COUNTING MULTIPLICITIES IN A HYPERSURFACE OVER FINITE FIELD

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Abstract. — In this paper, we consider a problem of counting multiplicities. We fix a counting function of multiplicity of rational points in a hypersurface of a projective space over a finite field, and we give an upper bound for the sum with respect to this counting function in terms of the degree of the hypersurface, the dimension and the cardinality of the finite field. This upper bound gives a description of the complexity of the singular locus of this hypersurface. In order to obtain this upper bound, we introduce a notion called intersection tree by intersection theory. We construct a sequence of intersections, such that the multiplicity of a singular rational point is equal to that of one of the irreducible components in these intersections. The multiplicities of these irreducible components constructed above are bounded by their multiplicities in the intersection tree.

***Résumé* (Comptage des multiplicités dans une hypersurface sur un corps fini)**

Dans cet article, on considère un problème de comptage de multiplicités. On fixe une fonction de comptage de multiplicités des points rationnels dans une hypersurface d'un espace projectif sur un corps fini, et on donne une majoration de la somme de cette fonction de comptage en terme du degré de l'hypersurface, de la dimension et du cardinal du corps fini. Cette majoration donne une description de la complexité du lieu singulier de cette hypersurface. Afin d'obtenir la majoration, on introduit une notion appelée arbre de l'intersection par la théorie de l'intersection. On construit une suite d'intersections, telle que la multiplicité d'un point rationnel singulier soit égale à celle d'une des composantes irréductibles dans les intersections. Les multiplicités des composantes irréductibles construites ci-dessus sont majorées par ses multiplicités dans l'arbre de l'intersection.

Keywords. arithmetic over finite fields, counting multiplicities, intersection tree, rational points.

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Contents

1. Introduction

In this paper, we consider the problem of counting multiplicities in a projective scheme over a finite field. Let X be a scheme of finite type over a finite field k, and we are interested in the counting problem of the form

$$\sum_{\xi\in X(k)} f(\mu_\xi(X)),$$

where $f(\cdot)$ is a polynomial and $\mu_\xi(X)$ is the multiplicity of the point ξ in X defined via the local Hilbert-Samuel function.

We fix a finite field $k=\mathbb{F}_q$, where q is a power of a prime number p (which is the characteristic of the field k). We consider the case where X is a closed subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$. There are many results on the lower and upper bounds of the number of the $\mathbb{F}_q$-points of X, which means that we take the counting function $f(\cdot)\equiv 1$ above. For this target, usually we can use analytic method or étale cohomology method.

If we take a non-trivial counting function, for example, we take $f(\mu_\xi(X))$ of the form $\mu_\xi(X)(\mu_\xi(X)-1)^t$, where t is a positive integer. In this case, these methods mentioned above seem to be difficult to use to solve them.

1.1. Known results. — Let X be a reduced projective plane curve. In this case, the singular locus of X is of dimension 0 if this curve is singular. Let δ be the degree of X, By Exercise 5-22 in Page 115 of [**7**], we have

$$\sum_{\xi\in X}\mu_\xi(X)\left(\mu_\xi(X)-1\right)\leqslant \delta(\delta-1), \tag{1}$$

which is obtained by Bézout Theorem in the intersection theory. More precisely, let g be the genus of the projective plane curve X. If X is geometrically integral, by Corollary 1 in Page 201 of [**7**], we have

$$g\leqslant\frac{(\delta-1)(\delta-2)}{2}-\sum_{\xi\in X}\frac{\mu_\xi(X)\left(\mu_\xi(X)-1\right)}{2}$$

by the Riemann-Roch Theorem over plane curves.

More generally, let $X\hookrightarrow\mathbb{P}^n_k$ be a projective hypersurface over an algebraically closed field k, whose singular locus is of dimension 0. By the method of Lefschetz pencils, a direct consequence of [**14**, Corollaire 4.2.1] gives

$$\sum_{\xi\in X}\mu_\xi(X)(\mu_\xi(X)-1)^{n-1}\leqslant\delta(\delta-1)^{n-1}.$$

But these conditions are too serious for a general problem of counting multiplicities. In particular, if the dimension of the singular locus is larger than or equal to 1, an upper bound of the left side must depend on the cardinality of k.

1.2. Principal result. — In this paper, we consider the problem of counting multiplicities in a scheme over a finite field. We take a counting function, and we will give an upper bound of this counting function taking the sum all over the rational points of a projective hypersurface. The principle result (Theorem 5.1) follows.

Theorem 1.1. — *Let X be a reduced hypersurface of degree δ in a projective space $\mathbb{P}^n_{\mathbb{F}_q}$, where $n \geqslant 2$ is a integer. Let s be the dimension of the singular locus of X. We have*

$$\sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1} \ll_n \delta^{n-s}\max\{\delta-1, q\}^s. \tag{2}$$

In Theorem 5.1, we clarify the implicit constant in the estimate (2). In Example 5.11, we will construct an example to show that the exponents of δ and $\max\{\delta-1,q\}$ in the estimate (2) are both optimal.

1.3. Motivation. — Let X be a pure dimensional reduced Noetherian scheme, since the regular locus X^{reg} is open dense in X, we have $\mathrm{codim}(X, X^{\mathrm{sing}}) \geqslant 1$, where X^{sing} denotes the singular locus of X.

Let X be a closed subscheme of $\mathbb{P}^n_k$. If we want to describe the complexity of the singular locus more precisely, it is not enough to only consider the dimension of X^{sing}. In fact, in order to describe the complexity of X^{sing}, we need to consider the dimension of X^{sing}, the degree of X^{sing} and the multiplicity of X^{sing} in X (or the multiplicities of singular points in X). Then we need to choose a convenable counting function of multiplicities $f(\cdot)$ such that $f(1)=0$.

By Theorem 1.1, when X is a reduced hypersurface of a projective space over a finite field, these three invariants mentioned above cannot be too large at the same time, which means the singular locus of X cannot be "too complicated". In Remark 5.12, we will explain why the counting function $\mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1}$ in the inequality (2) is a convenable choice. Then the inequality (2) is a convenable to describe the global complexity of the singular locus of X when q is large enough.

This work is motivated by the counting rational points problem in Diophantine geometry. More precisely, we consider a problem of counting rational points of bounded height in projective arithmetic varieties uniformly as in [**12**] for example. In [**24**, Theorem 3.2], in order to construct the auxiliary hypersurfaces by the determinant method, P. Salberger considers the multiplicities of a family of $\mathbb{F}_q$-rational points in $X \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{F}_p$ (this notion means the reduction of the Zariski closure of X in $\mathbb{P}^n_{\mathbb{Z}}$ at the prime p) in order to solve a conjecture of D. R. Heath-Brown [**12**, Conjecture 2]. In [**3, 4**], H. Chen generalized the determinant method of P. Salberger the Arakelov geometry. In order to apply the determinant method into the counting rational points problem, the inequality (2) will be useful to describe the density of rational points with large multiplicity. For example, in the proof of [**4**, Theorem 5.1] which is a conjecture of D. R. Heath-Brown in [**5**, Question 27], for a problem of counting rational points with bounded height of a projective plane curve, the inequality (1) is used to control the number of points with large multiplicities. If we want to generalize the determinant method to solve the same problem in the case of arithmetic hypersurfaces with higher dimension, the upper bound (2) above will be useful and important.

1.4. Principal tools. — Different from the classical methods, for example, the étale cohomology or the exponential sum, we will use the intersection theory to get a good control of multiplicities.

We consider a reduced hypersurface $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ whose singular locus is of dimension s. Let Y be an integral subscheme of X. Then there exists a sub dense set Y' of Y, such that for all point $\xi \in Y'$, we have $\mu_\xi(X) = \mu_Y(X)$. We find a family $\{X_i\}_{i=1}^{n-s-1}$ of hypersurfaces of $\mathbb{P}^n$ containing ξ such that $X, X_1, \ldots, X_{n-s-1}$ intersect properly and there exists an irreducible component Y of the intersection $X, X_1, \ldots, X_{n-s-1}$ containing ξ and satisfying $\mu_\xi(X) = \mu_Y(X)$. The construction of these hypersurface is involved by partial derivations (maybe higher orders) of the equation which defines X, and the construction can be done inductively. For this target, we introduce a notion called "intersection tree" by the language of graph theory, see §3.1 for the precise definition. An intersection tree is a labelled tree with weights generated by the intersections of X and its derivative hypersurfaces (see Definition 5.7), whose vertices are integral subschemes of X, the labels are derivative hypersurfaces, and the weights are intersections multiplicities corresponding to the vertex and its label.

Since X is a hypersurface, we can estimate the function $\mu_Y(X)(\mu_Y(X)-1)^{n-s-1}$ by the weights defined above. By Bézout Theorem (Theorem 2.2), the sum of weights can be bounded by the degree of X with respect to its universal bundle.

For a useful upper bound of the number of $\mathbb{F}_q$-rational points of a fixed irreducible component, we use the estimate in Proposition 2.17.

1.5. An analogue. — The taste of the method of intersection tree is very geometric, which does not use too much arithmetic information of the base field. Then we can consider an analogue over number fields. In [**30**], we consider a problem of counting multiplicities of rational or algebraic points with bounded height (naive height) in an arithmetic scheme. Let $X \hookrightarrow \mathbb{P}^n_{\mathbb{Q}}$ be a hypersurface of degree δ over the rational number field $\mathbb{Q}$, whose singular locus is of s. Let $S(X;B)$ be the set of the rational points of X with height less than or equal to B. In [**30**, Corollary 4.7], we prove

$$\sum_{\xi \in S(X;B)} \mu_\xi(X)\left(\mu_\xi(X)-1\right)^{n-s-1} \ll_n \delta^{n-s}\max\{\delta-1, B\}^{s+1},$$

where we clarify the implicit constant depending on n above in [**30**]. The exponents of δ and $\max\{\delta-1,B\}$ above are both optimal (see [**30**, Example 4.8]), since for any pure dimensional projective varieties X, we have $\#S(X;B) \ll_n \deg(X)\cdot B^{\dim(X)+1}$ (cf. [**30**, Theorem 3.3]), and the exponents of $\deg(X)$ and B are both optimal above. If we have enough information about the number of algebraic points with bounded height and degree, we can get a similar analogue.

1.6. Structure of the paper. — This paper is organized as following: in Section 2, we provide some useful notions on the local algebra and the intersection theory, and we will prove some useful results of the intersection theory and of counting objects over finite fields. They are preliminary results for the following work. In Section 3, we will introduce the definition of intersection tree in order to describe the series of intersections mentioned above. In Section 4, we will run a mathematical induction to prove a result which is an upper bound of the product of multiplicities by some weights in the intersection trees. In Section 5, we will construct some intersections in order to prove the inequality (2), and we will accomplish the proof.

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2. Arithmetic and geometric preliminaries

In this section, we will provide some preliminary results for this paper. Firstly, we introduce some fundamental notions on the local algebra and the intersection theory. Then we prove some useful results on algebraic geometry and counting objects over finite fields.

In all the paragraphs below, we recall some notions which we use in the definition of the intersection tree. Without specially mentioned, all rings are supposed to be commutative Noetherian rings with identity.

2.1. Length of a module. — Let A be a ring and M be a A-module. We call that M is of *finite length* if there exists a series of decreasing sub-module of M (called *composition series* of M)

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = \{0\}$$

such that every sub-quotient M_{i-1}/M_i is a simple A-module (i.e. isomorphic to a quotient module of A generated by a maximal ideal), where $i \in \{1, \ldots, n\}$. We can prove that the number n does not depend on the choice of the composition series. We call it the *length* of the module M, noted by $\ell_A(M)$, or by $\ell(M)$ for simplicity. The length of the zero-module est 0. We recall that, if A is an Artinian ring (i.e. a non-zero Noetherian ring whose every prime ideal is maximal), then each A-module of finite type is of finite length. We refer the readers to [**6**, §2.4] for more details.

2.2. Multiplicities of a module and a ring. — In this part, we recall some notions of multiplicities in the frame of commutative algebra.

Multiplicity of a module. — Let A be a ring whose dimension is larger than or equal to 1. Let d be an integer, $d \geqslant 1$, M be an A-module of finite type with $\dim_A(M) = d$, and $\mathfrak{a}$ be an ideal of A contained in the Jacobson radical of A such that the quotient ring $A/\mathfrak{a}$ is Artinian. For every non-negative integer m, let $H_{\mathfrak{a},M}(m) = \ell_{A/\mathfrak{a}}(\mathfrak{a}^m M/\mathfrak{a}^{m+1}M)$. There exists a polynomial $P_{\mathfrak{a},M}$ whose degree is less than or equal to $d-1$, such that $H_{\mathfrak{a},M}(m) = P_{\mathfrak{a},M}(m)$ for all m large enough. In the other words, there exists an integer $e_{\mathfrak{a},M} \geqslant 0$ such that

$$P_{\mathfrak{a},M}(m) = e_{\mathfrak{a},M}\frac{m^{d-1}}{(d-1)!} + o(m^{d-1}).$$

The integer $e_{\mathfrak{a},M}$ is called the *multiplicity* of M relative to the ideal $\mathfrak{a}$. When A is a local ring and $M \neq \{0\}$, we always have $e_{\mathfrak{a},M} > 0$ (cf. [**6**, Exercise 12.6]). If $M = A$, the number $e_{\mathfrak{a},A}$ is called the *multiplicity* of the ideal $\mathfrak{a}$ in A.

With the same notations as above, we consider the function $L_{\mathfrak{a},M}(m) = \ell_{A/\mathfrak{a}}(M/\mathfrak{a}^{m+1}M)$. There exists a polynomial $Q_{\mathfrak{a},M}$ whose degree is less than or equal to d, such that $Q_{\mathfrak{a},M}(m) = L_{\mathfrak{a},M}(m)$ for all m large enough. In addition, we have

$$Q_{\mathfrak{a},M}(m) = e_{\mathfrak{a},M}\frac{m^d}{d!} + o(m^d).$$

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of A contained in the Jacobson radical of A, such that $A/\mathfrak{a}$ and $A/\mathfrak{b}$ are Artinian. If $\mathfrak{a} \subseteq \mathfrak{b}$, by [**25**, Chap II, §3, a], we have $Q_{\mathfrak{a},M}(m) \geqslant Q_{\mathfrak{b},M}(m)$ when m is large enough. Then we have the inequality

$$e_{\mathfrak{a},M} \geqslant e_{\mathfrak{b},M}. \tag{3}$$

If A is a local ring, we can represent the multiplicity $e_{\mathfrak{a},M}$ as the local sum

$$e_{\mathfrak{a},M} = \sum_{\mathfrak{p}} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot e_{\mathfrak{a},A/\mathfrak{p}}, \tag{4}$$

where $\mathfrak{p}$ takes all over the set of minimal prime ideals of A such that $\dim(A) = \dim(A/\mathfrak{p})$ (see [**2**, Chap. VIII, §7, n° 1, Prop. 3] for a proof).

Multiplicity of a local ring. — Let A be a local ring, $\mathfrak{m}$ be its maximal ideal and $k = A/\mathfrak{m}$ be its residue field. The *multiplicity* of A is defined as the multiplicity of the maximal ideal $\mathfrak{m}$ in A. It is evident that we have $e_{\mathfrak{m},A} > 0$ (cf. [**6**, Exercise 12.6]).

We recall that the inequality $\dim(A) \leqslant \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is always verified (cf. [**16**, (12.J)]). If the equality $\dim(A) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ holds, we say that A ia a *regular local ring.* If A is a regular local ring, then $\bigoplus\limits_{i\geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is isomorphic to $\bigoplus\limits_{i\geqslant 0} \mathrm{Sym}^i_k(\mathfrak{m}/\mathfrak{m}^2)$ as graded k-algebras. In this case, the multiplicity of A is 1 (cf. [**16**, §14]). The inverse is not right in general: there exists a local ring of multiplicity 1 which is not regular (see Exercise 2.5 in Page 41 of [**23**] for a conter-example). This property is verified when $\operatorname{Spec} A$ is pure dimensional. We refer the readers to [**20**, (40.6)] for a proof.

2.3. Notions of the intersection theory. — In this part, we recall some notions of the classical intersection theory. The principle reference is [**26**] whose approach is equivalent to [**8**], see [**8**, Example 7.1.1] and Part e) in Page 84 of [**26**].

Multiplicity along a closed subscheme. — Let X be a local Noetherian scheme. If ξ is a point of X, we denote by $\mu_\xi(X)$ the multiplicity of the local ring $\mathcal{O}_{X,\xi}$. If Y is a closed integral subscheme of X whose generic point is η_Y, we denote by $\mathcal{O}_{X,Y}$ the local ring $\mathcal{O}_{X,\eta_Y}$ for simplicity, and we denote by $\mu_Y(X)$ the multiplicity of the local ring $\mathcal{O}_{X,Y}$.

