FINITE FREE CONVOLUTIONS VIA WEINGARTEN CALCULUS

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Abstract. We present an alternate approach to the finite free convolutions studied by A. Marcus, D. A. Spielman, and N. Srivastava, using Weingarten methods for integration on the unitary and orthogonal groups to directly compute the expected characteristic polynomials of certain randomly rotated matrices. Our methods also immediately yield the quadrature results for these convolution operations, in which the expectations are discretized by replacing the random rotations with random signed permutations.

1. Introduction

In [9], A. Marcus, D. A. Spielman, and N. Srivastava studied three convolution-type operations on polynomials of degree d that are defined in terms of expected characteristic polynomials of certain random $d \times d$ matrices. These operations are called *finite free convolutions*, due to their connection with free convolution in free probability theory, which is expounded upon in [7].

We use the following notation for characteristic polynomials: $c_x(A) = \det(xI - A)$ is the characteristic polynomial of the matrix A in the variable x. The finite free convolutions are defined in terms of expected characteristic polynomials of randomly rotated matrices:

Definition 1.1 ([9])**.** Let A and B be $d \times d$ complex matrices.

(1) If A and B are self-adjoint with characteristic polynomials $p(x)$ and $q(x)$ respectively, the *symmetric additive convolution* of $p(x)$ and $q(x)$ is defined by
$$p(x) \boxplus_d q(x) = \mathbb{E}_U c_x(A + UBU^*)$$
where U is a random $d \times d$ unitary matrix.

(2) If A and B are positive semidefinite with characteristic polynomials $p(x)$ and $q(x)$ respectively, the *symmetric multiplicative convolution* of $p(x)$ and $q(x)$ is defined by
$$p(x) \boxtimes_d q(x) = \mathbb{E}_U c_x(AUBU^*)$$
where U is a random $d \times d$ unitary matrix.

(3) If $p(x)$ and $q(x)$ are the characteristic polynomials of AA^* and BB^* respectively, the *asymmetric additive convolution* of $p(x)$ and $q(x)$ is defined by
$$p(x) \boxplus\!\!\boxplus_d q(x) = \mathbb{E}_{U,V} c_x((A + UBV)(A + UBV)^*)$$
where U and V are independent random $d \times d$ unitary matrices.

Of course, it is not clear *a priori* that these operations are well-defined. It is shown in [9] is that they are, as they only depend on $p(x)$ and $q(x)$, not on the matrices A and B. This is implicit in the following theorem:

Theorem 1.2 ([9, Theorems 1.2 & 1.5 & 1.8])**.** *Let $p(x)$ and $q(x)$ be polynomials of degree d, say $p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i$ and $q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i b_i$. Then we have the following formulae:*

$$p(x) \boxplus_d q(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} a_i b_j; \tag{1}$$

$$p(x) \boxtimes_d q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i \frac{a_i b_j}{\binom{d}{i}}; \tag{2}$$

$$p(x) \boxplus\!\!\boxplus_d q(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k \sum_{i+j=k} \left(\frac{(d-i)!(d-j)!}{d!(d-k)!}\right)^2 a_i b_j. \tag{3}$$

In this paper we present an alternate proof of Theorem 1.2, and we also use similar methods to the same ends with random *orthogonal* matrices instead of random *unitary* matrices. Our approach revolves around the Weingarten calculus for integration on the unitary and orthogonal groups; in particular, we take advantage of the connection between Weingarten functions and the combinatorial representation theory of symmetric groups. Using the orthogonality relations for irreducible characters of S_k in the unitary case and for the zonal spherical functions of the Gelfand pair (S_{2k}, H_k) in the orthogonal case, we obtain combinatorial identities for the relevant Weingarten functions which allow us to greatly reduce the apparent complexity of direct computations of the relevant expected characteristic polynomials.

A very interesting aspect of [9] is what one might call a *quadrature* phenomenon: the random unitary or orthogonal matrices in the definitions of the finite free convolutions can be replaced with random signed permutation matrices, thus discretizing these operations. They isolate a property that they call *minor-orthogonality*, which makes this quadrature work and which is satisfied by the group H_d of signed permutation matrices, also called the *hyperoctahedral* group. Our techniques also yield quadrature results, with a *quadrature property* which plays a similar role as minor-orthogonality.

The paper is organized as follows. In Section 2 we review some preliminaries on permutations and partitions, the combinatorial representation theory of S_k and the Gelfand pair (S_{2k}, H_k), and the Weingarten calculus for integration on the unitary and orthogonal groups. In Section 3 we isolate, from the definitions of the finite free convolutions, the computations we need to handle, and we arrive at our *quadrature property*. Sections 4, 5, and 6 are devoted to the proofs that U_d, O_d, and H_d, respectively, have this essential property. In the final Section 7 we finish the proof of Theorem 1.2 in terms of the quadrature property.

