ for all $t \in [t_N, \infty)$. This proves the boundedness of the trajectory $\theta(\cdot)$, and we recall Proposition 2.1 (2) to see it converges. By the choice of θ^*, we easily find that $\theta(t) \to \theta^*$. Furthermore, we insert $T = \infty$ to the above relations and obtain

$$\int_{t_N}^{\infty} \|\nabla f(\theta(s))\|ds < \frac{\sigma}{3} < \infty.$$

This implies that $\lim_{t\to\infty} \|\nabla f(\theta(t))\| = 0$. Recalling that $\hat{\theta}(t_k) \to \hat{\theta}^*$, so we have

$$\|\nabla f(\hat{\theta}^*)\| = \lim_{k\to\infty} \|\nabla f(\theta(t_k))\| = 0.$$

Hence, θ^* is an equilibrium. □

4.2. **Basin.** For a given configuration $\phi = (\phi_1, \phi_2, \dots, \phi_N)$,we introduce the indices

$$M \in \operatorname*{argmin}_i \phi_i \quad \text{and} \quad m \in \operatorname*{argmax}_i \phi_i,$$

then we have $\phi_M = \max_{1\leq i\leq N} \phi_i$ and $\phi_m := \min_{1\leq i\leq N} \phi_i$. Let $\phi(t)$ be the solution of (3.2). For time-varying configuration $\phi(t)$, the indices M and m depend on t and the extremal phase differences $\phi_M - \phi_m$ is Lipschitz continuous and piecewise differentiable. We will show that the set:

$$B_{con} = \left\{\phi \in \mathbb{R}^n : -\frac{\pi}{2} < \phi_m \leq \phi_M < \frac{\pi}{2}\right\}.$$

Lemma 4.1. *Let ϕ be the smooth solution to system* (3.2) *with initial condition $\phi_0 \in B_{con}$. Then ϕ_M is noninceasing and ϕ_m is nondecreasing for all $t \in [0, +\infty)$.*

Proof. • Step 1: Suppose for some $T \in (0, \infty]$ we have

$$-\frac{\pi}{2} < \phi_m(t) \leq \phi_M(t) < \frac{\pi}{2}, \quad t \in [0, T).$$

Then for $t \in [0, T)$, we have

$$\begin{aligned} -\frac{\pi}{2} < \frac{\phi_{M+1} - \phi_M}{2} \leq 0, \qquad 0 \leq \frac{\phi_{m+1} - \phi_m}{2} < \frac{\pi}{2}, \\ 0 \leq \frac{\phi_M - \phi_{M-1}}{2} < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < \frac{\phi_m - \phi_{m-1}}{2} \leq 0, \end{aligned}$$

and for any $1 \le i \le N$,

$$-\frac{\pi}{2} < \frac{\phi_{i+1} + \phi_i}{2} < \frac{\pi}{2}.$$

For $a.e.\, t \in [0, T)$ we can derive

$$\begin{aligned}
&\dot\phi_M(t) = (\sin\phi_{M+1} - \sin\phi_M) - (\sin\phi_M - \sin\phi_{M-1}) \\
&= 2\cos\left(\frac{\phi_{M+1}+\phi_M}{2}\right)\sin\left(\frac{\phi_{M+1}-\phi_M}{2}\right) - 2\cos\left(\frac{\phi_M+\phi_{M-1}}{2}\right)\sin\left(\frac{\phi_M-\phi_{M-1}}{2}\right) \le 0,
\end{aligned}$$

and

$$\begin{aligned}
&\dot\phi_m(t) = (\sin\phi_{m+1} - \sin\phi_m) - (\sin\phi_m - \sin\phi_{m-1}) \\
&= 2\cos\left(\frac{\phi_{m+1}+\phi_m}{2}\right)\sin\left(\frac{\phi_{m+1}-\phi_m}{2}\right) - 2\cos\left(\frac{\phi_m+\phi_{m-1}}{2}\right)\sin\left(\frac{\phi_m-\phi_{m-1}}{2}\right) \ge 0.
\end{aligned}$$

Therefore the continuous function ϕ_M is nonincreasing and ϕ_m is nondecreasing in the time interval $[0, T)$.

- Step 2: We define a set $\mathcal{T}$

$$\mathcal{T} := \left\{ T > 0 \,\middle|\, -\frac{\pi}{2} < \phi_m(t) \le \phi_M(t) < \frac{\pi}{2},\ \forall\, t \in [0, T) \right\}.$$

Since $-\frac{\pi}{2} < \phi_m(0) < \phi_M(0) < \frac{\pi}{2}$, by continuity there exists $\delta > 0$ such that

$$-\frac{\pi}{2} < \phi_m(t) \le \phi_M(t) < \frac{\pi}{2}, \quad t \in [0, \delta).$$

That is $\mathcal{T} \neq \emptyset$. Let $T_0 = \sup \mathcal{T}$. We next claim that $T_0 = \infty$. Suppose not, i.e., $T_0 < \infty$. Then by the definition of T_0, this yields

$$\text{either} \quad \liminf_{t \to T_0^-} \phi_M(t) = \frac{\pi}{2} \quad \text{or} \quad \limsup_{t \to T_0^-} \phi_m(t) = -\frac{\pi}{2}, \tag{4.6}$$

and

$$-\frac{\pi}{2} < \phi_m(t) \le \phi_M(t) < \frac{\pi}{2}, \quad \forall\, t \in [0, T_0).$$