Regular locus and singular locus. — Let X be a scheme. We denote by X^{reg} the set of the points of $\xi \in X$ such that $\mathcal{O}_{X,\xi}$ is a regular local ring, call the *regular locus* of X. If $X^{\mathrm{reg}} = X$, we say that X is a *regular scheme*. In the other side, let X^{sing} be the complementary $X \smallsetminus X^{\mathrm{reg}}$, called the *singular locus* of X. If X is of locally finite type over the spectre of a field, the set X^{reg} is Zariski open in X (cf. [**11**, Corollary 8.16, Chap. II]), and then the set of multiplicity 1 points is dense in X if X is irreducible and $X^{\mathrm{reg}} \neq \emptyset$.

Intersection multiplicity. — Let X be a Noetherian scheme. We call that X is *pure dimensional* if all the irreducible components of X have the same dimension.

Let k be a field. Let Y be a regular scheme of finite type over $\operatorname{Spec} k$ such that the canonical morphism $Y \to \operatorname{Spec} k$ is separated, and let $X_1, \ldots, X_r$ be pure dimensional closed subschemes of Y. We denote by $\Delta : Y \to Y^{\times_k r}$ the diagonal morphism. It is evident that the fiber product of $\Delta(Y)$ and $X_1 \times_k \cdots \times_k X_r$ sur $Y^{\times_k r}$ is isomorphic to the schematic intersection $\bigcap_{i=1}^r X_i$. Then we can consider $\bigcap_{i=1}^r X_i$ as a closed subscheme of $X_1 \times_k \cdots \times_k X_r$. Let $\mathcal{I}$ be ideal sheaf of $\mathcal{O}_{X_1 \times_k \cdots \times_k X_r}$ corresponding to $\bigcap_{i=1}^r X_i$.

Let M be an irreducible component of $\bigcap_{i=1}^r X_i$ considered to be a closed integral subscheme of Y. We denote by $\Delta(M)$ the integral closed subscheme of $X_1 \times_k \cdots \times_k X_r$ the image of M by the diagonal morphism (which is a closed immersion since Y is separated over $\operatorname{Spec} k$). Let η_M be the generic point $\Delta(M)$. The ideal $\mathcal{I}_{\eta_M}$ is called the *diagonal ideal* of the ring $\mathcal{O}_{X_1 \times_k \cdots \times_k X_r, \Delta(M)}$. We define the *intersection multiplicity* of $X_1, \ldots, X_r$ at M as the multiplicity of the ideal $\mathcal{I}_{\eta_M}$ in the local ring $\mathcal{O}_{X_1 \times_k \cdots \times_k X_r, \Delta(M)}$, noted by

$$i(M; X_1 \cdot \ldots \cdot X_r; Y).$$

If a closed integral subscheme N of Y is not a irreducible component of $X_1 \cap \cdots \cap X_r$, we define

$$i(N; X_1 \cdot \ldots \cdot X_r; Y) = 0$$

by convention. We refer the readers to Page 148 of [**29**] and Page 77 of [**26**] for more details about this definition (See Chapter 7 and Chapter 8 of [**8**] for another equivalent definition).

Proper components. — Let k be a field. Let Y be a regular separated k-scheme of finite type, and $X_1, \ldots, X_r$ be the closed pure dimensional subscheme of Y. We denote by $\mathcal{C}(X_1 \cdot \ldots \cdot X_r)$ the set of irreducible components of the schematic intersection $X_1 \cap \cdots \cap X_r$. In particular, if X is a closed pure dimensional subscheme of Y, then $\mathcal{C}(X)$ denotes the set of irreducible components of X. Without specially mentioned, all the irreducible component in $\mathcal{C}(X_1 \cdot \ldots \cdot X_r)$ or $\mathcal{C}(X)$ is considered as an integral closed subscheme of Y.

We recall that we have (cf. [**27**, Chap. III, Prop. 17])

$$\dim(M) \geqslant \dim(X_1) + \cdots + \dim(X_r) - (r-1)\dim(Y)$$

for every $M \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. We say that the schemes $X_1, \ldots, X_r$ *intersect properly* at M in Y, or equivalently that M is a *proper component* of the intersection $X_1 \cdot \ldots \cdot X_r$ in Y, if $M \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$ and the equality

$$\dim(M) = \dim(X_1) + \cdots + \dim(X_r) - (r-1)\dim(Y)$$

is verified. We say that $X_1, \ldots, X_r$ *intersect properly* if all element $M \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$ is a proper component of the intersection $X_1 \cdot \ldots \cdot X_r$ in Y.

2.4. Geometric preliminaries. — In this part, we will introduce some preliminary results of tastes of algebraic geometry.

Let Y be a regular separated k-scheme and L be an ample invertible $\mathcal{O}_Y$-module. If X is a closed subscheme of Y, we denote by $\deg_L(X)$ the degree of X with respect to the invertible $\mathcal{O}_Y$-module L, which is defined as $\deg\left(c_1(L)^{\dim(X)} \cap [X]\right)$.

Let k be a field, $n \geqslant 1$ be an integer, and E be a k-vector space of rank $n+1$. We define the projective space $\mathbb{P}(E)$ as the scheme which represents the functor from the category of commutative k-algebras to the category of sets, which sends each commutative k-algebra A to the set of quotient projective A-module of rank 1. In addition, we denote by $\mathbb{P}^n_k$ the projective space $\mathbb{P}(k^{n+1})$ for simplicity, or by $\mathbb{P}^n$ if there is no confusion over k. If L is the universal bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, the degree of X with respect to $\mathcal{O}_{\mathbb{P}(E)}(1)$ is noted by $\deg(X)$ for simplicity.

Commutativity and associativity of the intersections. — The intersection multiplicity satisfies the commutativity and the associativity by the following sense. We refer the readers to [**8**, Proposition 8.1.1] for a proof.

Theorem 2.1. — *Let X_1, X_2, X_3 be three separated regular pure dimensional closed subschemes of Y of finite type over* Spec k. *We have the following properties:*

- **(i). (commutativity) :** *for every $M \in \mathcal{C}(X_1 \cdot X_2) = \mathcal{C}(X_2 \cdot X_1)$, we have*

$$i(M; X_1 \cdot X_2; Y) = i(M; X_2 \cdot X_1; Y);$$

- **(ii). (associativity) :** *if X_1, X_2, X_3 intersect properly at $M \in \mathcal{C}(X_1 \cdot X_2 \cdot X_3)$, then we have:*

$$\begin{aligned} i(M; X_1 \cdot X_2 \cdot X_3; Y) &= \sum_{P \in \mathcal{C}(X_1 \cdot X_2)} i(M; P \cdot X_3; Y) \cdot i(P; X_1 \cdot X_2; Y) \\ &= \sum_{Q \in \mathcal{C}(X_2 \cdot X_3)} i(M; Q \cdot X_1; Y) \cdot i(Q; X_2 \cdot X_3; Y), \end{aligned}$$

see §2.3 for the notations of $\mathcal{C}(X_1 \cdot X_2 \cdot X_3)$, $\mathcal{C}(X_1 \cdot X_2)$ and $\mathcal{C}(X_2 \cdot X_3)$.

Bézout Theorem. — The Bézout in the frame of the intersection theory is a description of the complexity of a proper intersection in $\mathbb{P}(E)$ by the terms of degree with respect to the universal bundle.

***Theorem 2.2* (Bézout Theorem).** — *Let $X_1, \ldots, X_r$ be closed pure dimensional subschemes of $\mathbb{P}(E)$ which intersect properly. Then we have*

$$\sum_{Z \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Z; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) \deg(Z) = \deg(X_1) \cdots \deg(X_r).$$

We refer the readers to [**8**, Proposition 8.4] for more details, also see the equality (1) in Page 145 of [**8**].

Invariance under fields extensions. — Let X be a scheme over the field k, and k'/k be an extension of fields. We denote by $X_{k'}$ the fiber product $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$. In addition, let E be a k-vector space. We denote by $E_{k'}$ the k'-vector space $E \otimes_k k'$ for simplicity.

Let $X_1, \ldots, X_r$ be closed subschemes of $\mathbb{P}(E)$, $M \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$, and $M' \in \mathcal{C}(M_{k'})$ (see §2.3 for these notations).We will prove that $M' \in \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'})$ in Lemma 2.4. In addition, when k'/k is a finite Galois extension, we will study a relation between $i(M; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E))$ and $i(M'; X_{1,k'} \cdot \ldots \cdot X_{r,k'}; \mathbb{P}(E_{k'}))$. Let X be a closed subscheme of $\mathbb{P}(E)$, M be an integral closed subscheme of X, and $M' \in \mathcal{C}(M_{k'})$. We will obtain a relation between $\mu_M(X)$ and $\mu_{M'}(X_{k'})$ if k'/k is a finite Galois extension.

Proposition 2.3. — *Let X be a pure dimensional closed subscheme of $\mathbb{P}(E)$, and Z be an integral closed subscheme of X. Then we have*

$$\deg(X) = \sum_{X' \in \mathcal{C}(X)} \ell_{\mathcal{O}_{X,X'}}(\mathcal{O}_{X,X'}) \deg(X').$$

and

$$\mu_Z(X) = \sum_{X' \in \mathcal{C}(X)} \ell_{\mathcal{O}_{X,X'}}(\mathcal{O}_{X,X'}) \mu_Z(X').$$

Proof. — If we define the degree of a projective scheme by the multiplicity of an ideal (cf. [**11**, Chap. I, Proposition 7.5]), these two equalities are direct consequences of the equality (4). If we use the definition of degree of a pure dimensional projective scheme by the intersection number as above, we refer the readers to [**8**, Example 2.5.2 (b)] for a proof. □

Proposition 2.3 will be useful in the proof of the results below. At the same time, we also need the following lemma.

Lemma 2.4. — *Let k be a field. Let $X_1, \ldots, X_r$ be closed subschemes of $\mathbb{P}(E)$, and $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. Let k'/k be an extension of fields. Then for each irreducible component $Y' \in \mathcal{C}(Y_{k'})$, we have $Y' \in \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'})$. In addition, the canonical map*

$$\bigsqcup_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} \mathcal{C}(Y_{k'}) \to \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'})$$

is a bijection. In the other words, for every $Y' \in \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'})$, there exists a unique $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$ such that Y' is an irreducible component of $Y_{k'}$.

Proof. — By [**15**, Proposition 3.2.7], for every $Y' \in \mathcal{C}(Y_{k'})$, we have $\dim(Y') = \dim(Y_{k'}) = \dim(Y)$.

Let $Z' \in \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'})$. We consider the projection morphism $\pi' : \mathbb{P}(E_{k'}) \to \mathbb{P}(E)$. By definition, we have $\pi'(Z') \subseteq \bigcap_{i=1}^r X_i$, then we obtain that the scheme $\pi'(Z')$ is contained in an element in $\mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. By the fact that $Z' \subseteq \pi'(Z')_{k'}$, we have that Z' is contained in a $Y' \in \mathcal{C}(Y_{k'})$, where $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$.

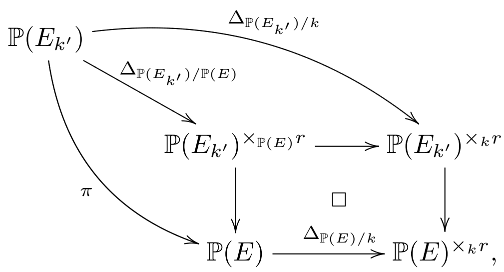
The morphism $\operatorname{Spec} k' \to \operatorname{Spec} k$ is finite and faithfully flat, so is the projection morphism $\pi : \mathbb{P}(E_{k'})^{\times_{k'} r} \to \mathbb{P}(E)^{\times_k r}$ (cf. [**9**, Corollaire 2.2.13 (i)]). Let $Y' \in \mathcal{C}(Y_{k'})$. Let η et η_0 be the generic points of $\Delta(Y)$ and $\Delta(Y')$ respectively, where these Δ denote the diagonal morphism. By [**9**, Proposition 2.3.4 (i)], the projection morphism π maps η_0 to η. If $Z' \in \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'})$ which is contained in $\Delta(Y')$, then we have $\pi(\Delta(Z')) = \Delta(Y)$. By [**9**, Proposition 2.3.4 (i)] again, we obtain that the codimension of Z' in $\mathbb{P}(E_{k'})$ is bounded by that of Y in $\mathbb{P}(E)$, from where we obtain $\dim(Z') \geqslant \dim(Y) = \dim(Y_{k'})$ since the base changes of algebraic schemes are Cartesian. So we have $Z' = Y'$. □

The following proposition is the invariance of the intersection multiplicity by a finite extension of fields. Some ideas of this proof are absorbed from [**21**].

Proposition 2.5. — *Let $X_1, \ldots, X_r$ be closed subschemes of $\mathbb{P}(E)$, and $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. Let k'/k be a finite Galois extension of fields. Then for each irreducible component $Y' \in \mathcal{C}(Y_{k'})$ (we have $Y' \in \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'})$ by Lemma 2.4), we have*

$$i(Y'; X_{1,k'} \cdot \ldots \cdot X_{r,k'}; \mathbb{P}(E_{k'})) = i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)).$$

Proof. — First, we consider the following diagram:



where $\Delta_{\mathbb{P}(E_{k'})/k}$, $\Delta_{\mathbb{P}(E_{k'})/\mathbb{P}(E)}$, and $\Delta_{\mathbb{P}(E)/k}$ are diagonal morphisms, and π is the canonical morphism obtained by the base change $\operatorname{Spec} k' \to \operatorname{Spec} k$.

By [**10**, Proposition (1.4.5), Chap. 0] and [**10**, Proposition (1.4.8), Chap. 0], the above diagram is commutative.

The extension k'/k is separable for it's a Galois extension, so the canonical morphism $\pi : \mathbb{P}(E_{k'}) \to \mathbb{P}(E)$ is finite étale. In addition, the morphism $\Delta_{\mathbb{P}(E_{k'})/\mathbb{P}(E)}$ is a section of the projection morphism (to an arbitrary coordinate)

$$\mathbb{P}(E_{k'})^{\times_{\mathbb{P}(E)} r} \to \mathbb{P}(E_{k'}),$$

where the above projection is étale and separated.

By [**18**, Corollary 3.12], for each closed subscheme of $\mathbb{P}(E_{k'})$, the morphism $\Delta_{\mathbb{P}(E_{k'})/\mathbb{P}(E)}$ is an isomorphism in all connected component of this closed subscheme. Then we obtain that for every integral closed subscheme M of $\mathbb{P}(E)$, and every $M' \in \mathcal{C}(M_{k'})$, the diagonal ideal of the ring $\mathcal{O}_{X_{1,k'}\times_k\cdots\times_k X_{r,k'},\Delta_{\mathbb{P}(E_{k'})/k}(M')}$ is a module obtained by the diagonal ideal of the ring $\mathcal{O}_{X_1\times_k\cdots\times_k X_r,\Delta_{\mathbb{P}(E)/k}(M)}$ by the scalar extension.

In addition, by [**10**, Proposition(1.4.8), Chap. 0], the diagram

$$\begin{array}{ccc}
\mathbb{P}(E_{k'}) & \xrightarrow{Id} & \mathbb{P}(E_{k'}) \\
\Big\downarrow{\scriptstyle \Delta_{\mathbb{P}(E_{k'})/k'}} & & \Big\downarrow{\scriptstyle \Delta_{\mathbb{P}(E_{k'})/k}} \\
\mathbb{P}(E_{k'})^{\times_{k'} r} & \longrightarrow & \mathbb{P}(E_{k'})^{\times_k r} \\
\Big\downarrow & \square & \Big\downarrow \\
\operatorname{Spec} k' & \longrightarrow & \operatorname{Spec}(k'^{\otimes_k r}),
\end{array}$$

is commutative, where $\Delta_{\mathbb{P}(E_{k'})/k'}$ and $\Delta_{\mathbb{P}(E_{k'})/k}$ are diagonal morphisms. Then we obtain that for all integral closed subscheme $M' \in \mathcal{C}(M_{k'})$ defined above, the diagonal ideal of the ring $\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'},\Delta_{\mathbb{P}(E_{k'})/k'}(M')}$ is a module obtained from the diagonal ideal of $\mathcal{O}_{X_{1,k'}\times_k\cdots\times_k X_{r,k'},\Delta_{\mathbb{P}(E_{k'})/k}(M')}$ by the scalar extension with respect to the base change above. As a consequence, the diagonal ideal of the ring $\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'},\Delta_{\mathbb{P}(E_{k'})/k'}(M')}$ is a module obtained from the diagonal ideal of the ring $\mathcal{O}_{X_1\times_k\cdots\times_k X_r,\Delta_{\mathbb{P}(E)/k}(M)}$ by the scalar extension with respect to the base change $\operatorname{Spec} k' \to \operatorname{Spec} k$.

Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_{X_1\times_k\cdots\times_k X_r}$ corresponding to the closed subscheme $X_1\cap\cdots\cap X_r$ via the diagonal morphism, and $\mathcal{I}'$ be the ideal sheaf of $\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'}}$ corresponding to the closed subscheme $X_{1,k'}\cap\cdots\cap X_{r,k'}$ via the diagonal morphism (voir §1.2.4 for the definition). We denote by Δ the diagonal morphisms defined above for simplicity. In addition, let η be the generic point of $\Delta(Y)$, and η' be the generic points of $\Delta(Y')$. By the above argument, we have

$$\mathcal{I}_\eta\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'},\Delta(Y')} = \mathcal{I}'_{\eta'}\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'},\Delta(Y')}$$

as the ideals of the ring $\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'},\Delta(Y')}$.

We can confirm that $\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'},\Delta(Y')}$ is a flat $\mathcal{O}_{X_1\times_k\cdots\times_k X_r,\Delta(Y)}$-module, since the canonical morphism

$$\mathcal{O}_{X_1\times_k\cdots\times_k X_r,\Delta(Y)} \hookrightarrow \mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'},\Delta(Y')} \tag{5}$$

is a composition of a fields extension and a localization. In addition, since the extension k'/k is separable (it is a Galois extension), the morphism (5) is étale.

We denote by $\kappa(Y)$ the residue field of the generic point of $\Delta(Y)$ viewed as a schematic point of $X_1\times_k\cdots\times_k X_r$, and by $\kappa(Y')$ the residue field of the generic point of $\Delta(Y')$ viewed as a schematic point of $X_{1,k'}\times_{k'}\cdots\times_{k'} X_{r,k'}$. Since the morphism

(5) is étale, by [**18**, Proposition 3.2(e)], we have the following Cartesian diagram:

$$\begin{array}{ccc}
\coprod\limits_{Y'\in\mathcal{C}(Y_{k'})} \operatorname{Spec}\kappa(Y') & \longrightarrow & \operatorname{Spec}\kappa(Y) \\
\downarrow & \square & \downarrow \\
\operatorname{Spec}\left(\mathcal{O}_{X_1\times_k\cdots\times_k X_r,\Delta(Y)}\otimes_k k'\right) & \longrightarrow & \operatorname{Spec}\mathcal{O}_{X_1\times_k\cdots\times_k X_r,\Delta(Y)} \\
\downarrow & \square & \downarrow \\
\operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k.
\end{array}$$

So we obtain the equality

$$\sum_{Y'\in\mathcal{C}(Y_{k'})}[\kappa(Y'):\kappa(Y)]=[k':k], \tag{6}$$

for this is an étale base change.

By [**25**, Chap. II, n° 5, f, coro. 2], we have

$$[k':k]e_{\mathcal{I}_\eta,\mathcal{O}_{X_1\times_k\cdots\times_k X_r,\Delta(Y)}}=\sum_{Y'\in\mathcal{C}(Y_{k'})}[\kappa(Y'):\kappa(Y)]e_{\mathcal{I}'_{\eta'},\mathcal{O}_{X_{1,k'}\times_{k'}\cdots\times_{k'}X_{r,k'},\Delta(Y')}}.$$

Then we have

$$\begin{aligned}&[k':k]i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))\\ =&\sum_{Y'\in\mathcal{C}(Y_{k'})}[\kappa(Y'):\kappa(Y)]i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))\end{aligned} \tag{7}$$

by the definition of intersection multiplicitiy (see §2.3 for the definition).

Since $X_{i,k'}$ is $\mathrm{Gal}(k'/k)$-invariant for every $i=1,\ldots,r$, and all elements in $\mathcal{C}(Y_{k'})$ are in the same Galois orbit by [**19**, Proposition A.14] for the extension k'/k is Galois, the function $i(\cdot\,;X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))$ is constant over $\mathcal{C}(Y_{k'})$. So by the equalities (6) and (7), we get the assertion.

□

Proposition 2.6. — *Let X be a closed subscheme of $\mathbb{P}(E)$, and Y be an integral closed subscheme of X. Let k'/k be a finite Galois extension of fields. For every $Y'\in\mathcal{C}(Y_{k'})$, we have*

$$\mu_Y(X)=\mu_{Y'}(X_{k'}).$$

Proof. — We will use a method which is similar to that of the proof of Proposition 2.5. By [**15**, Proposition 3.2.7], for every irreducible component $Y'\in\mathcal{C}(Y_{k'})$, we have $\dim(Y')=\dim(Y)$. All the $Y'\in\mathcal{C}(Y_{k'})$ are isomorphic as k'-schemes by [**19**, Proposition A.14]. In addition, the ring $\mathcal{O}_{X_{k'},Y'}$ is a flat $\mathcal{O}_{X,Y}$-module, for the canonical morphism

$$\mathcal{O}_{X,Y}\hookrightarrow\mathcal{O}_{X_{k'},Y'} \tag{8}$$

is a composition of an fields extension and a localization. In addition, since k'/k is a separable extension (it's a Galois extension), the morphism (8) is étale.

We denote by $\kappa(Y)$ the residue field of the ring $\mathcal{O}_{X,Y}$, and by $\kappa(Y')$ the residue of the ring $\mathcal{O}_{X_{k'},Y'}$. By [**18**, Proposition 3.2(e)], We have the following Carstain diagram:

$$\begin{array}{ccc} \coprod\limits_{Y'\in\mathcal{C}(Y_{k'})} \operatorname{Spec}\kappa(Y') & \longrightarrow & \operatorname{Spec}\kappa(Y) \\ \downarrow & \square & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{X,Y}\otimes_k k') & \longrightarrow & \operatorname{Spec}\mathcal{O}_{X,Y} \\ \downarrow & \square & \downarrow \\ \operatorname{Spec}k' & \longrightarrow & \operatorname{Spec}k. \end{array}$$

Then we have the equality

$$\sum_{Y'\in\mathcal{C}(Y_{k'})}[\kappa(Y'):\kappa(Y)]=[k':k], \tag{9}$$

for the base change is étale.

Let $\mathfrak{m}_{\mathcal{O}_{X,Y}}$ be the maximal ideal of the ring $\mathcal{O}_{X,Y}$, and $\mathfrak{m}_{X_{k'},Y'}$ be the maximal ideal of the ring $\mathcal{O}_{X_{k'},Y'}$. Then we have $\mathfrak{m}_{X_{k'},Y'}=\mathcal{O}_{X_{k'},Y'}\mathfrak{m}_{\mathcal{O}_{X,Y}}$ since the morphism (8) is étale. By [**25**, Chap. II, n° 5, f, coro. 2], we have

$$[k':k]e_{\mathfrak{m}_{X,Y},\mathcal{O}_{X,Y}}=\sum_{Y'\in\mathcal{C}(Y_{k'})}[\kappa(Y'):\kappa(Y)]e_{\mathfrak{m}_{X_{k'},Y'},\mathcal{O}_{X_{k'},Y'}}.$$

Then we have

$$[k':k]\mu_Y(X)=\sum_{Y'\in\mathcal{C}(Y_{k'})}[\kappa(Y'):\kappa(Y)]\mu_{Y'}(X_{k'}). \tag{10}$$

Since $X_{k'}$ is $\mathrm{Gal}(k'/k)$-invariant for each $i=1,\ldots,r$, and all the elements in $\mathcal{C}(Y_{k'})$ are in the same Galois orbit by [**19**, Proposition A.14] for the extension k'/k is Galois, then the function $\mu_{(\cdot)}(X_{k'})$ is constant over $\mathcal{C}(Y_{k'})$. So by the equalities (9) and (10), we have the assertion.

□

Comparison of multiplicities. — We define a *closed k-linear subscheme of* $\mathbb{P}(E)$ (or *closed linear subscheme of* $\mathbb{P}(E)$ for simplicity if there is no confusion with the base field) of dimension d as every $n-d$ complete intersection of k-hyperplanes of $\mathbb{P}(E)$. We can prove that it is an integral closed subscheme of $\mathbb{P}(E)$ of degree 1 with respect to the universal bundle.

***Definition 2.7* (Cylinder).** — Let X be a pure dimensional closed subscheme of $\mathbb{P}(E)$ of dimension d, where $d<n=\mathrm{rk}_k(E)-1$, and P be a point in $X(k)$. Let L be a closed subscheme of $\mathbb{P}(E)$. We say that X and L *only intersect at the neighbourhood of* P if L contains P and every irreducible component of $X\cap L$ passing P is exactly $\{P\}$. In the rest part of the definition, we fix a k-linear closed subscheme L of $\mathbb{P}(E)$ such that X and L only intersect at the neighbourhood of P.

Next, we define a rational map $\phi:\mathbb{P}(E)\times_k\mathbb{P}(E)\dashrightarrow\mathbb{P}(E)$. The point $P\in\mathbb{P}(E)(k)$ corresponds to a surjective homomorphism $E\to\mathcal{O}_{\mathbb{P}(E)}(1)|_P$. Let $H_P=$

$\ker(E \to \mathcal{O}_{\mathbb{P}(E)}(1)|_P)$. We fix a k-linear injective map $\psi : k \to E$. We denote by $U_\psi = \mathbb{P}(E) \smallsetminus V(\psi)$, where $V(\psi)$ is the hyperplane defined by the k-linear map ψ. We suppose that $V(\psi)$ does not contain neither the point P nor the generic point of X.

If R is a k-algebra, then $U_\psi(R)$ is the set of R-linear maps $f : E \otimes_k R \to R$ such that the composition of morphisms

$$R \xrightarrow{\psi\otimes \mathrm{Id}} E \otimes_k R \xrightarrow{f} R$$

is the identity map of R. This set is a functorial bijection (over R) to the set of R-linear maps from H_P to R. Then we can identify the k-scheme U_ψ to the affine space $\mathbb{A}(H_P)$. The affine coordinate from the point $P \in U_\psi$ to $U_\psi(H_P)$ is $0 \in H_P^\vee$.

The affine space $\mathbb{A}(H_P)$ is a group scheme by considering the canonical additive rule ϕ

$$\phi : \mathbb{A}(H_P) \times_k \mathbb{A}(H_P) \to \mathbb{A}(H_P)$$

which maps every point (a, b) to $a + b$.

The Zariski closure Y of $\phi(X \times_k L)$ in $\mathbb{P}(E)$, which is of dimension $m + d$ (see Remark 2.8 below for a proof), is called the *cylinder* passing X of the direction L relatively to P. We remark that the rational class of ϕ and so the cylinder does not depend on the choice of ψ.

Remark 2.8. — We are going to prove that the dimension of the cylinder defined in Definition 2.7 is $m+d$. With all the notations in Definition 2.7, for $\dim(X \times_k L) = m + d$, we have $\dim(Y) \leqslant m + d$ (cf. [**15**, Corollary 3.3.14]).

For the inverse inequality, we take a closed k-linear subscheme L' of $\mathbb{P}(E)$ of dimension $n - m$ which only intersects L in $\mathbb{P}(E)$ at the point $\{P\}$. The morphism $\phi|_{(U_\psi \cap L) \times_k (U_\psi \cap L')} : (U_\psi \cap L) \times_k (U_\psi \cap L') \to U_\psi$ is a isomorphism of schemes. Then we can construct a $\overline{k}$-morphism $\theta : X_{\overline{k}} \to L'_{\overline{k}}$, such that $\dim(\theta(X)) = d$ is the inverse image of all k-point of $\theta(X)$ with respect to θ is a finite set. Then we can take a subset X' of $X_{\overline{k}}$ of dimension d such that $\theta : X' \to L'_{\overline{k}}$ is a bijection. Then the morphism $\phi_{\overline{k}}|_{(X' \cap U_\psi) \times_k (L \cap U_\psi)}$ is an immersion, then we have $\dim(\phi(X' \times_{\overline{k}} L_{\overline{k}})) = m + d$. In addition, we have $\phi(X' \times_{\overline{k}} L_{\overline{k}}) \subseteq Y_{k'}$ by definition.

We have proved that $Y_{\overline{k}}$ contains a subscheme of dimension $m+d$. Since $\dim(Y) = \dim(Y_{\overline{k}})$ by [**15**, Proposition 3.2.7], we obtain the inequality $\dim(Y) \geqslant m + d$, which terminates the proof.

With all the above notations and definitions, we have the following proposition.

Proposition 2.9. — *Let U be a closed integral subscheme of $\mathbb{P}(E)$ such that $U^{\mathrm{reg}}(k) \neq \emptyset$. Let m be an integer satisfying $\dim(U) < m < n + \dim(U)$. We fix a point $P \in U^{\mathrm{reg}}(k)$. Then exists a cylinder U_1 of dimension $n + \dim(U) - m$ whose direction is defined by a k-linear close subscheme L of $\mathbb{P}(E)$ of dimension $n - m$ passing P such that, for all pure dimensional closed subscheme V of dimension m of $\mathbb{P}(E)$ which contains U, if L intersects V properly at the point P, then the cylinder U_1 intersects V properly at U. In addition, we have*

$$\mu_U(V) = i(U; U_1 \cdot V; \mathbb{P}(E))$$

and

$$\mu_Q(V) = \mu_U(V)$$

for all $Q \in U^{\mathrm{reg}}(k)$. *See §2.2 for the notation of* $\mu_U(V)$.

We refer the readers to the second paragraph of [**26**, Chap. II §6, n° 2, b)] for a proof of Proposition 2.9. In the proof, the author of [**26**] used the condition $U^{\mathrm{reg}}(k) \neq \emptyset$ implicitly without a precise statement.

Definition 2.10. — Let X be a scheme. We say that a property depending on a point of X *holds for almost all the points of* X if there exists a dense sub-set U of X, such that this property holds for all the points in U.

If the scheme X is irreducible, and X^{reg} is dense in X while $X^{\mathrm{reg}} \neq \emptyset$. We have the following consequence of Proposition 2.9.

Corollary 2.11. — *Let X be a closed subscheme of $\mathbb{P}(E)$. Let Y and Z be two integral closed subschemes of X, where $Z \subseteq Y$ and $Z^{\mathrm{reg}}(k) \neq \emptyset$. Then we have $\mu_Y(X) \leqslant \mu_Z(X)$. In addition, for almost all point P in Y, we have $\mu_P(X) = \mu_Y(X)$.*

We refer the readers to [**26**, Chap. II §6, n° 2, c)] for a proof of Corollary 2.11.

Next, we will compare the intersection multiplicity of a family of schemes at an irreducible component and the product of multiplicities of this irreducible in this family of schemes. In [**26**, Chap. II §6, n° 2, e)], the author of [**26**] proved Proposition 2.14. But in the proof, the author of [**26**] used the condition that this irreducible component is geometrically integral without a precise statement. Here we do not need to suppose this condition, and we can prove it for the case that the base field is perfect.

For this target, first we introduce the following lemma.

Lemma 2.12. — *Let X be an integral closed subscheme of $\mathbb{P}(E)$. If the set $X^{\mathrm{reg}}(k) \neq \emptyset$, then X is geometrically integral.*

Proof. — We need to prove that X is both geometrically reduced and geometrically irreducible.

First, we prove that X is geometrically irreducible. Soit $\xi \in X^{\mathrm{reg}}(k)$. For every fields extension k'/k, let $\xi' = \xi \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$. Then by the Jacobian criterion (cf. [**15**, Theorem 4.2.19]), we have

$$\mu_{\xi'}(X_{k'}) = \mu_\xi(X) = 1,$$

for the rank of Jacobian matrix at a rational point is invariant under the extension of fields. In addition, if the extension k'/k is Galois, the point ξ' is $\mathrm{Gal}(k'/k)$-invariant. So for each irreducible component $X' \in \mathcal{C}(X_{k'})$, we have $\xi' \in X'$.

By Proposition 2.3, for every Galois extension k'/k, we have the equality

$$\sum_{X' \in \mathcal{C}(X_{k'})} \ell_{\mathcal{O}_{X_{k'},X'}}(\mathcal{O}_{X_{k'},X'})\mu_{\xi'}(X') = \mu_{\xi'}(X_{k'}) = 1.$$

So we obtain $\#\mathcal{C}(X_{k'}) = 1$ and $\ell_{\mathcal{O}_{X_{k'},X'}}(\mathcal{O}_{X_{k'},X'}) = 1$ for the $X' \in \mathcal{C}(X_{k'})$. The assertion $\#\mathcal{C}(X_{k'}) = 1$ means that $X_{k'}$ is irreducible. Then X is geometrically irreducible.

Next, we are going to prove that X is geometrically reduced. If the extension k'/k is separable, then by [**15**, Corollary 3.2.14], the scheme X is geometrically reduced.

If k'/k is not separable, then the field k is not perfect. We suppose that the characteristic of k is p. In this case, we can device the extension into a composition of a separable extension and a purely inseparable extension. For the purely inseparable part, we can device it into a composition of some purely inseparable extensions of degree p. Then we need to prove that if k'/k is a purely inseparable extension with $[k' : k] = p$, the scheme $X_{k'}$ is reduced. Since the question is local, we can suppose that X is affine. Let $X = \operatorname{Spec} A$, where A is a ring containing k.