2. Preliminaries

In this section we review some necessary preliminaries from combinatorics and random matrix theory. We mostly follow the notation of [4].

2.1. **Elementary combinatorics.** For an integer partition $\lambda \vdash k$, it is convenient to write $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots)$ where $m_i(\lambda)$ is the multiplicity of i in λ, and $\ell(\lambda) = \sum_{i\geq 1} m_i(\lambda)$ is the length of λ. Write $\mu_\sigma \vdash k$ for the cycle type of a permutation $\sigma \in S_k$, and for $\rho \vdash k$, write $z_\rho := \prod_{i\geq 1} i^{m_i(\rho)} m_i(\rho)!$. Let us collect some properties that we will use:

Lemma 2.1. *Let $\rho \vdash k$.*

(1) *The number of permutations $\sigma \in S_k$ with $\mu_\sigma = \rho$ is $\frac{k!}{z_\rho}$.*
(2) *We have $z_{2\rho} = 2^{\ell(\rho)} z_\rho$.*
(3) *For $\sigma \in S_k$, we have $\operatorname{sgn}(\sigma) = (-1)^{k-\ell(\mu_\sigma)}$.*

Proof. For (1), see e.g. [10, Proposition 1.3.2]. For (2), we have

$$z_{2\rho} = \prod_{i\geq 1} (2i)^{m_i(\rho)} m_i(\rho)! = 2^{\ell(\rho)} \prod_{i\geq 1} i^{m_i(\rho)} m_i(\rho)! = 2^{\ell(\rho)} z_\rho$$

since $\ell(\rho) = \sum_{i\geq 1} m_i(\rho)$. For (3), a cycle of length i is a product of $i-1$ transpositions:

$$(j_1, \ldots, j_i) = (j_1, j_2) \cdots (j_{i-1}, j_i).$$

So if $\mu_\sigma = (1^{m_1}, 2^{m_2}, \ldots)$, then σ can be written as a product of

$$\sum_{i\geq 1} (i-1) m_i = \sum_{i\geq 1} i m_i - \sum_{i\geq 1} m_i = k - \ell(\mu_\sigma)$$

transpositions, which gives $\operatorname{sgn}(\sigma) = (-1)^{k-\ell(\mu_\sigma)}$. □

The number of permutations in S_k with exactly i cycles is called the *unsigned Stirling number of the first kind* with parameters k and i and is denoted by $c(k,i)$. These numbers are related to the k-th *rising factorial*, which is

$$x^{(k)} = x(x+1)\cdots(x+k-1),$$

by the following lemma:

Lemma 2.2. *We have*

$$\sum_{i=0}^{k} c(k,i) x^i = x^{(k)}.$$

Proof. See e.g. [10, Proposition 1.3.7]. □

Recall that a *partition* of a set X is a collection of disjoint subsets, called *blocks* of the partition, whose union is X. Let $P(k)$ be the set of partitions of the set $[k] = \{1, \ldots, k\}$ and for $\pi \in P(k)$, write $|\pi|$ for the number of blocks of π. Write $P_2(k)$ for the set of partitions of $[k]$ whose blocks are all of size 2; note that this is only non-empty when k is even.

The ordering of $P(k)$ is defined by letting $\pi \leq \sigma$ if each block of π is a subset of a block of σ. This ordering makes $P(k)$ a lattice. For $\mathbf{i} : [k] \to [d]$, write $\ker(\mathbf{i})$ for the element of $P(k)$ whose blocks are the equivalence classes of the relation defined by $s \sim t$ if and only if $\mathbf{i}(s) = \mathbf{i}(t)$; in other words $\ker(\mathbf{i})$ is the partition of $[k]$ whose blocks are the "level sets" of the multi-index $\mathbf{i}$. With this notation, $\pi \leq \ker(\mathbf{i})$ if and only if $\mathbf{i}(s) = \mathbf{i}(t)$ whenever s and t are in the same block of π; in other words the multi-index $\mathbf{i}$ labels the blocks of π in a consistent way.

There is a natural embedding of $P_2(2k)$ into S_{2k} which we will use extensively as in [4]: each $\pi \in P_2(2k)$ can be written uniquely in the form

$$\{\{\pi(1), \pi(2)\}, \ldots, \{\pi(2k-1), \pi(2k)\}\}$$

with $\pi(2i-1) < \pi(2i)$ for $1 \leq i \leq k$ and $\pi(1) < \cdots < \pi(2k-1)$, and the embedding is

$$\pi \mapsto \begin{pmatrix} 1 & \cdots & 2k \\ \pi(1) & \cdots & \pi(2k) \end{pmatrix} \in S_{2k}. \tag{4}$$

2.2. **Combinatorial representation theory.** The representation theory of symmetric groups is well known to be described by the combinatorics of integer partitions and Young diagrams; a nice reference on this theory is [2]. In particular, the irreducible representations of S_r are canonically labeled, say as V^λ, by the integer partitions $\lambda \vdash r$. Write χ^λ for the character of V^λ, and χ^λ_ρ for the value of χ^λ on the conjugacy class in S_r labeled by $\rho \vdash r$.