By the analysis in Step 1 we get

$$\phi_m(0) \le \phi_m(t) \le \phi_M(t) \le \phi_M(0), \quad t \in [0, T_0).$$

This implies

$$\lim_{t \to T_0^-} \phi_M(t) \le \phi_M(0) < \frac{\pi}{2} \quad \text{and} \quad \lim_{t \to T_0^-} \phi_m(t) \ge \phi_m(0) > -\frac{\pi}{2},$$

which contradicts (4.6). This proves that $T_0 = \infty$ and we conclude that ϕ_M is nonincreasing and ϕ_m is nondecreasing for all $t \ge 0$. □

Theorem 4.2. *Let ϕ be the smooth solution to system (3.2) with initial condition $\phi_0 \in B_{con}$ which satisfies*

$$\sum_{i=1}^{N} \phi_{i0} = 2k\pi \quad \text{for some } k \text{ with} -\frac{N}{4} < k < \frac{N}{4}.$$

Then we have

$$\lim_{t \to \infty} \phi(t) = \frac{2k\pi}{N} \mathbb{1}_N.$$

Furthermore, the convergence is exponentially fast.

Proof. Step 1: We refine the estimate in Lemma 4.1 and claim that: (i) ϕ_m is strictly increasing; (ii) ϕ_M is strictly decreasing. We will prove (i) and ignore the proof for (ii) since they are similar. Note that

$$\dot{\phi}_m(t) = (\sin\phi_{m+1} - \sin\phi_m) - (\sin\phi_m - \sin\phi_{m-1}) \geq 0.$$

If ϕ_m is not strictly increasing, then we can find some open interval I such that

$$(\sin\phi_{m+1} - \sin\phi_m) - (\sin\phi_m - \sin\phi_{m-1}) = 0 \quad \text{on } I.$$

Then it follows from the graph of sinusoidal function on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that we should have

$$\text{either} \quad \phi_m = \phi_{m-1} = \phi_{m+1} \quad \text{or} \quad \phi_{m+1} < \phi_m < \phi_{m-1}.$$

The second case certainly contradicts the definition of ϕ_m. So we should have

$$\phi_m = \phi_{m-1} = \phi_{m+1} \quad \text{on } I.$$

This implies that for $t \in I$

$$0 = \dot{\phi}_m = \dot{\phi}_{m+1} = (\sin\phi_{m+2} - \sin\phi_{m+1}) - (\sin\phi_{m+1} - \sin\phi_m) = \sin\phi_{m+2} - \sin\phi_{m+1}.$$

Using similar argument we can obtain

$$\phi_l = \phi_m, \qquad \forall\, t \in I, \forall\, 1 \leq l, m \leq N,$$

which implies that ϕ is an equilibrium. However, this contradicts the initial condition which is not an equilibrium. Thus, ϕ_m is strictly increasing.

Step 2: Following the result in Step 1, we have

$$-\frac{\pi}{2} < \phi_m(0) < \phi_m(t) \leq \phi_M(t) < \phi_M(0) < \frac{\pi}{2}.$$

This implies that $\phi_m(t)$ and $\phi_M(t)$ converge as $t \to \infty$ and

$$-\frac{\pi}{2} < \phi_m(0) < \lim_{t\to\infty} \phi_m(t) \leq \lim_{t\to\infty} \phi_M(t) < \phi_M(0) < \frac{\pi}{2}. \tag{4.7}$$

By Theorem 4.1 and Theorem 3.1 we see that the solution of (2.1) or (3.2) converges to some phase-locking with ϕ_i being α or $\pi - \alpha$ for some $\alpha \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Therefore, we have

$$\lim_{t\to\infty} \phi_m(t), \lim_{t\to\infty} \phi_M(t) \in \{\alpha, \pi - \alpha\}.$$

Since

$$\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Leftrightarrow \pi - \alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

we invoke the relation (4.7) to find that for any $1 \leq i \leq N$,

$$\lim_{t\to\infty} \phi_i(t) = \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

On the other hand, we have a conservation law that

$$\sum_{i=1}^N \phi_i(t) = \sum_{i=1}^N \phi_i(0) = 2k\pi, \quad t > 0.$$

Thus, we have

$$N\alpha = \lim_{t\to\infty} \sum_{i=1}^N \phi_i(t) = 2k\pi, \quad \text{i.e.,} \quad \alpha = \frac{2k\pi}{N}.$$

That is, $\phi(t)$ converges to the splay-state $\frac{2k\pi}{N}\mathbb{1}_N$ as $t \to \infty$.

Finally, we use Theorem 3.3 and Proposition 2.2 to conclude the exponential rate (see Remark 3.4). □

Remark 4.1. *(1) If ϕ_0 satisfies the framework in Theorem 4.2 with $k \neq 0$, then Theorem 4.2 tells that the trajectory exponentially converges to a splay state.*

(2) If the initial configuration satisfies

$$\phi_i^0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \forall\, i \in 1, 2, \ldots, N, \quad \textit{and} \quad \sum_{i=1}^{N} \phi_i^0 = 0$$

then the constant k in Theorem 4.2 equals 0 . As a consequence, we get the following result

$$N\alpha = \lim_{t\to\infty} \sum_{i=1}^{N} \phi_i(t) = 0.$$

That is, $\alpha = 0$. In other words, the trajectory exponentially converges to a phase sync state.

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