Since X has a regular k-rational point, then we take $\xi \in X^{\mathrm{reg}}(k)$, and we denote by $\mathfrak{m}_\xi$ the maximal ideal of the ring $\mathcal{O}_{X,\xi}$. Then we have $\widehat{A}_{\mathfrak{m}_\xi} = \widehat{\mathcal{O}}_{X,\xi} \cong k[[T_1, \ldots, T_d]]$ (cf. [**16**, (28.J)]), where $d = \dim(X)$. Let $\xi' = \xi \times_{\operatorname{Spec} k} \operatorname{Spec} k'$, then we have $\widehat{\mathcal{O}}_{X_{k'},\xi'} \cong k'[[T_1, \ldots, T_d]]$ for ξ' is regular in $X_{k'}$. So we have the following commutative diagram:

$$\begin{array}{ccc} A & \hookrightarrow & k[[T_1, \ldots, T_d]] \\ \downarrow & & \downarrow \\ A \otimes_k k' & \hookrightarrow & k'[[T_1, \ldots, T_d]]. \end{array}$$

The ring $k'[[T_1, \ldots, T_d]]$ is integral, then the ring $A \otimes_k k'$ is also integral, which must be reduced. So we obtain that X is geometrically reduced. Then we proved the desired result. □

Remark 2.13. — The proof of Lemma 2.12 absorbs some ideas from [**22**, Lemma 10.1], but the condition of Lemma 2.12 is weaker.

Proposition 2.14. — *We suppose that k is a perfect field. Let $X_1, \ldots, X_r$ be pure dimensional closed subschemes of $\mathbb{P}(E)$ and $M \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. Then we have*

$$i(M; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) \geqslant \prod_{i=1}^{r} \mu_M(X_i).$$

Proof. — First, we suppose that $M^{\mathrm{reg}}(k) \neq \emptyset$. In this case, by Lemma 2.12, the scheme M is geometrically integral. Then we obtain that the scheme $M^{\times_k r}$ is also geometrically integral by [**9**, (4.6.5) (ii)].

The intersection multiplicity $i(M; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E))$ is the multiplicity of an ideal of the local ring $\mathcal{O}_{X_1 \times_k \cdots \times_k X_r, \Delta(M)}$ which is containd in the maximal ideal of $\mathcal{O}_{X_1 \times_k \cdots \times_k X_r, \Delta(M)}$. By the inequality (3), we obtain

$$i(M; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) \geqslant \mu_{\Delta(M)}(X_1 \times_k \cdots \times_k X_r).$$

In addition, the scheme $\Delta(M)$ is geometrically integral and it has a regular k-rational point. By the fact that $\Delta(M) \subseteq M^{\times_k r}$, we have

$$\mu_{\Delta(M)}(X_1 \times_k \cdots \times_k X_r) \geqslant \mu_{M^{\times_k r}}(X_1 \times_k \cdots \times_k X_r)$$

by Corollary 2.11.

Let U_1 and U_2 be two closed geometrically integral schemes of Y_1 et Y_2 respectively, where Y_1 et Y_2 are two closed subschemes of $\mathbb{P}(E)$. By [**9**, (4.6.5) (ii)], the scheme $U_1\times_k U_2$ is geometrically integral. In this case, the scheme $U_1\times_k U_2$ is a closed integral subscheme of $Y_1\times_k Y_2$, from where we obtain $\mathcal{O}_{Y_1\times_k Y_2, U_1\times_k U_2}\cong \mathcal{O}_{Y_1,U_1}\otimes_k\mathcal{O}_{Y_2,U_2}$. By [**25**, Chap. VI, n° 1, d, prop. 1], we get

$$\mu_{U_1\times_k U_2}(Y_1\times_k Y_2)=\mu_{U_1}(Y_1)\mu_{U_2}(Y_2).$$

Then we have

$$\begin{aligned}\mu_{M^{\times_k r}}(X_1\times_k\cdots\times_k X_r) &= \mu_M(X_1)\cdot\mu_{M^{\times_k(r-1)}}(X_2\times_k\cdots\times_k X_r)\\ &= \cdots\\ &= \prod_{i=1}^r \mu_M(X_i),\end{aligned}$$

which proves the assertion.

Next, we will prove the case where k is a perfect field and $M\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$. Let k'/k be a finite Galois extension of fields such that for every irreducible component $M'\in\mathcal{C}(M_{k'})$, the scheme M' contains at least one regular k'-rational point. By Lemma 2.12, every $M'\in\mathcal{C}(M_{k'})$ is geometrically integral. By the above argument, if we fix a $M'\in\mathcal{C}(M_{k'})\subseteq\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})$ (by Lemma 2.4), we have

$$i(M';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))\geqslant\prod_{i=1}^r\mu_{M'}(X_{i,k'}).$$

By Proposition 2.5, we have

$$i(M;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))=i(M';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'})).$$

By Proposition 2.6, we have

$$\mu_M(X_i)=\mu_{M'}(X_{i,k'}).$$

Then we prove the assertion. □

2.5. Counting objects over a finite field. — In this paragraph, we will provide some estimates on the number of certain objects over finite fields.

Let k be a field and V be a k-vector space of finite rank. We denote by $\mathrm{Gr}(r,V^\vee)$ the Grassmannian which classifies all the vector sub-spaces of dimension r of V. Let k'/k be an extension of fields, and we denote by $\mathrm{Gr}(r,V^\vee)(k')$ the set of k'-rational points of $\mathrm{Gr}(r,V^\vee)$. We denote by $\mathrm{Gr}_k(r,n)$ the Grassmannian $\mathrm{Gr}(r,(k^n)^\vee)$, or by $\mathrm{Gr}(r,n)$ if there is no confusion over the base field k. In particular, we have $\mathrm{Gr}_k(n-1,n)\cong\mathbb{P}^{n-1}_k$.

Lemma 2.15. — *With all the above notations, let $\mathbb{F}_q$ be the finite field with cardinality q. Then we have*

$$\#\,\mathrm{Gr}_{\mathbb{F}_q}(r,n)(\mathbb{F}_q)=\frac{\prod\limits_{t=1}^n(q^{t-1}+q^{t-2}+\cdots+1)}{\prod\limits_{t=1}^r(q^{t-1}+q^{t-2}+\cdots+1)\cdot\prod\limits_{t=1}^{n-r}(q^{t-1}+q^{t-2}+\cdots+1)}.$$

In particular, we have

$$\mathbb{P}^n_{\mathbb{F}_q}(\mathbb{F}_q) = q^n + \cdots + 1.$$

We refer the readers to [**28**, Proposition 1.7.2] for a proof of the Lemma 2.15.

Let k'/k be an extension of fields, E be k-vector space of finite rank, and $\phi : X \hookrightarrow \mathbb{P}(E_{k'})$ be a closed immersion. We have the following commutative diagram:

$$\begin{array}{ccccc} X & \overset{\phi}{\hookrightarrow} & \mathbb{P}(E_{k'}) & \xrightarrow{\pi} & \mathbb{P}(E) \\ & & \big\downarrow & \square & \big\downarrow \\ & & \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k. \end{array}$$

Definition 2.16. — We denote by $X_\phi(k)$ the sub-set of $X(k')$ of the $\xi \in X(k')$ (considered as k'-morphism from $\operatorname{Spec} k'$ to X) whose composition with the canonical morphism $X \to \mathbb{P}(E)$ gives a k-point of $\mathbb{P}(E)$ with the value in k' which is induced from a k-rational point of $\mathbb{P}(E)$. In another words, we define $X_\phi(k) = X(k') \cap \pi^{-1}(\mathbb{P}(E)(k))$. If there is no confusion with the immersion ϕ, we denote by $X(k)$ the set $X_\phi(k)$ for simplicity.

When k is a finite field, we have the following result for estimating the cardinality of the set $X_\phi(k)$ when X is pure dimensional.

Proposition 2.17. — *Let $k/\mathbb{F}_q$ be an extension of fields, E be a k-vector space of finite rank, and $\phi : X \hookrightarrow \mathbb{P}(E)$ be a closed immersion. We suppose that X is pure dimensional of dimension d. Then*

$$\#X_\phi(\mathbb{F}_q) \leqslant \deg(X) \# \mathbb{P}^d_{\mathbb{F}_q}(\mathbb{F}_q).$$

We refer the readers to Page 236 of [**17**]. Proposition 2.17 is a direct consequence of this argument.

Let k be a field, and $X_1, \ldots, X_r$ be k-schemes such that $\bigcap_{i=1}^r X_i(k) \neq \emptyset$. If $P \in \bigcap_{i=1}^r X_i(k)$, and every irreducible component of the intersection of $X_1 \cap \cdots \cap X_r$ passing P is exact $\{P\}$, we say that $X_1, \ldots, X_r$ *only intersect at the neighbourhood of* P.

The following proposition is used to determine whether there exists a k-linear closed subscheme of $\mathbb{P}(E)$ which only intersect a family of pure dimensional schemes with fixed dimension at the neighbourhood of this point.

Proposition 2.18. — *Let $U_1, \ldots, U_r$ be closed pure dimensional subschemes of $\mathbb{P}(E)$. We suppose that $\bigcap_{i=1}^r U_i(k) \neq \emptyset$ and $\dim(U_i) = d < n = \operatorname{rk}_k(E) - 1$ for all $i = 1, \ldots, r$. Let $P \in \bigcap_{i=1}^r U_i(k)$. If the inequality*

$$\#k \geqslant \deg(U_1) + \cdots + \deg(U_r)$$

is verified, then there exists at least one k-linear closed subscheme of $\mathbb{P}(E)$ of dimension less than to equal to $n - d$ which only intersect every U_i at the neighbourhood of P.

Proof. — If there exists a closed k-linear subscheme L of $\mathbb{P}(E)$ of dimension $n-d$ which intersect every one of $U_1,\ldots,U_r$ properly at the point P, then for each closed k-linear subscheme of $\mathbb{P}(E)$ passing P contained in L, it only intersects $U_1,\ldots,U_r$ at the neighbourhood of P. Then we need to prove that there exists a closed k-linear subscheme L' of $\mathbb{P}(E)$ of dimension $n-d$ such that $\{P\}$ is a proper component of the intersection $L'\cdot U_1\cdot\ldots\cdot U_r$ in $\mathbb{P}(E)$.

We denote by $\mathscr{H}_P$ the set of projective k-hyperplanes passing the point P, then we have $\mathscr{H}_P=\mathrm{Gr}(n-1,E^\vee)(k)$. First, we will prove that we can find an $H_1\in\mathscr{H}_P$ which intersects all the U_i properly. For a fixed U_i, its irreducible components are contained in at most $\deg(U_i)$ closed k-linear subschemes of $\mathbb{P}(E)$ of dimension d. In addition, for a fixed closed k-linear subscheme of $\mathbb{P}(E)$ of dimension d, there exists $\#\,\mathrm{Gr}(n-d-1,n-d)(k)$ hyperplanes which contain this closed k-linear subscheme of $\mathbb{P}(E)$. If k is a finite field, then $\#\,\mathrm{Gr}(m,n)(k)$ is calculated in Lemma 2.15, then we can confirm that we have the inequality

$$\begin{aligned}\#\mathscr{H}_P &= \#\,\mathrm{Gr}(n-1,n)(k)\\ &> (\deg(U_1)+\cdots+\deg(U_r))\,\#\,\mathrm{Gr}(n-d-1,n-d)(k),\end{aligned}$$

when $\#k\geqslant r\geqslant 1$ et $\#k\geqslant 2$. So there always exists such a hyperplane H_1.

If k is infinite, there always exists a hyperplane $H_1\in\mathscr{H}_P$ which satisfies that the schemes $U_1,\ldots,U_r,H_1$ intersect properly at an irreducible component containing the point P.

If we have already found the hyperplanes $H_1,\ldots,H_{t-1}\in\mathscr{H}_P$, such that the schemes $U_i,H_1,\ldots,H_{t-1}$ intersect properly for all $i=1,\ldots,r$, where $1\leqslant t\leqslant d$. By Bézout Theorem (Theorem 2.2), we obtain that there exist at most $\deg(U_i)$ elements in $\mathcal{C}(H_1\cdot H_2\cdot\ldots\cdot H_{t-1}\cdot U_i)$, where each element is of dimension $d-t+1$. In addition, every element in $\mathcal{C}(H_1\cdot H_2\cdot\ldots\cdot H_{t-1}\cdot U_i)$ is contained in at most one closed k-linear subscheme of of $\mathbb{P}(E)$ of dimension $d-t+1$, where $i=1,\ldots,r$. If k is a finite, By Proposition 2.15, we have

$$\begin{aligned}&\#\,\mathrm{Gr}(n-t,n-t+1)(k)\\ >\ &(\deg(U_1)+\cdots+\deg(U_r))\,\#\,\mathrm{Gr}(n-d-1,n-d)(k),\end{aligned}$$

when $\#k\geqslant r\geqslant 1$, $\#k\geqslant 2$ et $t\leqslant d$. Donc on peut trouver un sous-schéma k-linéaire fermé de $\mathbb{P}(E)$ de dimension $n-t$ passant par P contenu dans $H_1\cap\cdots\cap H_{t-1}$, qui intersecte tous les éléments dans $\mathcal{C}(H_1\cdot H_2\cdot\ldots\cdot H_{t-1}\cdot U_i)$ proprement pour tout $i=1,\ldots,r$.

Every closed k-linear subscheme of $\mathbb{P}(E)$ passing P contained in $H_1\cap\cdots\cap H_{t-1}$ can be lifted to a hyperplane in $\mathscr{H}_P$. We lift whis closed k-linear closed subscheme of $\mathbb{P}(E)$ to $H_t\in\mathscr{H}_P$ such that $H_1\cap\cdots\cap H_{t-1}\cap H_t$ is a complete intersection.

If k is infinite, there always exists a hyperplane $H_t\in\mathscr{H}_P$ which satisfies that the projective schemes $U_1,\ldots,U_r,H_1,\ldots,H_{t-1},H_t$ intersect properly at an irreducible component containing the point P.

So we can find a series of elements $H_1,H_2,\ldots,H_d\in\mathscr{H}_P$, such that the schemes $H_1,H_2,\ldots,H_d,U_i$ intersect properly at the point P for all $i=1,\ldots,r$. The closed k-linear subscheme of $\mathbb{P}(E)$ defined by the complete intersection of $H_1,H_2,\ldots,H_d$ intersects all the U_i properly at the point P, where $i=1,\ldots,r$. □

3. Intersection tree

In this paragraph, we introduce the notion of intersection tree in the frame of graph theory, which will be used in the estimate of counting multiplicities. This construction is valid in a frame of general projective regular schemes over a field equipped with an ample line bundle. In this paragraph, we fix a field k.

3.1. Definition. — Let $\delta \geqslant 1$ be an integer, Y be a regular separated k-scheme and $\mathscr{L}$ be an ample invertible $\mathcal{O}_Y$-module. We call a directed rooted tree $\mathscr{T}$ with labelled vertices and weighted edges an *intersection tree of level* δ over Y, if it satisfies the following conditions:

1. the vertices of $\mathscr{T}$ are the occurrences of integral closed subschemes of Y (an integral closed subscheme of Y can appear several times in a tree);
2. to each vertex X of $\mathscr{T}$ is attached a label, which is a pure dimensional closed subscheme of Y;
3. a vertex of $\mathscr{T}$ is a leaf if and only if its label is the empty closed subscheme;
4. if X is a vertex of $\mathscr{T}$ which is not a leaf, then
   — its label $\widetilde{X}$ satisfies the inequality $\deg_{\mathscr{L}}(\widetilde{X}) \leqslant \delta$ and the closed subschemes X and $\widetilde{X}$ intersect properly in Y;
   — the children of X are precisely the irreducible components of the intersection product $X \cdot \widetilde{X}$ in Y;
   — for each child Z of X, the edge ℓ which links X and Z is attached with a weight $w(\ell)$ which equals the intersection multiplicity $i(Z; X \cdot \widetilde{X}; Y)$.

For every fixed intersection tree $\mathscr{T}$, we call any of the complete sub-trees of $\mathscr{T}$ an *intersection sub-tree*, which is necessarily an intersection tree.

Weight of a vertex. — Let Y be a regular separated scheme over $\operatorname{Spec} k$, equipped with an ample invertible sheaf $\mathscr{L}$, and $\mathscr{T}$ be an intersection tree over Y. For each vertex X of $\mathscr{T}$, we define the *weight* of X as the product of the weights of all edges in the path which links the root of $\mathscr{T}$ and the vertex X, denoted as $w_{\mathscr{T}}(X)$. If X is the root of an intersection tree, we define $w_{\mathscr{T}}(X) = 1$ for convenience.

Weight of an integral closed subscheme. — Let Z be an integral closed subscheme of Y. We define the *weight* of Z relative to the tree $\mathscr{T}$ as the sum of the weights of all the occurrences of Z as vertices of $\mathscr{T}$, noted by $W_{\mathscr{T}}(Z)$. If Z does not appear in the tree $\mathscr{T}$ as a vertex, for convenience the weight $W_{\mathscr{T}}(Z)$ is defined to be 0. Let Z be a vertex in the intersection tree $\mathscr{T}$. When we write $W_{\mathscr{T}}(Z)$, the symbol Z is considered as an integral closed subscheme of Y. In other words, we count all the occurrences of the subscheme Z in the intersection tree $\mathscr{T}$.