2.2.1. *The symmetric group and Schur functions.* An important general feature of the representation theory of finite groups is that the characters of irreducible representations are orthogonal, and for S_k this takes the following form:

Theorem 2.3 (Orthogonality relations for χ^λ)**.** *For $\lambda, \mu \vdash k$, we have*

$$\frac{1}{|S_k|} \sum_{\sigma \in S_k} \chi^\lambda(\sigma)\chi^\mu(\sigma) = \sum_{\rho \vdash k} \frac{1}{z_\rho} \chi^\lambda_\rho \chi^\mu_\rho = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}.$$

Proof. See e.g. [2, Corollary 1.3.7], or any textbook on representation theory. □

The graded algebra whose k-th component is the space of class functions on S_k is identified with the graded algebra of symmetric functions, see e.g. [6, Section I.7], and the symmetric functions s_λ corresponding to the irreducible characters χ^λ are called the *Schur functions.* We just need a particular value of s_λ:

Proposition 2.4 (Hook-content formula)**.** *We have*

$$s_\lambda(1^d) = \frac{\chi^\lambda(1)}{k!} \prod_{(i,j) \in \lambda} (d + j - i)$$

for $\lambda \vdash k$ and $k \leq d$.

Proof. See e.g. [2, Theorem 4.3.3]. □

2.2.2. *The Gelfand pair* (S_{2k}, H_k). If G is a finite group and K is a subgroup of G, recall that (G, K) is called a *Gelfand pair* if the trivial representation of K induces a multiplicity-free representation of G. Write H_k for the centralizer of $(1, 2) \cdots (2k-1, 2k) \in S_{2k}$; this is called the *hyperoctahedral group* and has order $2^k k!$. The hyperoctahedral group H_k may be alternately described as the group of signed permutations of k letters, as the symmetry group of a k-dimensional hypercube, or as the wreath product $S_2 \wr S_k$. It is well known (see e.g. [6, VII.2.2]) that (S_{2k}, H_k) is a Gelfand pair.

Associated to a Gelfand pair is its family of *zonal spherical functions*. These are defined by taking the characters of the irreducible representations contained in $\mathrm{Ind}_K^G(\mathrm{triv})$ and averaging them over K, which we will make more precise below. By [6, VII.2.4] the irreducible representations of S_{2k} contained in $\mathrm{Ind}_{H_k}^{S_{2k}}(\mathrm{triv})$ are precisely the ones labeled by $2\lambda = (2\lambda_1, 2\lambda_2, \ldots)$ for $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash k$, so we make the following concrete definition.

Definition 2.5 (Zonal spherical functions)**.** For $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash k$, define $\omega^\lambda : S_{2k} \to \mathbb{C}$ by

$$\omega^\lambda(\sigma) = \frac{1}{|H_k|} \sum_{\zeta \in H_k} \chi^{2\lambda}(\sigma\zeta)$$

for $\sigma \in S_{2k}$. This is called the *zonal spherical function* of the Gelfand pair (S_{2k}, H_k) corresponding to λ.

Let us single out the values of a particular ω^λ:

Lemma 2.6. *For $\rho \vdash k$, we have*

$$\omega_\rho^{1^k} = \frac{(-1)^{k-\ell(\rho)}}{2^{k-\ell(\rho)}}.$$

Proof. See e.g. [6, Example VII.2.2.b]. □

In the analogy between χ^λ and ω^λ, the relation of χ^λ with the conjugacy classes of S_k corresponds to the relation of ω^λ with the double cosets of H_k in S_{2k}:

Definition 2.7. For $\sigma \in S_{2k}$, define a graph $\Gamma(\sigma)$ as follows:

- the vertices are $1, \ldots, 2k$;
- the edges connect $2i-1$ with $2i$ and $\sigma(2i-1)$ with $\sigma(2i)$ for $1 \le i \le k$.

The connected components of $\Gamma(\sigma)$ are cycles of even lengths, and dividing those lengths by 2, we get an integer partition $\Xi(\sigma)$ of k which is called the *coset type* of σ.

By e.g. [6, VII.2.1.i], the coset type labels the double cosets of H_k in S_{2k}. So for $\rho \vdash k$ we write H_ρ for the corresponding double coset and then $S_{2k} = \bigsqcup_{\rho \vdash k} H_\rho$. By [6, VII.2.3] the cardinality of a double coset H_ρ is

$$|H_\rho| = \frac{|H_k|^2}{z_{2\rho}} = \frac{|H_k|^2}{2^{\ell(\rho)} z_\rho}.$$

Clearly the zonal spherical functions are constant on double cosets, so we write ω_ρ^λ for the value of ω^λ on H_ρ. For us, the critical property of the zonal spherical functions is that we still have the orthogonality relations analogous to Theorem 2.3:

Theorem 2.8 (Orthogonality relations for ω^λ)**.** *For $\lambda, \mu \vdash k$, we have*

$$\sum_{\rho \vdash k} \frac{1}{z_{2\rho}} \omega^\lambda_\rho \omega^\mu_\rho = \begin{cases} \frac{h(2\lambda)}{|H_k|^2} & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases},$$

where $h(2\lambda)$ is the product of the hook lengths in 2λ.

Proof. See e.g. [6, VII.2.15']. □

We need an analogue of the identification of the irreducible characters χ^λ with the Schur functions s_λ. To this end, one identifies the graded algebra whose k-th component is the space of functions on S_{2k} which are constant on the double cosets of H_k, with the graded algebra of symmetric functions. Then, the *zonal polynomial* Z_λ is the symmetric function corresponding to ω^λ in this identification. Again, we only need a particular value of Z_λ; write $(x)_k := x(x-1)\cdots(x-k+1)$ for the k-th *falling factorial.*

Proposition 2.9. *We have*

$$Z_\lambda(1^d) = \prod_{(i,j)\in\lambda} (d + 2j - i - 1)$$

for $\lambda \vdash k$ and $k \leq d$. In particular, we have $Z_{1^k}(1^d) = (d)_k$.

Proof. See e.g. [6, VII.2.25]. □

2.3. **Random matrices and the Weingarten calculus.** We denote by U_d and O_d the compact groups of $d \times d$ unitary and orthogonal matrices, respectively; a *random $d \times d$ unitary* or *orthogonal matrix* is a random element of U_d or O_d, respectively, sampled according to the groups' respective Haar probability measures. For $1 \leq i, j \leq d$ and $G = U_d$ or $G = O_d$, let $u_{ij} : G \to \mathbb{C}$ be the (i,j)-th matrix coordinate function, i.e. the function which picks out the (i,j)-th entry of a unitary or orthogonal matrix respectively.

The *Weingarten calculus* is a family of combinatorial techniques for integration over certain classical matrix groups, named after D. Weingarten due to his pioneering work [11] which concerned the group SO_d. Ideas which first appeared there were systematized and developed for the unitary group by Collins in [3], and later for the orthogonal and symplectic groups by Collins-Śniady in [5]. One may prefer to take the perspective of e.g. Banica-Speicher in [1], in which the Weingarten calculus follows from the construction of combinatorial models of the representation categories of so-called *easy* groups.

2.3.1. *Integration on the unitary group.* The main theorem on integration over U_d is the following, which is due to Collins and Śniady in [3, 5]. The matrix $\mathrm{Wg}^U_{k,d}$, indexed by S_{2k}, is constructed from the invariant theory of U_d.

Theorem 2.10 (Unitary Weingarten calculus)**.** *For $k, k' \geq 1$ and $\mathbf{i}, \mathbf{j} : [k] \to [d]$ and $\mathbf{j}', \mathbf{j}' : [k'] \to [d]$, the integral*

$$\int_{U_d} u_{\mathbf{i}(1)\mathbf{j}(1)} \cdots u_{\mathbf{i}(k)\mathbf{j}(k)} \overline{u_{\mathbf{i}'(1)\mathbf{j}'(1)}} \cdots \overline{u_{\mathbf{i}'(k')\mathbf{j}'(k')}} \, dU$$

is

$$\sum_{\substack{\pi,\sigma\in S_k\\ \mathbf{i}=\mathbf{i}'\circ\pi\\ \mathbf{j}=\mathbf{j}'\circ\sigma}} \mathrm{Wg}^U_{k,d}(\pi,\sigma)$$

when $k = k'$, *and it is* 0 *otherwise.*

We single out an expression for Wg^U in terms of the characters of S_{2k} which we use in a critical way:

Theorem 2.11 ([5, Proposition 2.3])**.** *For* $\pi, \sigma \in S_{2k}$, *we have*

$$\mathrm{Wg}^U_{k,d}(\pi,\sigma) = \frac{1}{(k!)^2}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le d}} \frac{\chi^\lambda(1)^2}{s_\lambda(1^d)}\chi^\lambda(\pi^{-1}\sigma).$$

In particular, $\mathrm{Wg}^U_{k,d}(\pi,\sigma)$ *only depends on the cycle type of* $\pi^{-1}\sigma$.

2.3.2. *Integration on the orthogonal group.* Now let us state the main theorem on integration over O_d, which is due to Collins-Śniady in [5, Corollary 3.4]. The matrix $\mathrm{Wg}^O_{k,d}$, indexed by $P_2(2k)$, is constructed from the invariant theory of O_d, although in a somewhat different way from the unitary case.