Example 3.1. — We will give an example of the operation in Theorem 3.2. For convention we suppose $\operatorname{char}(k) \neq 2$ in this example. We take $\mathbb{P}(E) = \mathbb{P}^4_k = \operatorname{Proj}(k[T_0, T_1, T_2, T_3, T_4])$ as the base scheme. Let

$$X_1 = \operatorname{Proj}(k[T_0, T_1, T_2, T_3, T_4]/(T_4)),$$

and

$$X_2 = \mathrm{Proj}\left(k[T_0, T_1, T_2, T_3, T_4]/(T_3(T_0^2T_1 - T_2^3 + T_2^2T_1))\right).$$

Then we have $\deg(X_1) = 1$ et $\deg(X_2) = 4$. The schemes X_1 et X_2 intersect properly in $\mathbb{P}^4_k$. The intersection of X_1 and X_2 has two irreducible components, noted by Y_1 et Y_2. Let

$$Y_1 = \mathrm{Proj}\left(k[T_0, T_1, T_2, T_3, T_4]/(T_0^2T_1 - T_2^3 + T_2^2T_1, T_4)\right)$$

be an element in $\mathcal{C}(X_1 \cdot X_2)$, and

$$Y_2 = \mathrm{Proj}\left(k[T_0, T_1, T_2, T_3, T_4]/(T_3, T_4)\right)$$

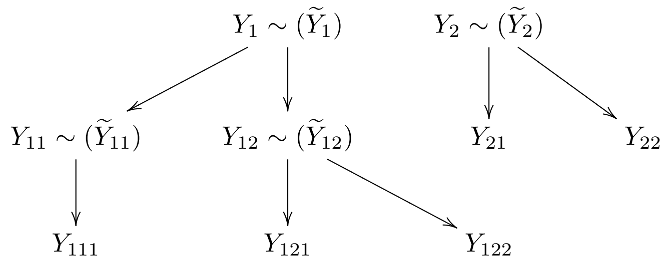
be the other element in $\mathcal{C}(X_1 \cdot X_2)$. Then by definition, we have

$$i(Y_1; X_1 \cdot X_2; \mathbb{P}^4_k) = 1,\ \deg(Y_1) = 3;$$

and

$$i(Y_2; X_1 \cdot X_2; \mathbb{P}^4_k) = 1,\ \deg(Y_2) = 1.$$

We are going to construct two intersection trees below whose roots are Y_1 and Y_2 respectively.



We suppose that the label of Y_1 is the hypersurface

$$\widetilde{Y}_1 = \mathrm{Proj}\left(k[T_0, T_1, T_2, T_3, T_4]/(T_1T_3)\right),\ \deg(\widetilde{Y}_1) = 2;$$

and the label of Y_2 is

$$\widetilde{Y}_2 = \mathrm{Proj}\left(k[T_0, T_1, T_2, T_3, T_4]/(T_2, T_0(T_1 + T_0))\right),\ \deg(\widetilde{Y}_2) = 2.$$

Then we can confirm that the intersection of Y_1 and $\widetilde{Y}_1$ and the intersection of Y_2 and $\widetilde{Y}_2$ are proper.

Next, we consider the intersection tree with the root Y_2. In fact, it has two irreducible components, noted by Y_{21} et Y_{22}. By definition, we obtain

$$Y_{21} = [0 : 1 : 0 : 0 : 0],\ i(Y_{21}; Y_2 \cdot \widetilde{Y}_2; \mathbb{P}^4_k) = 1;$$

and

$$Y_{22} = [1 : -1 : 0 : 0 : 0],\ i(Y_{22}; Y_2 \cdot \widetilde{Y}_2; \mathbb{P}^4_k) = 1.$$

For the tree whose root is Y_1, the set $\mathcal{C}(Y_1 \cdot \widetilde{Y}_1)$ has two elements, noted by Y_{11} et Y_{12} respectively. We suppose

$$Y_{11} = \mathrm{Proj}\left(k[T_0, T_1, T_2, T_3, T_4]/(T_1, T_2, T_4)\right)$$

and

$$Y_{12} = \mathrm{Proj}\left(k[T_0, T_1, T_2, T_3, T_4]/(T_0^2T_1 - T_2^3 + T_2^2T_1, T_3, T_4)\right)$$

Then we have

$$i(Y_{11};Y_1\cdot\widetilde{Y}_1;\mathbb{P}^4_k)=3,\ \deg(Y_{11})=1;$$

and

$$i(Y_{12};Y_1\cdot\widetilde{Y}_1;\mathbb{P}^4_k)=1,\ \deg(Y_{12})=3.$$

The equality $i(Y_{11};Y_1\cdot\widetilde{Y}_1;\mathbb{P}^4_k)=3$ is by the fact that the local ring at Y_{11} is Cohen-Macaulay, and by [**8**, Proposition 7.1], this intersection multiplicity is equal to $\ell(\mathcal{O}_{Y_1\cap\widetilde{Y}_1,Y_{11}})$, which is 3. Let

$$\widetilde{Y}_{11}=\operatorname{Proj}\left(k[T_0,T_1,T_2,T_3,T_4]/(T_0+T_3)\right),\ \deg(\widetilde{Y}_{11})=1$$

be the label of Y_{11}, and

$$\widetilde{Y}_{12}=\operatorname{Proj}\left(k[T_0,T_1,T_2,T_3,T_4]/(T_2)\right),\ \deg(\widetilde{Y}_{12})=1$$

be the label of Y_{12}. Then we have that the intersection of Y_{11} and $\widetilde{T}_{11}$ have one irreducible component, and the intersection of Y_{12} and $\widetilde{Y}_{12}$ has two irreducible components, noted by Y_{121} and Y_{122}. In addition, we have

$$Y_{111}=[1:0:0:-1:0],\ i(Y_{111};Y_{11}\cdot\widetilde{Y}_{11};\mathbb{P}^4_k)=1,$$

and

$$Y_{121}=[0:1:0:0:0],\ i(Y_{121};Y_{12}\cdot\widetilde{Y}_{12};\mathbb{P}^4_k)=2,$$

and

$$Y_{122}=[1:0:0:0:0],\ i(Y_{122};Y_{12}\cdot\widetilde{Y}_{12};\mathbb{P}^4_k)=1$$

by definition directly.

Let $M=[0:1:0:0:0]$. We can confirm that $Y_{121}=Y_{21}=M$ satisfies the conditions in Theorem 3.2 considered as two integral subschemes of $\mathbb{P}^4_k$. In this example, the left hand side of the inequality (11) is equal to

$$\begin{aligned}&i(Y_1;X_1\cdot X_2;\mathbb{P}^4_k)i(Y_{12};Y_1\cdot\widetilde{Y}_1;\mathbb{P}^4_k)i(Y_{121};Y_{12}\cdot\widetilde{Y}_{12};\mathbb{P}^4_k)\\&+i(Y_2;X_1\cdot X_2;\mathbb{P}^4_k)i(Y_{21};Y_2\cdot\widetilde{Y}_2;\mathbb{P}^4_k)\\=\ &3.\end{aligned}$$

In addition, as the hypersurface X_1 is regular, we have

$$\mu_M(X_1)=1;$$

by considering the Taylor expansion of the equation defining the hypersurface X_2, we obtain

$$\mu_M(X_2)=3.$$

Then the right hand side of the inequality (11) is equal to

$$\mu_M(X_1)\mu_M(X_2)=3.$$

So we have the inequality

$$\begin{aligned}&i(Y_1;X_1\cdot X_2;\mathbb{P}^4_k)i(Y_{12};Y_1\cdot\widetilde{Y}_1;\mathbb{P}^4_k)i(Y_{121};Y_{12}\cdot\widetilde{Y}_{12};\mathbb{P}^4_k)\\&+i(Y_2;X_1\cdot X_2;\mathbb{P}^4_k)i(Y_{21};Y_2\cdot\widetilde{Y}_2;\mathbb{P}^4_k)\\\geqslant\ &\mu_M(X_1)\mu_M(X_2),\end{aligned}$$

which is an example of Theorem 3.2.

3.2. Estimate of the weighs of intersection trees. — Let $\{X_i\}_{i=1}^r$ be a family of pure dimensional closed subschemes of $\mathbb{P}(E)$ which intersect properly in $\mathbb{P}(E)$ (see §2.3 for the definition). We will state the following theorem, which can be considered as a upper bound of local multiplicities of $X_1, \ldots, X_r$ by function of intersection trees.

Theorem 3.2. — *We suppose that k is a perfect field. Let $\{X_i\}_{i=1}^r$ be a family of pure dimensional closed subscheme of $\mathbb{P}(E)$ which intersect properly in $\mathbb{P}(E)$. For each irreducible component $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$, let $\mathscr{T}_Y$ be an intersection tree whose root is Y. We consider a vertex M in these intersection trees $\{\mathscr{T}_Y\}_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}$ satisfying: for every vertex Z in $\{\mathscr{T}_Y\}_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}$, if M is a proper subscheme of Z, then there exists a descent of Z which is occurrence of M as schemes. Then the following inequality is satisfied:*

$$\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} W_{\mathscr{T}_Y}(M) i\left(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)\right) \geqslant \mu_M(X_1)\cdots\mu_M(X_r), \tag{11}$$

where $\mu_M(X_i)$ denotes the local multiplicity of X_i at the generic point of M.

We recall that the *depth* of a vertex is defined as the length of the chain which links this vertex and the root of the tree. In addition, the *depth* of a tree is defined as the maximal value of all the depths of its vertices.

We will prove Theorem 3.2 in the next section.

4. Proof of Theorem 3.2

This section is focus on a proof of Theorem 3.2. Let k be a perfect field, and $X_1, \ldots, X_r$ be closed pure dimensional subschemes of $\mathbb{P}(E)$ which intersect properly. For all $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$, we construct an intersection tree $\mathscr{T}_Y$ of level $\delta = \max\limits_{i\in\{1,\ldots,r\}}\{\deg(X_i)\}$ whose root is Y. The strategy consists of a mathematical induction on the maximal depth of the intersection trees $\mathscr{T}_Y$ (see §3.1 for the definition). Let M be a vertex of these intersection trees $\mathscr{T}_Y$. We suppose that M satisfies the following conditions: for every vertex Z of these intersection trees $\{\mathscr{T}_Y\}_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}$, if M is a proper subscheme of Z, then there exists a descendant of Z which is an occurrence of M as a scheme. The aim of this paragraph is to prove the inequality (11) reproduced below:

$$\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} W_{\mathscr{T}_Y}(M) i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) \geqslant \mu_M(X_1)\cdots\mu_M(X_r).$$

Definition 4.1. — Let s be a positive integer. With all the notations above, we define $\mathcal{C}_s$ as the set of the vertices of depth s in these intersection trees $\mathscr{T}_Y$, where $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. In addition, we define $\mathcal{C}_* = \bigcup\limits_{s\geqslant 0} \mathcal{C}_s$.

For every positive integer s, we define a sub-set of $\mathcal{C}_s$ as below.

Definition 4.2. — Let s be a positive integer. We define $\mathcal{Z}_s$ as the sub-set of $\mathcal{C}_s$ of the elements N which satisfy the following conditions: for every vertex Z of the intersection trees $\{\mathscr{T}_Y\}_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}$, if N is a proper subscheme of Z, then there exists a descendant of N which is an occurrence of Z as a scheme. In addition, we define $\mathscr{Z}_* = \bigcup_{s\geqslant 0} \mathscr{Z}_s$.

By definition, we have $\mathscr{Z}_0 = \mathcal{C}_0 = \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. We will prove Theorem 3.2 for the vertices in the set $\mathscr{Z}_*$.

The principle idea of this proof of Theorem 3.2 follows: if $M \in \mathscr{Z}_0$, the left hand side of the inequality (11) is a intersection multiplicity at M, then Theorem 3.2 is concluded by Proposition 2.14. If $M \in \mathscr{Z}_* \smallsetminus \mathscr{Z}_0$, as k is a perfect field, the intersection multiplicity and the multiplicity of point in a scheme verify the invariance under an Galois extension of fields by Proposition 2.5 and Proposition 2.6. First we fix a finite Galois extension of fields k'/k such that $M^{\mathrm{reg}}(k') \neq \emptyset$ and the cardinality of k' is large enough. Then we can construct an auxiliary k'-scheme such that the intersection of $X_{1,k'}, \ldots, X_{r,k'}$ and this scheme is proper at one of the irreducible component of $M_{k'}$ (in fact, the auxiliary scheme is a cylinder passing this irreducible component, whose existence is assured when k' is large enough, see Definition 2.9 for the definition of cylinder). Next, we prove that the left hand side of the inequality (11) is larger than or equal to the intersection multiplicity of $X_{1,k'}, \ldots, X_{r,k'}$ and the above auxiliary k'-scheme at this irreducible component of $M_{k'}$. By the comparison between the multiplicity of this intersection product at this irreducible component of $M_{k'}$ and the multiplicity of M in $X_1, \ldots, X_r$ and in the auxiliary scheme (Proposition 2.14), we obtain this result.

Proof of Theorem 3.2. — In this proof, the irreducible component $M \in \mathscr{Z}_*$ is the same as that in the statement of Theorem 3.2.

Step 1: the depth of vertex is zero. - If $M \in \mathscr{Z}_0 = \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$, the for every $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$, we have $W_{\mathscr{T}_Y}(M) = 0$ or 1. Then the assertion of Theorem 3.2 is a direct consequence of Theorem 2.2 and Proposition 2.14, which prove that the intersection multiplicity of the intersection product of $X_1 \cdot \ldots \cdot X_r$ at an irreducible component is larger than or equal to the product of the multiplicities of this irreducible component in $X_1, \ldots, X_r$.

Step 2: the depth of vertex is strictly larger than zero. - If $M \in \mathscr{Z}_* \smallsetminus \mathscr{Z}_0$, we will prove the following statement.

Proposition 4.3. — *Let $n = \mathrm{rk}_k(E) - 1$. Let k'/k be a finite Galois extension of fields, such that*

$$\# k' \geqslant \delta^{\sum_{i=1}^{r} \dim(X_i) - (r-1)(n-1)}$$

and $M^{\mathrm{reg}}(k') \neq \emptyset$. Then for each irreducible component $M' \in \mathcal{C}(M_{k'})$, there exists a cylinder $M^0_{k'}$ in $\mathbb{P}(E_{k'})$, such that $M' \in \mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'} \cdot M^0_{k'})$ and the schemes $X_{1,k'}, \ldots, X_{r,k'}, M^0_{k'}$ intersect properly at the irreducible component M', and

$$\begin{aligned} &\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) W_{\mathscr{T}_Y}(M) \\ \geqslant\ & i(M'; X_{1,k'} \cdot \ldots \cdot X_{r,k'} \cdot M^0_{k'}; \mathbb{P}(E_{k'})). \end{aligned}$$

If we admit Proposition 4.3, by Proposition 2.14, we obtain

$$\begin{aligned} i(M';X_{1,k'}\cdot\ldots\cdot X_{r,k'}\cdot M^0_{k'};\mathbb{P}(E_{k'})) &\geqslant \mu_{M'}(X_{1,k'})\cdots\mu_{M'}(X_{r,k'})\mu_{M'}(M^0_{k'})\\ &\geqslant \mu_{M'}(X_{1,k'})\cdots\mu_{M'}(X_{r,k'})\\ &= \mu_M(X_1)\cdots\mu_M(X_r), \end{aligned}$$

where the equality is due to Proposition 2.6. Then we prove the assertion of Theorem 3.2.

In order to prove Proposition 4.3, we run a mathematical induction on the maximal depth s of these $\mathscr{T}_Y$, where $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$.

Step 2-1: the case of $s=1$. **-** First, we will prove the case of $s=1$. We suppose that $\mathscr{Z}_*\smallsetminus\mathscr{Z}_0\neq\emptyset$. In this case, we are going to prove the following lemma.

Lemma 4.4. — *Let $M\in\mathscr{Z}$, and k'/k be a finite Galois extension of fields such that $\#k'\geqslant\deg(X_1)\cdots\deg(X_r)$ et $M^{\mathrm{reg}}(k')\neq\emptyset$. For every $M'\in\mathcal{C}(M_{k'})$, there exists a cylinder $M^0_{k'}$ in $\mathbb{P}(E_{k'})$, such that $M'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'}\cdot M^0_{k'})$ and the schemes $X_{1,k'},\ldots,X_{r,k'},M^0_{k'}$ intersect properly at the irreducible component M'. In addition, the equality*

$$\begin{aligned}&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))W_{\mathscr{T}_Y}(M)\\ \geqslant\ & i(M';X_{1,k'}\cdot\ldots\cdot X_{r,k'}\cdot M^0_{k'};\mathbb{P}(E_{k'}))\end{aligned}$$

is verified.