Theorem 2.12 (Orthogonal Weingarten calculus)**.** *For* $k \geq 1$ *and* $\mathbf{i}, \mathbf{j} : [2k] \to [d]$, *we have*

$$\int_{O_d} u_{\mathbf{i}(1)\mathbf{j}(1)}\cdots u_{\mathbf{i}(2k)\mathbf{j}(2k)}\, dU = \sum_{\substack{\pi,\sigma\in P_2(2k)\\ \pi\le\ker(\mathbf{i})\\ \sigma\le\ker(\mathbf{j})}} \mathrm{Wg}^O_{k,d}(\pi,\sigma)$$

where the integral is with respect to the Haar probability measure of O_d.

Again, we require an expression for Wg^O in terms of the representation theory of S_{2k}; here, the analogue of Theorem 2.11 is in terms of the zonal spherical functions of the Gelfand pair (S_{2k}, H_k). Recall that the value of $\chi^{2\lambda}$ at 1, which is the dimension of the irreducible representation of S_{2k} labeled by 2λ, is

$$\chi^{2\lambda}(1) = |\{\text{standard Young tableaux of shape } 2\lambda\}| = \frac{(2k)!}{h(2\lambda)}$$

where the last equality is by the hook-length formula [2, Theorem 4.2.14].

Theorem 2.13 ([4, Theorem 3.1])**.** *For* $\pi, \sigma \in P_2(2k)$, *we have*

$$\mathrm{Wg}^O_{k,d}(\pi,\sigma) = \frac{2^k k!}{(2k)!}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le d}} \frac{\chi^{2\lambda}(1)}{Z_\lambda(1^d)}\omega^\lambda(\pi^{-1}\sigma)$$

where $P_2(2k)$ *is embedded into* S_{2k} *as in* (4). *In particular,* $\mathrm{Wg}^O_{k,d}(\pi,\sigma)$ *only depends on the coset type* $\Xi(\pi^{-1}\sigma)$ *of* $\pi^{-1}\sigma$.

3. Quadrature property

To isolate the computations we need to make, let us just dive in to the symmetric additive case and see what Haar integrals must be handled. For $A \in M_d(\mathbb{C})$, we have

$$c_x(A) = \det(xI - A) = \sum_{k=0}^{d} (-1)^k x^{d-k} \mathsf{e}_k(A)$$

where e_k is the k-th elementary symmetric function, and it is evaluated at the eigenvalues of A. For $S, T \subseteq [d]$, write $A(S,T)$ for the submatrix of A consisting of the rows indexed by S and the columns indexed by T; notice that

$$\mathsf{e}_k(A) = \sum_{\substack{S \subseteq [d] \\ |S|=k}} \det(A(S,S)).$$

We can assume without loss of generality that A and B are diagonal: since they are self-adjoint, they can be diagonalized, say as $A = V_A D_A V_A^*$ and $B = V_B D_B V_B^*$ for some unitary V_A and V_B and some diagonal D_A and D_B. Then we have

$$\begin{aligned}\chi_x(A + UBU^*) &= \chi_x(V_A D_A V_A^* + UV_B D_B V_B^* U^*) \\ &= \chi_x(D_A + V_A^* U V_B D_B V_B^* U^* V_A) \\ &= \chi_x(D_A + (V_A^* U V_B) D_B (V_A^* U V_B)^*)\end{aligned}$$

and by invariance of Haar measure, we have

$$\mathbb{E}_U \chi_x(A + UBU^*) = \mathbb{E}_U \chi_x(D_A + UD_B U^*).$$

So write $A = \operatorname{diag}(a_1, \ldots, a_d)$ and $B = \operatorname{diag}(b_1, \ldots, b_d)$, and $W := A + UBU^*$, so for $S \subseteq [d]$ with $|S| = k$ we have

$$\begin{aligned}&\det(W(S,S)) \\ &= \sum_{\sigma \in \mathrm{Sym}(S)} \operatorname{sgn}(\sigma) \prod_{i \in S} \left(a_i \delta_{i,\sigma(i)} + \sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right) \\ &= \sum_{\sigma \in \mathrm{Sym}(S)} \operatorname{sgn}(\sigma) \sum_{R \subseteq S} \left(\left(\prod_{i \in R} a_i \delta_{i,\sigma(i)} \right) \left(\prod_{i \in S \setminus R} \left(\sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right) \right) \right) \\ &= \sum_{R \subseteq S} \sum_{\sigma \in \mathrm{Sym}(S \setminus R)} \operatorname{sgn}(\sigma) \left(\prod_{i \in R} a_i \right) \left(\prod_{i \in S \setminus R} \left(\sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right) \right).\end{aligned}$$

Switching the product and sum, we have

$$\begin{aligned}&\mathbb{E}_U \left(\prod_{i \in S \setminus R} \left(\sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right) \right) \\ &= \sum_{\mathbf{p}: S \setminus R \to [d]} \mathbb{E}_U \left(\prod_{i \in S \setminus R} u_{i\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}} \right) \left(\prod_{i \in S \setminus R} b_{\mathbf{p}(i)} \right)\end{aligned}$$

and putting this back into the sum above, we get

$$\begin{aligned}&\mathbb{E}_U \det(W(S,S))\\&= \sum_{R\subseteq S}\left(\prod_{i\in R} a_i\right)\sum_{\mathbf{p}:S\setminus R\to[d]}\left(\prod_{i\in S\setminus R} b_{\mathbf{p}(i)}\right)\\&\qquad\sum_{\sigma\in\mathrm{Sym}(S\setminus R)}\mathrm{sgn}(\sigma)\mathbb{E}_U\left(\prod_{i\in S\setminus R} u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\right).\end{aligned}$$

So we want to look at

$$\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\mathbb{E}_U\left(\prod_{i=1}^k u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\right)$$

for $\mathbf{p}:[k]\to[d]$, for $0\le k\le d$.

Definition 3.1. Let us say that a compact subgroup $G\le U_d$ satisfies the *quadrature property* if

$$\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\int_G\prod_{i=1}^k u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\,dU = \begin{cases}\frac{(d-k)!}{d!} & \text{if } \mathbf{p} \text{ is injective}\\ 0 & \text{otherwise}\end{cases}$$

for $\mathbf{p}:[k]\to[d]$, for all $0\le k\le d$.

4. The unitary case

In this section we show that U_d itself has the quadrature property.

Theorem 4.1. *For $0\le k\le d$ and $\mathbf{p}:[k]\to[d]$, we have*

$$\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\int_{U_d}\prod_{i=1}^k u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\,dU = \begin{cases}\frac{(d-k)!}{d!} & \textit{if } \mathbf{p} \textit{ is injective}\\ 0 & \textit{otherwise}\end{cases}. \tag{5}$$

To prove this we will use Theorem 2.11 to reduce the computation to the following simple lemma:

Lemma 4.2. *We have*

$$\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\chi^\lambda(\sigma) = \begin{cases}k! & \textit{if } \lambda=1^k\\ 0 & \textit{otherwise}\end{cases}$$

for $\lambda\vdash k$.

Proof. Observe that χ^{1^k} is just the sign character of S_k. So if $\lambda=1^k$, we have

$$\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\chi^{1^k}(\sigma) = \sum_{\sigma\in S_k} 1 = k!$$

and otherwise, if $\lambda\ne 1^k$, then by Theorem 2.3 we have

$$\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\chi^\lambda(\sigma) = \sum_{\sigma\in S_k}\chi^{1^k}(\sigma)\chi^\lambda(\sigma) = 0$$

so we are done. □

Proof of Theorem 4.1. By Theorem 2.10 we have

$$\begin{aligned}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\int_{U_d}\prod_{i=1}^{k}u_{i\mathbf{p}(i)}\overline{u_{\sigma(i)\mathbf{p}(i)}}\,dU &= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\sum_{\substack{\pi,\tau\in S_k\\ \mathrm{Id}=\sigma\circ\pi\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}(\pi,\tau)\\ &= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau).\end{aligned}$$

If $\mathbf{p}$ is not injective, say there are some $i,j\in[k]$ with $i\neq j$ and $\mathbf{p}(i)=\mathbf{p}(j)$, we want to identify pairs of summands which cancel each other out, i.e. for each $\sigma\in S_k$ we want a corresponding $\sigma'\in S_k$ with $\operatorname{sgn}(\sigma')=-\operatorname{sgn}(\sigma)$ and

$$\sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau)=\sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}((\sigma')^{-1},\tau).$$

To this end let $\sigma'=\sigma\cdot(i,j)$, so that $\operatorname{sgn}(\sigma')=-\operatorname{sgn}(\sigma)$. Moreover, since $\mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau)$ only depends on the cycle type of $\sigma\tau$, we have

$$\begin{aligned}\sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}((\sigma')^{-1},\tau) &= \sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}((i,j)\sigma^{-1},\tau)\\ &= \sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}(\sigma^{-1},(i,j)\tau)\\ &= \sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau)\end{aligned}$$

as the condition $\mathbf{p}=\mathbf{p}\circ\tau$ is invariant under translation of τ by (i,j). Thus we have shown that when $\mathbf{p}$ is not injective, the summands in Eq. (5) cancel each other out and the sum is 0.