Proof. — For every $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$, we denote by $\widetilde{Y}$ the label of Y in the intersection tree considered as a scheme. By the definition in §3.1, we have

$$W_{\mathscr{T}_Y}(M)=i(M;Y\cdot\widetilde{Y};\mathbb{P}(E)). \tag{12}$$

In fact, as $s=1$, if M appears in the descendants of Y, it appears only once. In addition, we have

$$i(M;Y\cdot\widetilde{Y};\mathbb{P}(E))\geqslant\mu_M(Y)\mu_M(\widetilde{Y})\geqslant\mu_M(Y) \tag{13}$$

by Proposition 2.14. By Proposition 2.6, we have

$$\mu_M(Y)=\mu_{M'}(Y_{k'}). \tag{14}$$

As k is a perfect field, the scheme $Y_{k'}$ is reduced by [**15**, Proposition 3.2.7]. So $\mathcal{O}_{Y_{k'},Y'}$ is a reduced Artinian local ring, which is a field (cf. [**1**, Proposition 8.9], the maximal ideal of $\mathcal{O}_{Y_{k'},Y'}$ is empty). So we have $\ell_{\mathcal{O}_{Y_{k'},Y'}}(\mathcal{O}_{Y_{k'},Y'})=1$. By Proposition 2.3, we have

$$\mu_{M'}(Y_{k'})=\sum_{Y'\in\mathcal{C}(Y_{k'})}\mu_{M'}(Y'). \tag{15}$$

So we obtain

$$\begin{aligned}&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))W_{\mathscr{T}_Y}(M)\\ \geqslant&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))\sum_{Y'\in\mathcal{C}(Y_{k'})}\mu_{M'}(Y')\end{aligned}$$

by the inequalities (12), (13), (14), and (15).

On the other hand, we have

$$\begin{aligned}&\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})}\deg(Y')\\ \leqslant&\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})} i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))\deg(Y')\\ =&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}\sum_{Y'\in\mathcal{C}(Y_{k'})} i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))\deg(Y')\\ =&\deg(X_{1,k'})\cdots\deg(X_{r,k'})\\ =&\deg(X_1)\cdots\deg(X_r),\end{aligned}\tag{16}$$

where the first equality is obtained from Proposition 2.5, the second equality comes from Bézout Theorem (Theorem 2.2), and the last equality is gotten from the fact that the degree of a closed subscheme of $\mathbb{P}(E)$ is invariant under the fields extension. So we have

$$\#k'\geqslant\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})}\deg(Y')$$

by the inequality (16).

We denote by $\mathcal{D}(M)$ the sub-set of $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$ such that M appears as a descendant of Y in the intersection tree $\mathscr{T}_Y$ considered as a scheme.

The component M admits a regular k'-point. Since k is perfect, by Proposition 2.3,the component M' admits a k'-rational point P of multiplicity 1, which is regular for $M_{k'}$ is pure dimensional. By Proposition 2.18, we obtain that there exists a closed k'-linear subscheme of $\mathbb{P}(E_{k'})$ of dimension $n-\dim(Y)=n-\dim(Y_{k'})$ which intersects all the $Y'\in\bigcup\limits_{Y\in\mathcal{D}(M)}\mathcal{C}(Y_{k'})$ properly at the point P or at some components which do not contain P. In this case, this closed k'-linear subscheme of $\mathbb{P}(E_{k'})$ only intersects M' at the neighbourhood of this regular k'-point of M'. By Proposition 2.9, we can find a cylinder $M^0_{k'}$ of dimension $n-\dim(Y)+\dim(M)=n-\dim(Y')+\dim(M')$ whose direction is defined by this closed k'-linear subscheme of $\mathbb{P}(E_{k'})$, such that it intersects all the $Y'\in\bigcup\limits_{Y\in\mathcal{D}(M)}\mathcal{C}(Y_{k'})$ properly at the component M' or at some irreducible components which do not contain M'. In addition, we hve

$$\mu_{M'}(Y')=i(M';Y'\cdot M^0_{k'};\mathbb{P}(E_{k'}))$$

for every irreducible component $Y'\in\mathcal{C}(Y_{k'})$, $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$.

By Lemma 2.4 and Proposition 2.5, we have

$$i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))=i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'})), \tag{17}$$

where $Y'\in\mathcal{C}(Y_{k'})$. So we obtain

$$\begin{aligned}&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))\sum_{Y'\in\mathcal{C}(Y_{k'})}\mu_{M'}(Y')\\ =&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}\sum_{Y'\in\mathcal{C}(Y_{k'})} i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))i(M';Y'\cdot M^0_{k'};\mathbb{P}(E_{k'}))\end{aligned}$$

by the equality (17). By the definition of $\mathscr{Z}_*$ (Definition 4.2), the irreducible components in $\mathcal{C}(X_1\cdot\ldots\cdot X_r)\smallsetminus\mathcal{D}(M)$ do not contain M. So the cylinder $M^0_{k'}$ does not intersect the irreducible components of the intersection $X_{1,k'}\cdot\ldots\cdot X_{r,k'}$ in $\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})\smallsetminus\{N\in\mathcal{C}(Y_{k'})|\ Y\in\mathcal{D}(M)\}$ at the component M'. Then by the associativity of proper intersection ((ii) of Proposition 2.1), we have

$$\begin{aligned}&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}\sum_{Y'\in\mathcal{C}(Y_{k'})} i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))i(M';Y'\cdot M^0_{k'};\mathbb{P}(E_{k'}))\\ =&\ i(M';X_{1,k'}\cdot\ldots\cdot X_{r,k'}\cdot M^0_{k'};\mathbb{P}(E_{k'})).\end{aligned}$$

This is the end of the proof of Lemma 4.4, which proves Proposition 4.3 for the case of $s=1$. □

Step 2-2: from the case where the maximal depth is $s-1$ to the case where maximal depth is s. - In order to prove the Proposition 4.3, we run a mathematical induction on the maximal depth of the intersection trees $\mathscr{T}_Y$, where $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$. We recall that we take a finite Galois extension k'/k such that

$$\#k'\geqslant\delta^{\sum\limits_{i=1}^{r}\dim(X_i)-(r-1)(n-1)}$$

et que $M^{\mathrm{reg}}(k')\neq\emptyset$, où $n=\mathrm{rk}_k(E)-1$.

Proof of Proposition 4.3. — In this proof, we keep all the notations in the proof of Lemma 4.4. We run a mathematical induction on the maximal depth s of the intersection trees $\mathscr{T}_Y$, où $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$. The case of $s=1$ is proved in Lemma 4.4.

In an intersection tree, a child of a vertex is of codimension larger than or equal to 1 in this vertex, so we obtain that the maximal value of s is $\dim(Y)$. For every $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$, we have $\dim(Y)=\sum\limits_{i=1}^{r}\dim(X_i)-(r-1)n$.

Now we suppose that the assertion is proved for the case where the maximal depth of $\mathscr{T}_Y$ is $s-1$ for all the $Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)$. Next, we prove the case where the maximal depth of all these $\{\mathscr{T}_Y\}_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}$ is s. For all $N\in\mathcal{C}_*$, we denote by $\widetilde{N}$ the label of N in the intersection tree $\mathscr{T}_Y$.

For all the $N\in\mathcal{C}_*$, by the condition $\#k'\geqslant\delta^{\sum\limits_{i=1}^{r}\dim(X_i)-(r-1)(n-1)}$, we obtain the inequality

$$\#k'\geqslant\deg(N)\deg(\widetilde{N})$$

by Bézout Theorem (Theorem 2.2). So by Lemma 4.4, we can use the induction hypothesis to all the intersection sub-trees of $\{\mathscr{T}_Y\}_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}$, whose roots are the vertices in $\mathcal{C}_1$. By the induction hypothesis and Definition 4.2, for every $Y\in\mathcal{D}(M)$, we can find a cylinder Z_Y in $\mathbb{P}(E_{k'})$ of dimension $n-\dim(Y)+\dim(M)$, such that $Y_{k'}$, $\widetilde{Y}_{k'}$ and Z_Y intersect properly at M', and

$$W_{\mathscr{T}_Y}(M)=\sum_{Y'\in\mathcal{C}(Y\cdot\widetilde{Y})} i(Y';Y\cdot\widetilde{Y};\mathbb{P}(E))W_{\mathscr{T}_{Y'}}(M)\geqslant i(M';Y_{k'}\cdot\widetilde{Y}_{k'}\cdot Z_Y;\mathbb{P}(E_{k'})), \tag{18}$$

where $\mathscr{T}_{Y'}$ is the intersection sub-tree whose root is $Y'\in\mathcal{C}(Y\cdot\widetilde{Y})$.

Next, we will estimate the intersection multiplicity $i(M';Y_{k'}\cdot\widetilde{Y}_{k'}\cdot Z_Y;\mathbb{P}(E_{k'}))$. Since k est un corps parfait, le schéma $Y_{k'}$ est réduit. D'où l'on a

$$\begin{aligned}
W_{\mathscr{T}_Y}(M) &\geqslant i(M';Y_{k'}\cdot\widetilde{Y}_{k'}\cdot Z_Y;\mathbb{P}(E_{k'}))\\
&\geqslant \mu_{M'}(Y_{k'})\mu_{M'}(\widetilde{Y}_{k'})\mu_{M'}(Z_Y)\quad\text{(la proposition 2.14)}\\
&\geqslant \mu_{M'}(Y_{k'})\\
&= \sum_{Y'\in\mathcal{C}(Y_{k'})}\mu_{M'}(Y')\quad\text{(la proposition 2.3)}
\end{aligned}\tag{19}$$

by the inequality (18), for the ring $\mathcal{O}_{Y_{k'},Y'}$ is a reduced Artinian local ring, which have to be a field (cf. [**1**, Proposition 8.9], the maximal ideal of $\mathcal{O}_{Y_{k'},Y'}$ est empty). So we obtain

$$\begin{aligned}
&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))W_{\mathscr{T}_Y}(M)\\
\geqslant&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))\sum_{Y'\in\mathcal{C}(Y_{k'})}\mu_{M'}(Y')\\
=&\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)}\sum_{Y'\in\mathcal{C}(Y_{k'})} i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))\mu_{M'}(Y')\\
=&\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})} i(Y';X_{1,k'}\cdot\ldots\cdot X_{r,k'};\mathbb{P}(E_{k'}))\mu_{M'}(Y')
\end{aligned}$$

by the inequality (19), Lemma 2.4 and Proposition 2.5.

On the other hand, by the inequality (16), we have

$$\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})}\deg(Y')\leqslant\#k'.$$

The component M admits a regular k'-point. Since k is perfect, by Proposition 2.3, the component M' admits a k'-rational point P of multiplicity 1, which is regular for $M_{k'}$ is pure dimensional. By Proposition 2.18, we obtain that there exists a closed k'-linear subscheme of $\mathbb{P}(E_{k'})$ of dimension $n-\dim(Y)=n-\dim(Y_{k'})$ which intersect all $Y'\in\bigcup\limits_{Y\in\mathcal{D}(M)}\mathcal{C}(Y_{k'})$ properly at the point P or at the components which do not contain P. In this case, this closed k'-linear subscheme of $\mathbb{P}(E_{k'})$ only intersect M' at the neighbourhood of P. By Proposition 2.9, we can find a cylinder $M_{k'}^0$ of dimension $n-\dim(Y)+\dim(M)=n-\dim(Y')+\dim(M')$ whose direction if defined by this closed k'-linear subscheme of $\mathbb{P}(E_{k'})$, such that it intersects all the $Y' \in \bigcup\limits_{Y\in\mathcal{D}(M)} \mathcal{C}(Y_{k'})$ properly at the component M' or at the components which do not contain M'. In addition, we have

$$\mu_{M'}(Y') = i(M'; Y' \cdot M^0_{k'}; \mathbb{P}(E_{k'}))$$

for every irreducible $Y' \in \mathcal{C}(Y_{k'})$. So we obtain

$$\begin{aligned}&\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})} i(Y'; X_{1,k'} \cdot \ldots \cdot X_{r,k'}; \mathbb{P}(E_{k'}))\mu_{M'}(Y')\\ =&\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})} i(Y'; X_{1,k'} \cdot \ldots \cdot X_{r,k'}; \mathbb{P}(E_{k'}))i(M'; Y' \cdot M^0_{k'}; \mathbb{P}(E_{k'}))\end{aligned}$$

by the equality (17). By the definition of $\mathcal{Z}_*$ (Definition 4.2), the irreducible components in $\mathcal{C}(X_1 \cdot \ldots \cdot X_r) \smallsetminus \mathcal{D}(M)$ do not contain M. So the cylinder $M^0_{k'}$ does not intersect the irreducible components of the intersection $X_{1,k'} \cdot \ldots \cdot X_{r,k'}$ in $\mathcal{C}(X_{1,k'} \cdot \ldots \cdot X_{r,k'}) \smallsetminus \{N \in \mathcal{C}(Y_{k'})|\ Y \in \mathcal{D}(M)\}$ at the component M'. So by the associativity of the proper intersection ((ii) of Proposition 2.1), we have

$$\begin{aligned}&\sum_{Y'\in\mathcal{C}(X_{1,k'}\cdot\ldots\cdot X_{r,k'})} i(Y'; X_{1,k'} \cdot \ldots \cdot X_{r,k'}; \mathbb{P}(E_{k'}))i(M'; Y' \cdot M^0_{k'}; \mathbb{P}(E_{k'}))\\ =\ & i(M'; X_{1,k'} \cdot \ldots \cdot X_{r,k'} \cdot M^0_{k'}; \mathbb{P}(E_{k'})).\end{aligned}$$

Then we prove Proposition 4.3 for the case where the maximal depth is s. This is the end of the proof of Proposition 4.3. □

A consequence of Theorem 3.2. — We have proved Theorem 3.2. Next, we are going to deduce a consequence of Theorem 3.2, which gives a global upper bound of the multiplicities of the vertices in $\mathcal{Z}_*$. This upper bound will be useful in the proof of the principle theorem (Theorem 5.1 below).

Definition 4.5. — Let s be a positive integer. We denote by $\mathcal{C}'_s$ (resp. $\mathcal{Z}'_s$, $\mathcal{C}'_*$, et $\mathcal{Z}'_*$) the set of labels of $\mathcal{C}_s$ (resp. $\mathcal{Z}_s$, $\mathcal{C}_*$, et $\mathcal{Z}_*$), see Definition 4.1 and Definition 4.2 for the definitions of $\mathcal{C}_s$, $\mathcal{Z}_s$, $\mathcal{C}_*$, et $\mathcal{Z}_*$.

With all the above notations, if all the non-empty labels in $\mathscr{T}_Y$ have the same dimension, for all the vertices in $\mathcal{Z}_*$, we have the following corollary which is a global description of their multiplicities in $X_1, \ldots, X_r$.

Proposition 4.6. — *With all the notations and conditions in Theorem 3.2, we suppose that all the non-empty elements in $\mathcal{C}'_*$ have the same dimension. Then we have*

$$\sum_{Z\in\mathcal{Z}_s}\left(\prod_{i=1}^{r}\mu_Z(X_i)\right)\deg(Z) \leqslant \prod_{i=1}^{r}\deg(X_i)\prod_{j=0}^{s-1}\max_{\widetilde{Y}\in\mathcal{C}'_j}\{\deg(\widetilde{Y})\}.$$

In particular, if $s = 0$, we define $\prod\limits_{j=0}^{s-1}\max\limits_{Y'\in\mathcal{C}'_j}\{\deg(Y')\} = 1$.

Proof. — Since all these $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$ are of the same dimension and their labels are also of the same dimension, those vertices of depth 1 in these $\mathscr{T}_Y$ are of the same dimension because Y intersects its label properly for every $Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)$. By the same argument as above, for a fixed position integer s, the vertices in $\mathcal{C}_s$ are of the same dimension.

In order to prove this proposition, first we run a mathematical induction on the depth s to prove the inequality.

$$\sum_{Z \in \mathcal{C}_s} \left(\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) W_{\mathscr{T}_Y}(Z) \right) \deg(Z)$$
$$\leqslant \prod_{i=1}^{r} \deg(X_i) \prod_{j=0}^{s-1} \max_{\widetilde{Y} \in \mathcal{C}'_j} \{\deg(\widetilde{Y})\}.$$

By Bézout Theorem (Theorem 2.2), we have

$$\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y_i; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) \deg(Y) = \deg(X_1) \deg(X_2) \cdots \deg(X_r),$$

which proves the case of $s = 0$.