If $\mathbf{p}$ is injective, then the only $\tau\in S_k$ with $\mathbf{p}=\mathbf{p}\circ\tau$ is $\tau=1$ so by Theorem 2.11 we have

$$\begin{aligned}&\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\sum_{\substack{\tau\in S_k\\ \mathbf{p}=\mathbf{p}\circ\tau}}\mathrm{Wg}^U_{k,d}(\sigma^{-1},\tau)\\ &= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\mathrm{Wg}^U_{k,d}(\sigma^{-1},1)\\ &= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\frac{1}{(k!)^2}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\leq d}}\frac{\chi^\lambda(1)^2}{s_\lambda(1^d)}\chi^\lambda(\sigma)\\ &= \frac{1}{(k!)^2}\sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\leq d}}\left(\frac{k!\chi^\lambda(1)^2}{\chi^\lambda(1)\prod_{(i,j)\in\lambda}(d+j-i)}\right)\left(\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\chi^\lambda(\sigma)\right)\\ &= \left(\frac{\chi^{1^k}(1)}{k!\prod_{1\leq i\leq k}(d+1-i)}\right)k!\end{aligned}$$

$$= \frac{1}{\prod_{1\le i\le k}(d-i+1)} = \frac{(d-k)!}{d!}$$

and we are done. □

5. The orthogonal case

In this section we show that O_d has the quadrature property.

Theorem 5.1. *For $0 \le k \le d$ and $\mathbf{p} : [k] \to [d]$, we have*

$$\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \int_{O_d} \prod_{i=1}^{k} u_{i\mathbf{p}(i)} u_{\sigma(i)\mathbf{p}(i)}\, dU = \begin{cases} \frac{(d-k)!}{d!} & \text{if } \mathbf{p} \text{ is injective} \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

To prove this we will use Theorem 2.13 to reduce the computation to the following lemma:

Lemma 5.2. *We have*

$$\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\omega^\lambda_{\mu_\sigma} = \begin{cases} \frac{(k+1)!}{2^k} & \text{if } \lambda = 1^k \\ 0 & \text{otherwise} \end{cases}.$$

for $\lambda \vdash k$.

Proof. If $\lambda = 1^k$, then by (3) in Lemma 2.1, Lemma 2.2, and Lemma 2.6, we have

$$\begin{aligned}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\omega^{1^k}_{\mu_\sigma} &= \sum_{\sigma\in S_k} (-1)^{k-\ell(\mu_\sigma)} \frac{(-1)^{k-\ell(\mu_\sigma)}}{2^{k-\ell(\mu_\sigma)}} \\ &= \frac{1}{2^k} \sum_{\sigma\in S_k} 2^{\ell(\mu_\sigma)} \\ &= \frac{1}{2^k} \sum_{i=1}^{k} 2^i c(k,i) \\ &= \frac{2^{(k)}}{2^k} \\ &= \frac{(k+1)!}{2^k}.\end{aligned}$$

On the other hand, if $\lambda \neq 1^k$, then by Lemma 2.1, Lemma 2.6, and Theorem 2.8, for any $\lambda \neq 1^k$ we have

$$\begin{aligned}0 &= \sum_{\rho\vdash k} \frac{1}{z_{2\rho}} \omega^\lambda_\rho \omega^{1^k}_\rho \\ &= \sum_{\rho\vdash k} \frac{1}{z_{2\rho}} \omega^\lambda_\rho \frac{(-1)^{k-\ell(\rho)}}{2^{k-\ell(\rho)}} \\ &= \frac{1}{2^k} \sum_{\rho\vdash k} (-1)^{k-\ell(\rho)} 2^{\ell(\rho)} \frac{1}{z_{2\rho}} \omega^\lambda_\rho \\ &= \frac{1}{2^k k!} \sum_{\rho\vdash k} (-1)^{k-\ell(\rho)} \frac{k!}{z_\rho} \omega^\lambda_\rho\end{aligned}$$

$$= \frac{1}{2^k k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega^\lambda_{\mu_\sigma}$$

thus $\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega^\lambda_{\mu_\sigma} = 0$. □

Proof of Theorem 5.1. By Theorem 2.12, with $\mathbf{i}_\sigma := (1, \sigma(1), \ldots, k, \sigma(k))$ and $\mathbf{pp} := (\mathbf{p}(1), \mathbf{p}(1), \ldots, \mathbf{p}(k), \mathbf{p}(k))$, we have

$$\begin{aligned}\sum_{\sigma \in S_k} \int_{O_d} \prod_{i=1}^k u_{i\mathbf{p}(i)} u_{\sigma(i)\mathbf{p}(i)} \, dU &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\pi, \tau \in P_2(2k) \\ \pi \le \ker(\mathbf{i}_\sigma) \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\pi, \tau) \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau)\end{aligned}$$

since the condition $\pi \le \ker(\mathbf{i}_\sigma)$ forces equality. If $\mathbf{p}$ is not injective, say there are some $i \neq j$ with $\mathbf{p}(i) = \mathbf{p}(j)$, we want to identify pairs of summands which cancel each other out, i.e. for each $\sigma \in S_k$ we want a corresponding $\sigma' \in S_k$ with $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$ and

$$\sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau) = \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_{\sigma'}), \tau).$$