Now we suppose that the case of depth $s - 1$ is proved. For the case of depth s, we have

$$\prod_{i=1}^{r} \deg(X_i) \prod_{j=0}^{s-1} \max_{\widetilde{Y} \in \mathcal{C}'_j} \{\deg(\widetilde{Y})\}$$
$$\geqslant \max_{\widetilde{Y} \in \mathcal{C}'_{s-1}} \{\deg(\widetilde{Y})\} \sum_{Z \in \mathcal{C}_{s-1}} \left(\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) W_{\mathscr{T}_Y}(Z) \right) \deg(Z)$$

by Bézout Theorem (Theorem 2.2). For every vertex Z, we denote by $\widetilde{Z}$ the label of Z. So we obtain

$$\max_{\widetilde{Y} \in \mathcal{C}'_{s-1}} \{\deg(\widetilde{Y})\} \sum_{Z \in \mathcal{C}_{s-1}} \left(\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) W_{\mathscr{T}_Y}(Z) \right) \deg(Z)$$
$$\geqslant \sum_{Z \in \mathcal{C}_{s-1}} \left(\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) W_{\mathscr{T}_Y}(Z) \right) \deg(Z) \deg(\widetilde{Z})$$
$$= \sum_{Z \in \mathcal{C}_{s-1}} \left(\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) W_{\mathscr{T}_Y}(Z) \right) \cdot \sum_{Z' \in \mathcal{C}(Z \cdot \widetilde{Z})} i(Z'; Z \cdot \widetilde{Z}; \mathbb{P}(E)) \deg(Z')$$
$$= \sum_{Z' \in \mathcal{C}_s} \left(\sum_{Y \in \mathcal{C}(X_1 \cdot \ldots \cdot X_r)} i(Y; X_1 \cdot \ldots \cdot X_r; \mathbb{P}(E)) W_{\mathscr{T}_Y}(Z') \right) \deg(Z'),$$

which proves the case of depth s.

Next, we need to prove the inequality

$$\sum_{Z\in\mathscr{Z}_s}\left(\prod_{i=1}^{r}\mu_Z(X_i)\right)\deg(Z)$$
$$\leqslant \sum_{Z\in\mathcal{C}_s}\left(\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))W_{\mathscr{T}_Y}(Z)\right)\deg(Z).$$

For a fixed $Z\in\mathscr{Z}_s$, By Theorem 3.2, we obtain

$$\sum_{Y\in\mathcal{C}(X_1\cdot\ldots\cdot X_r)} i(Y;X_1\cdot\ldots\cdot X_r;\mathbb{P}(E))W_{\mathscr{T}_Y}(Z)\geqslant \mu_Z(X_1)\cdots\mu_Z(X_r).$$

By Definition 4.2, the set $\mathscr{Z}_s$ is a sub-est of $\mathcal{C}_s$ for each $s\geqslant 0$. So we obtain the result. □

5. Estimate of multiplicities in a hypersurface

The following result is an upper bound of a counting of multiplicities in a reduced projective hypersurface over the finite field $\mathbb{F}_q$. This upper bound can be considered as a description of the complexity of the singular locus of this reduced projective hypersurface.

Theorem 5.1. — *Let* $X\hookrightarrow\mathbb{P}^n_{\mathbb{F}_q}$ *be a reduced projective hypersurface of degree* δ, *where* $\dim(X^{\mathrm{sing}})=s$. *Then we have*

$$\begin{aligned}\sum_{\xi\in X(\mathbb{F}_q)}\mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1} \leqslant\ & \delta(\delta-1)^{n-s-1}(q^s+q^{s-1}+\cdots+1)+\\ & \delta(\delta-1)^{n-s}(q^{s-1}+q^{s-2}+\cdots+1)+\cdots\\ & +\delta(\delta-1)^{n-1},\end{aligned}$$

where $\mu_\xi(X)$ *is the multiplicity of* ξ *in* X *(see §2.3 for the definition).*

Before the proof of Theorem 5.1, we need to introduce some special properties on the multiplicity of a point in a hypersurface section, and introduce a method to construct some useful intersection trees for this counting multiplicities problem.

5.1. Multiplicities in a hypersurface section. — Let k be a field, and $f\in k[T_0,\ldots,T_n]$ be a non-zero homogeneous polynomial of degree δ. We say that the scheme

$$X=\mathrm{Proj}\,(k[T_0,\ldots,T_n]/(f))$$

is a *projective hypersurface* (or *hypersurface* for simplicity) of $\mathbb{P}^n_k$ defined by the polynomial f. We can prove that X is a closed subscheme of degree δ of $\mathbb{P}^n_k$ (cf. [**11**, Proposition 7.6, Chap. I]).

We are going to introduce some special properties on the multiplicity of a point in a projective hypersurface.

Proposition 5.2 **([13], Example 2.70 (2)).** — *Let X be a hypersurface of $\mathbb{P}^n_k$ defined by a non-zero homogeneous polynomial f, $\xi \in X(\overline{k})$, and $\mathfrak{m}_\xi$ be the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^n_k,\xi}$. Let $H_\xi(s)$ be the local Hilbert-Samuel function of X at the point ξ (see §2.2 for the definition). If the image of f in $\mathcal{O}_{\mathbb{P}^n_k,\xi}$ appears in the set $\mathfrak{m}_\xi^r \smallsetminus \mathfrak{m}_\xi^{r+1}$. Then we have*

$$H_\xi(s) = \binom{n+s-1}{s} - \binom{n+s-r-1}{s-r}.$$

In particular, we have $\mu_\xi(X) = r$.

Let $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ be an index, we define $|I| = i_0 + \cdots + i_n$. Let $g(T_0, \ldots, T_n)$ be a non-zero homogeneous polynomial of degree δ, then we can expand the polynomial $g(T_0+S_0, T_1+S_1, \ldots, T_n+S_n) \in k[T_0, T_1, \ldots, T_n, S_0, S_1, \ldots, S_n]$ as

$$\begin{aligned} & g(T_0+S_0, \ldots, T_n+S_n) \\ = \; & g(T_0, \ldots, T_n) + \sum_{\alpha=1}^{\delta} \sum_{\substack{I=(i_0,\ldots,i_n)\in\mathbb{N}^{n+1} \\ |I|=\alpha}} g^I(T_0, \ldots, T_n) S_0^{i_0} \cdots S_n^{i_n}, \end{aligned}$$

where $g^I(T_0, \ldots, T_n)$ is a homogeneous polynomial of degree $\delta - |I|$ or zero. We denote by $\mathcal{D}^\alpha(g)$ the set of the polynomials $g^I(T_0, \ldots, T_n)$ defined above, where $|I| = \alpha \geqslant 1$.

For an integer $1 \leqslant \alpha \leqslant \delta$, we define $\mathcal{T}^\alpha(g)$ as the k-vector space generated by the elements in $\mathcal{D}^\alpha(g)$. For every non-zero $g \in \mathcal{T}^\alpha(g)$ non-nul, g defines a projective hypersurface of degree $\delta - \alpha$ of $\mathbb{P}^n_k$.

In addition, we define $\mathcal{D}^0(g) = \{g\}$ and $\mathcal{T}^0(g) = k \cdot g$.

Remark 5.3. — The elements in $\mathcal{D}^1(g)$ are those

$$\frac{\partial g}{\partial T_0}, \frac{\partial g}{\partial T_1}, \ldots, \frac{\partial g}{\partial T_n},$$

which are homogeneous polynomials of degree $\delta - 1$ or zero. If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > \delta$, these elements in $\mathcal{D}^\alpha(g)$ have the form of

$$\frac{1}{i_0! \cdots i_n!} \cdot \frac{\partial^{i_0+\cdots+i_n} g(T_0, \ldots, T_n)}{\partial T_0^{i_0} \cdots \partial T_n^{i_n}},$$

where $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ is an index with $i_0 + \cdots + i_n = \alpha$. In addition, the k-vector space $\mathcal{T}^\alpha(g)$ is the space of directional derivatives of order α of $g(T_0, \ldots, T_n)$.

With all the above notations, we have a direct consequence of Proposition 5.2 below.

Corollary 5.4. — *Let $X \hookrightarrow \mathbb{P}^n_k$ be the projective hypersurface defined by a homogeneous polynomial $f \neq 0$ of degree δ, $\xi \in X(\overline{k})$, and α be an integer such that $0 \leqslant \alpha \leqslant \mu_\xi(X) - 1$. Then for every non-zero $g \in \mathcal{T}^\alpha(f)$, the point ξ is contained in the hypersurface defined by g. There exists a non-zero $g' \in \mathcal{T}^{\mu_\xi(X)}(f)$, such that ξ is not contained in the hypersurface defined by g'.*

Proof. — Let $\xi = [a_0 : \cdots : a_n]$. By Proposition 5.2, the image of f in the local ring $\mathcal{O}_{\mathbb{P}^n_k,\xi}$ is in the set $\mathfrak{m}_\xi^{\mu_\xi(X)} \smallsetminus \mathfrak{m}_\xi^{\mu_\xi(X)+1}$, which means that this image is in $\mathfrak{m}_\xi^{\mu_\xi(X)}$ but not in $\mathfrak{m}_\xi^{\mu_\xi(X)+1}$. The fact that the image in $\mathfrak{m}_\xi^{\mu_\xi(X)}$ means that for every polynomial $f^I(T_0,\ldots,T_n)$ defined above with $0 \leqslant |I| \leqslant \mu_\xi(X)-1$, we have $f^I(a_0,\ldots,a_n)=0$. The fact that the image is not in $\xi \notin \mathfrak{m}_\xi^{\mu_\xi(X)+1}$ means that there exists a polynomial $f^I(T_0,\ldots,T_n)$ with $|I| = \mu_\xi(X)$ such that $f^I(a_0,\ldots,a_n) \neq 0$. Then we have the assertion. □

A direct consequence of Corollary 5.4 is below.

Corollary 5.5. — *Let $X \hookrightarrow \mathbb{P}^n_k$ be the projective hypersurface defined by a homogeneous polynomial f of degree δ, and $\eta \in X$ be a schematic point. For an integer $\alpha \in \{0,\ldots,\delta\}$, let X' be the hypersurface of $\mathbb{P}^n_k$ defined by a non-zero element $g \in \mathcal{T}^\alpha(f)$, where $\alpha < \mu_\eta(X)$. Then the multiplicity $\mu_\eta(X')$ is at least $\mu_\eta(X)-\alpha$. In addition, there exists at least an element in $\mathcal{T}^\alpha(f)$ which defines a hypersurface X'' of $\mathbb{P}^n_k$, such that the multiplicity $\mu_\eta(X'')$ is equal to $\mu_\xi(X)-\alpha$.*

Proof. — Let $Z = \overline{\{\eta\}}$ considered as an integral scheme, and $\xi \in Z^{\mathrm{reg}}(\overline{k})$.By Corollay 2.11, we have $\mu_\xi(X) = \mu_\eta(X)$. Since $Z^{\mathrm{reg}}(\overline{k})$ is dense in Z (cf. [**11**, Corollary 8.16, Chap. II]), we have the assertion. □

Remark 5.6. — By Corollary 5.4, if $X \hookrightarrow \mathbb{P}^n_k$ is a hypersurface defined by a non-zero homogeneous polynomial of degree δ, the multiplicity of a closed point in X is at most δ.

Definition 5.7. — We say that the projective hypersurface defined by $g \in \mathcal{T}^\alpha(f)$ is a *derivative hypersurface* of order α of the hypersurface defined by f.

5.2. Construction of intersection trees from a hypersurface. — In order to study the counting multiplicities problem in a hypersurface, we need to construct some intersection trees originated from this hypersurface. We can study the multiplicity of a rational point by the upper bound of vertices in the constructed intersection trees. In this part, let k be a field, X be a k-scheme, and k'/k be an extension of fields. We denote by $X_{k'}$ the k'-scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ for simplicity.

First, we introduce the following lemma, which will be used in the construction of the roots of these intersection trees.

Lemma 5.8. — *Let k be a field, and $g \in k[T_0,\ldots,T_n]$ be a non=-zero homogeneous polynomial. We denote by $V(g)$ the projective hypersurface of $\mathbb{P}^n_k$ defined by g. Let $f \neq 0$ be a homogeneous polynomial of degree δ. If the dimension of the singular locus of $V(f)$ is s, where $0 \leqslant s \leqslant n-2$. Then there exists a finite extension k'/k and a family of $g_1,\ldots,g_{n-s-1} \in \mathcal{T}^1(f) \otimes_k k'$, such that*

$$\dim\left(V(f)_{k'} \cap V(g_1) \cap \cdots \cap V(g_{n-s-1})\right) = s.$$

In another words, the scheme $V(f)_{k'} \cap V(g_1) \cap \cdots \cap V(g_{n-s-1})$ is a complete intersection.

Proof. — Since $V(f)$ has singular points, the degree of f is larger than or equal to 2. First, we suppose that k' is an algebraic closure of the field k, then the cardinality of k' is infinite. If we prove this assertion for such a field k', there exists a finite extension of the field k which satisfies the requirement, too. In the rest part of this proof, all the schemes which we considere are over this algebraic closure of the field k.

By Jacobian criterion (cf. [**15**, Theorem 4.2.19]), we have

$$\dim\left(V(f)\cap\bigcap_{g\in\mathcal{T}^1(f)}V(g)\right)=\dim\left(V(f)_{k'}\cap\bigcap_{g\in\mathcal{T}^1(f)\otimes_k k'}V(g)\right)=s.$$

We denote by V_t the scheme

$$V(f)_{k'}\cap V(g_1)\cap\cdots\cap V(g_t)$$

for simplicity. For every $t\in\{0,1,\ldots,n-s-1\}$, we will prove that there exists $g_1,\ldots,g_t\in\mathcal{T}^1(f)\otimes_k k'$ (if $t=0$, we define that the set of these $\{g_1,\ldots,g_t\}$ is empty), such that V_t is a complete intersection. If we have the above assertion, we prove the original result.

We run a mathematical induction on the integer t defined above, where $0\leqslant t\leqslant n-s-1$. Since $V_0=V(f)_{k'}$ is a hypersurface which is definitely a complete intersection, the case of $t=0$ is proved by definition directly.

If we have already found the $g_1,\ldots,g_t\in\mathcal{T}^1(f)\otimes_k k'$, such that V_t is a complete intersection, where $0\leqslant t\leqslant n-s-2$. Then for every $U\in\mathcal{C}(V_t)$, we have $\dim(U)=n-t-1$.

If for every $h\in\mathcal{T}^1(f)\otimes_k k'$, there always exists a $U\in\mathcal{C}(V_t)$, such that $U\subseteq V(h)$. Then we obtain

$$U\subsetneq V(f)_{k'}\cap\left(\bigcap_{g\in\mathcal{T}^1(f)\otimes_k k'}V(g)\right),$$

which contradicts with $\dim(V(f)_{k'}^{\mathrm{sing}})=s<n-t-1=\dim(U)$.

Then for every $U\in\mathcal{C}(V_t)$, we can find a $g_U\in\mathcal{T}^1(f)\otimes_k k'$, such that

$$U\nsubseteq V(g_U).$$

We define

$$L(U)=\{h\in\mathcal{T}^1(f)\otimes_k k'|U\subseteq V(h)\}.$$

Then in this case, for every $U\in\mathcal{C}(V_t)$, $L(U)$ is a proper k'-vector sub-space of $\mathcal{T}^1(f)\otimes_k k'$. Since the cardinality of k' is infinite and the cardinality of $\mathcal{C}(V_t)$ is finite, there exists a vector $h\in\mathcal{T}^1(f)\otimes_k k'$, such that

$$h\notin\bigcup_{U\in\mathcal{C}(V_t)}L(U).$$

Then for every $U\in\mathcal{C}(V_t)$, we have $U\nsubseteq V(h)$. So $V(h)\cap V_t$ is a complete intersection.

So for every $0\leqslant t\leqslant n-s-1$, we can find $g_1,\ldots,g_t$ which satisfy the requirement. This is the end of the proof. □

Let $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ be the projective hypersurface defined by the non-zero homogeneous polynomial f of degree δ, whose singular locus is of dimension larger that or equal to zero. Let $\mathbb{F}_{q^m}/\mathbb{F}_q$ be a finite extension such that we can find a series of non-zero $g_1, \ldots, g_{n-s-1} \in \mathcal{T}^1(f) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ which satisfy that $X_{\mathbb{F}_{q^m}}, V(g_1), \ldots, V(g_{n-s-1})$ is a complete intersection. The extension $\mathbb{F}_{q^m}/\mathbb{F}_q$ is Galois, for $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) = (\mathbb{Z}/m\mathbb{Z}, +)$. By Lemma 5.8, these $g_1, \ldots, g_{n-s-1} \in \mathcal{T}^1(f) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ exist when the integer m is large enough. Let $\xi \in X(\mathbb{F}_q)$, and $\xi' = \xi \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$. Then we have $\mu_\xi(X) = \mu_{\xi'}(X_{\mathbb{F}_{q^m}})$ by Proposition 2.6.

We denote by $X_{i,\mathbb{F}_{q^m}}$ the hypersurface $V(g_i)$ defined by g_i over $\mathbb{F}_{q^m}$, where $i = 1, \ldots, n-s-1$. By Jacobian criterion (cf. [**15**, Theorem 4.2.19]), we obtain $X^{\mathrm{sing}}_{\mathbb{F}_{q^m}} \subseteq X_{\mathbb{F}_{q^m}} \cap X_{1,\mathbb{F}_{q^m}} \cap \cdots \cap X_{n-s-1,\mathbb{F}_{q^m}}$.

For every integral subscheme Y of $X_{\mathbb{F}_{q^m}}$, we denote by $Y^{(a)}$ the locus of the points in Y whose multiplicities are equal to $\mu_Y(X_{\mathbb{F}_{q^m}})$, and by $Y^{(b)}$ the locus of points in Y whose multiplicities are larger than or equal to $\mu_Y(X_{\mathbb{F}_{q^m}})+1$. In addition, we denote by $Y^{(a)}(\mathbb{F}_q)$ (*resp.* $Y^{(b)}(\mathbb{F}_q)$) the set of $\mathbb{F}_{q^m}$-rational point of $Y^{(a)}$ (*resp.* $Y^{(b)}$) which appear in the inverse images of the elements in $\mathbb{P}^n_{\mathbb{F}_q}(\mathbb{F}_q)$ with respect to the closed immersion from Y in $\mathbb{P}^n_{\mathbb{F}_{q^m}}$ under the base change $\mathbb{P}^n_{\mathbb{F}_{q^m}} \to \mathbb{P}^n_{\mathbb{F}_q}$ (see Definition 2.16). So we have $Y(\mathbb{F}_q) = Y^{(a)}(\mathbb{F}_q) \bigsqcup Y^{(b)}(\mathbb{F}_q)$.

By Corollary 2.11, we obtain that $Y^{(a)}$ is dense in Y si $Y^{(a)} \neq \emptyset$, and $Y^{(b)}$ is of dimension less than or equal to $\dim(Y) - 1$.

Next, we construct a family of intersection tree $\{\mathscr{T}_Y\}$, where $Y \in \mathcal{C}(X_{\mathbb{F}_{q^m}} \cdot X_{1,\mathbb{F}_{q^m}} \cdot \ldots \cdot X_{n-s-1,\mathbb{F}_{q^m}})$. The root of the intersection tree $\mathscr{T}_Y$ is Y.

In order to construct the vertices of depth larger than or equal to 1, let U be a vertex already constructed in these intersection trees $\{\mathscr{T}_Y\}$. We consider the vertex U as an integral scheme. We need to consider the properties of $U(\mathbb{F}_q)$, where $U(\mathbb{F}_q)$ is defined in Definition 2.16. If $U^{(b)}(\mathbb{F}_q) = \emptyset$, the vertex U is a leaf in these intersection trees.

If $U^{(b)}(\mathbb{F}_q) \neq \emptyset$, then we have $\mu_U(X_{\mathbb{F}_{q^m}}) < \delta$. By Corollary 5.4, we can find a $h \in \mathcal{T}^{\delta-\mu_U(X_{\mathbb{F}_{q^m}})}(f) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$, such that the hypersurface defined by the polynomial h intersects U properly. Of course we have $\deg(h) \leqslant \delta - 1$. In this case, we define $V(h)$ as the label of U.

The weights of edges are the intersection multiplicities of the respective intersections.

For the above construction, all the labels mentioned above are of dimension $n-1$, so the vertices in $\mathcal{C}_w$ are of dimension $n-w-2$, where $1 \leqslant w \leqslant n-2$ is an integer.

The following lemma is a property of the set $\mathcal{Z}_*$ (see Definition 4.2), which will be useful in the proof of Theorem 5.1. This is the reason why we define the sub-set $\mathcal{Z}_*$ of $\mathcal{C}_*$.

Lemma 5.9. — *With all the notations and constructions above, for every* $\xi \in X^{\mathrm{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)$, *there exists at least one* $Z \in \mathcal{Z}_*$ *such that* $\xi \in Z^{(a)}(\mathbb{F}_q)$, *where* $\mathcal{Z}_*$ *is defined in Definition 4.2.*

Proof. — Let $Y \in \mathcal{C}(X_{\mathbb{F}_{q^m}} \cdot X_{1,\mathbb{F}_{q^m}} \cdot \ldots \cdot X_{n-s-1,\mathbb{F}_{q^m}})$. By the above construction of intersection trees $\mathscr{T}_Y$, for every $\xi \in X^{\text{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)$, we have $\xi \in Y(\mathbb{F}_q)$ for at least one $Y \in \mathcal{C}(X_{\mathbb{F}_{q^m}} \cdot X_{1,\mathbb{F}_{q^m}} \cdot \ldots \cdot X_{n-s-1,\mathbb{F}_{q^m}})$.

Let $\mathcal{C}_m$ be as in Definition 4.1. If $\mathcal{C}_{n-2} \neq \emptyset$, the vertices in $\mathcal{C}_{n-2}$ are some rational points, which must be regular. If $\mathcal{C}_t = \emptyset$ but $\mathcal{C}_{t-1} \neq \emptyset$, the for each $U \in \mathcal{C}_{t-1}$, we have $U^{(b)}(\mathbb{F}_q) = \emptyset$. So for every $\xi \in X^{\text{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)$, there always exists one $Y \in \mathcal{C}_w$, such that $\xi \in Y^{(a)}(\mathbb{F}_q)$.

For a fixed $\xi \in X^{\text{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)$, we take the minimal value w such that there exists a $Y \in \mathcal{C}_w$ verifying $\xi \in Y^{(a)}(\mathbb{F}_q)$. If there exists such a $Y \in \mathcal{Z}_w$, we have the desired assertion. If not, for each $Y \in \mathcal{C}_w$ which satisfies $\xi \in Y^{(a)}(\mathbb{F}_q)$, we always have $Y \notin \mathcal{Z}_w$. Then we can find the maximal positive integer w' which satisfies the following conditions: $w' < w$, and there exists a $Y_0 \in \mathcal{C}_{w'}$ such that $Y \subsetneq Y_0$ but Y does not appear in the descendants of Y_0. If $\xi \in Y_0^{(a)}(\mathbb{F}_q)$, it contradicts with that w is minimal. If $\xi \in Y_0^{(b)}(\mathbb{F}_q)$, then we have $\mu_Y(X_{\mathbb{F}_{q^m}}) = \mu_\xi(X_{\mathbb{F}_{q^m}}) \geqslant \mu_{Y^0}(X_{\mathbb{F}_{q^m}}) + 1$. By the above construction of intersection trees, Y is a descendant of Y_0, which contradicts with the choice of w' is maximal.

To sum up, we prove the assertion. □

5.3. Proof of Theorem 5.1. — With all the preparations above, we are going to prove Theorem 5.1.

Proof of Theorem 5.1. — We take the constuction of the intersection trees whose roots are the elements in $\mathcal{C}(X_{\mathbb{F}_{q^m}} \cdot X_{1,\mathbb{F}_{q^m}} \cdot \ldots \cdot X_{n-s-1,\mathbb{F}_{q^m}})$ in §5.2. By Proposition 2.6, since $\mathbb{F}_q$ is a perfect field, we have $\mu_\xi(X) = \mu_{\xi'}(X_{\mathbb{F}_{q^m}})$, where $\xi \in X(\mathbb{F}_q)$ and $\xi' = \xi \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\mathbb{F}_{q^m}$.

So we obtain

$$\begin{aligned}(20)\qquad &\sum_{\xi\in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1}\\ =&\sum_{\xi\in X^{\text{sing}}(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1}\\ =&\sum_{\xi\in X^{\text{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)} \mu_\xi(X_{\mathbb{F}_{q^m}})(\mu_\xi(X_{\mathbb{F}_{q^m}})-1)^{n-s-1},\end{aligned}$$

where the notation $X^{\text{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)$ is introduced in Definition 2.16.

By Lemma 5.9, for every $\xi \in X^{\text{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)$, we can find a $Z \in \mathcal{Z}_*$ such that $\xi \in Z^{(a)}(\mathbb{F}_q)$. So we obtain

$$\begin{aligned}(21)\qquad &\sum_{\xi\in X^{\text{sing}}_{\mathbb{F}_{q^m}}(\mathbb{F}_q)} \mu_\xi(X_{\mathbb{F}_{q^m}})(\mu_\xi(X_{\mathbb{F}_{q^m}})-1)^{n-s-1}\\ \leqslant&\sum_{t=0}^{s}\sum_{Z\in\mathcal{Z}_t}\sum_{\xi\in Z^{(a)}(\mathbb{F}_q)} \mu_\xi(X_{\mathbb{F}_{q^m}})(\mu_\xi(X_{\mathbb{F}_{q^m}})-1)^{n-s-1}.\end{aligned}$$

By Corollary 5.5, for each $Z \in \mathcal{Z}_*$, we obtain the inequality

$$\mu_Z(X_{\mathbb{F}_{q^m}}) - 1 \leqslant \mu_Z(X_{i,\mathbb{F}_{q^m}}),$$

is verified for all $i = 1, \dots, n-s-1$. So we have the inequality

$$\mu_Z(X_{\mathbb{F}_{q^m}})(\mu_Z(X_{\mathbb{F}_{q^m}}) - 1)^{n-s-1} \leqslant \mu_Z(X_{\mathbb{F}_{q^m}})\mu_Z(X_{1,\mathbb{F}_{q^m}}) \cdots \mu_Z(X_{n-s-1,\mathbb{F}_{q^m}}). \tag{22}$$

By Proposition 4.6 and the inequality (22), we have

$$\begin{aligned} &\sum_{Z \in \mathcal{Z}_t} \mu_Z(X_{\mathbb{F}_{q^m}})(\mu_Z(X_{\mathbb{F}_{q^m}}) - 1)^{n-s-1} \deg(Z) \\ \leqslant &\sum_{Z \in \mathcal{Z}_t} \mu_Z(X_{\mathbb{F}_{q^m}})\mu_Z(X_{1,\mathbb{F}_{q^m}}) \cdots \mu_Z(X_{n-s-1,\mathbb{F}_{q^m}}) \deg(Z) \leqslant \delta(\delta-1)^{n-s+t-1} \end{aligned} \tag{23}$$

for every $t = 0, \dots, s$, since all the labels in $\mathcal{C}'_*$ are of degree less than or equal to $\delta - 1$.

With the inequalities (21) and (23), we have

$$\begin{aligned} &\sum_{t=0}^{s} \sum_{Z \in \mathcal{Z}_t} \sum_{\xi \in Z^{(a)}(\mathbb{F}_q)} \mu_\xi(X_{\mathbb{F}_{q^m}})(\mu_\xi(X_{\mathbb{F}_{q^m}}) - 1)^{n-s-1} \\ = &\sum_{t=0}^{s} \sum_{Z \in \mathcal{Z}_t} \mu_Z(X_{\mathbb{F}_{q^m}})(\mu_Z(X_{\mathbb{F}_{q^m}}) - 1)^{n-s-1} \# Z^{(a)}(\mathbb{F}_q) \\ \leqslant &\sum_{t=0}^{s} \sum_{Z \in \mathcal{Z}_t} \mu_Z(X_{\mathbb{F}_{q^m}})(\mu_Z(X_{\mathbb{F}_{q^m}}) - 1)^{n-s-1} \# Z(\mathbb{F}_q) \\ \leqslant &\sum_{t=0}^{s} \sum_{Z \in \mathcal{Z}_t} \left(\mu_Z(X_{\mathbb{F}_{q^m}})(\mu_Z(X_{\mathbb{F}_{q^m}}) - 1)^{n-s-1} \deg(Z) \# \mathbb{P}^{s-t}(\mathbb{F}_q)\right) \\ = &\sum_{t=0}^{s} \# \mathbb{P}^{s-t}(\mathbb{F}_q) \left(\sum_{Z \in \mathcal{Z}_t} \mu_Z(X_{\mathbb{F}_{q^m}})(\mu_Z(X_{\mathbb{F}_{q^m}}) - 1)^{n-s-1} \deg(Z) \right) \\ \leqslant &\ \delta(\delta-1)^{n-s-1} \# \mathbb{P}^s(\mathbb{F}_q) + \delta(\delta-1)^{n-s} \# \mathbb{P}^{s-1}(\mathbb{F}_q) \\ &+ \cdots + \delta(\delta-1)^{n-1}, \end{aligned} \tag{24}$$

where the inequality in the third line is verified by Proposition 2.17, and the last inequality holds by Lemma 2.15.

By the inequalities (20), (21) and (24), we have the results. □

Remark 5.10. — If $n = 2$, by the similar method to the proof of Theorem 5.1, we obtain the inequality (1) , where we can consider all the closed point of this plane curve essentially. By Theorem 5.1, we have

$$\begin{aligned} \sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X) - 1)^{n-s-1} &\leqslant (s+1)^2 \delta(\delta-1)^{n-s-1} \max\{\delta - 1, q\}^s \\ &\ll_n \delta^{n-s} \max\{\delta - 1, q\}^s \end{aligned}$$

as $s \leqslant n - 2$, which is of the form of Theorem 1.1.

Example 5.11. — Let $X' \hookrightarrow \mathbb{P}^2_{\mathbb{F}_q}$ be a reduced plane curve of degree δ defined by the homogeneous equation $f(T_0, T_1, T_2) = 0$ which only has one f_q-rational singular point of multiplicit δ. Then we can consider $f(T_0, T_1, T_2)$ as a homogeneous polynomial of degree δ in $\mathbb{F}_q[T_0, \ldots, T_n]$. So the homogeneous equation $f(T_0, T_1, T_2) = 0$ defines a reduced hypersurface of degree δ of $\mathbb{P}^n_{\mathbb{F}_q}$ $(n \geqslant 2)$, noted by X this hypersurface. Let $[a_0 : a_1 : a_2]$ be the projective coordinate of this singular point of X'. Then we have

$$\begin{aligned} X^{\text{sing}}(\mathbb{F}_q) &= \{[x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathbb{F}_q}(\mathbb{F}_q) |\ x_0 = a_0, x_1 = a_1, x_2 = a_2\} \cup \\ &\quad \{[x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathbb{F}_q}(\mathbb{F}_q) |\ x_0 = x_1 = x_2 = 0\}, \end{aligned}$$

where all the singular $\mathbb{F}_q$-rational points are of multiplicity δ. Then for the hypersurface X, we obtain

$$\begin{aligned} \sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X) - 1) &= \delta(\delta - 1)q^{n-2} + \delta(\delta - 1)(q^{n-3} + \cdots + 1) \\ &= \delta(\delta - 1)(q^{n-2} + \cdots + 1) \\ &\sim_n \delta^2 q^{n-2}. \end{aligned}$$

Then the order of δ and the order of q in Theorem 5.1 are both optimal pour the case where q is large enough and $\dim(X^{\text{sing}}) = n - 2$.

Remark 5.12. — Let X be a hypersurface of $\mathbb{P}^n_{\mathbb{F}_q}$, where $\dim(X^{\text{sing}}) = s$. By Theorem 5.1, we obtain

$$\begin{aligned} & \sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X) - 1) \\ \leqslant\ & \sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X) - 1)^2 \leqslant \cdots \leqslant \sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X) - 1)^{n-s-1} \\ \leqslant\ & \delta(\delta - 1)^{n-s-1}(q^s + q^{s-1} + \cdots + 1) + \\ & \delta(\delta - 1)^{n-s}(q^{s-1} + q^{s-2} + \cdots + 1) + \cdots + \delta(\delta - 1)^{n-1}. \end{aligned}$$

So we obtain that for every $t \in \{1, \ldots, n - s - 1\}$, we have

$$\sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X) - 1)^t \ll_n \delta^{n-s} q^s$$

when $q \geqslant \delta - 1$.

Let t be an integer with $t \geqslant n - s$, we can construct an example (Example 5.11 for instance), such that

$$\sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X) - 1)^t \sim_n \delta^{t+1} q^s$$

when $q \geqslant \delta - 1$.

Let $f(T) \in \mathbb{R}[T]$ be a polynomial of degree $n - s$, which satisfies $f(1) = 0$ and $f(x) > 0$ for all $x \geqslant 2$. So there exists a constant $C_f > 0$ depending on the polynomial $f(T)$, such that the inequality

$$f(x) \leqslant C_f x(x - 1)^{n-s-1}$$

is verified for every $x \geqslant 1$. Then we have

$$\sum_{\xi \in X(\mathbb{F}_q)} f(\mu_\xi(X)) \leqslant C_f \sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1} \ll_{n,f} \delta^{n-s} \max\{\delta-1, q\}^s.$$

Then the choice of the counting function

$$\mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1}$$

is convenable for describing the complexity of the singular locus of X, where $\xi \in X(\mathbb{F}_q)$.

In order to generalize Theorem 5.1 to the case where X is a general projective scheme, we propose the following conjecture.

Conjecture 5.13. — *Let X be a pure dimensional reduced closed subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$ which is of dimension d and degree δ. If the dimension of its singular locus is s, then we have*

$$\sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X)-1)^{d-s} \ll_n \delta^{d-s+1} q^s.$$

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