To this end let $\sigma' = (i,j)\sigma$, which obviously satisfies $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$. Moreover, we have $\ker(\mathbf{j}_{\sigma'}) = (i,j) \ker(\mathbf{j}_\sigma)$ in the embedding (4), so with $\tau' = (i,j)\tau$, $\tau^{-1} \ker(\mathbf{j}_{\sigma'})$ and $(\tau')^{-1} \ker(\mathbf{j}_\sigma)$ have the same coset type. Since the condition $\tau \le \ker(\mathbf{pp})$ is invariant under translation of τ by (i,j), by Theorem 2.13 we have

$$\begin{aligned}\sum_{\substack{\tau \in P_2 2k \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_{\sigma'}), \tau) &= \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau') \\ &= \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau).\end{aligned}$$

Thus we have shown that when $\mathbf{p}$ is not injective, the summands in Eq. (6) cancel each other out and the sum is 0.

If $\mathbf{p}$ is injective, then the condition $\tau \le \ker(\mathbf{pp})$ forces equality, so since $\chi^{21^k}(1) = \frac{(2k)!}{k!(k+1)!}$ and $Z_{1^k}(1^d) = (d)_k$, by Theorem 2.13 and Lemma 5.2, we have

$$\begin{aligned}&\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\tau \in P_2(2k) \\ \tau \le \ker(\mathbf{pp})}} \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \tau) \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \mathrm{Wg}^O_{k,d}(\ker(\mathbf{i}_\sigma), \ker(\mathbf{pp})) \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{\chi^{2\lambda}(1)}{Z_\lambda(1^d)} \omega^\lambda_{\mu_\sigma}\end{aligned}$$

$$= \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\frac{2^k k!}{(2k)!}\frac{\chi^{21^k}(1)}{Z_{1^k}(1^d)}\omega^{1^k}_{\mu_\sigma} + \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\frac{2^k k!}{(2k)!}\sum_{\substack{\lambda\vdash k\\ \lambda\neq 1^k}}\frac{\chi^{2\lambda}(1)}{Z_\lambda(1^d)}\omega^{\lambda}_{\mu_\sigma}$$

$$= \frac{2^k}{(k+1)!(d)_k}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\omega^{1^k}_{\mu_\sigma} + \sum_{\substack{\lambda\vdash k\\ \lambda\neq 1^k}}\frac{2^k k!\chi^{2\lambda}(1)}{(2k)!Z_\lambda(1^d)}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\omega^{\lambda}_{\mu_\sigma}$$

$$= \frac{1}{(d)_k} = \frac{(d-k)!}{d!}$$

and we are done. □

6. The hyperoctahedral case

In this brief section we show the elementary proof that group H_d of signed permutation matrices has the quadrature property. This is similar to e.g. [9, Lemma 2.6].

Theorem 6.1. *For $0 \le k \le d$ and $\mathbf{p} : [k] \to [d]$, we have*

$$\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\int_{H_d}\prod_{i=1}^k u_{i\mathbf{p}(i)}u_{\sigma(i)\mathbf{p}(i)}\,dU = \begin{cases}\frac{(d-k)!}{d!} & \text{if } \mathbf{p} \text{ is injective}\\ 0 & \text{otherwise}\end{cases}.$$

Proof. We have

$$\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\int_{H_d}\prod_{i=1}^k u_{i\mathbf{p}(i)}u_{\sigma(i)\mathbf{p}(i)}\,dU$$

$$= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\frac{1}{2^d d!}\sum_{\epsilon_1,\dots,\epsilon_d\in\{\pm1\}}\sum_{\tau\in S_d}\prod_{i=1}^k(\epsilon_i\delta_{\mathbf{p}(i)=\tau(i)})(\epsilon_{\sigma(i)}\delta_{\mathbf{p}(i)=\tau(\sigma(i))})$$

and the non-zero summands are the ones with $\mathbf{p}(i) = \tau(i)$ and $\mathbf{p}(i) = \tau(\sigma(i))$ for $1 \le i \le k$. If $\mathbf{p}$ is not injective, then there is no $\tau \in S_d$ with $\mathbf{p}(i) = \tau(i)$ for $1 \le i \le k$, so the sum is 0. On the other hand, if $\mathbf{p}$ is injective, there are $(d-k)!$ permutations $\tau \in S_d$ with $\mathbf{p}(i) = \tau(i)$ for $1 \le i \le k$, i.e. $\tau \in S_{d-k}$; similarly the condition $\mathbf{p}(i) = \tau(\sigma(i))$ forces $\sigma(i) = i$ for $1 \le i \le k$, i.e. $\sigma = 1$. So the sum above becomes

$$\frac{1}{2^d d!}\sum_{\epsilon_1,\dots,\epsilon_d\in\{\pm1\}}\sum_{\tau\in S_{d-k}}\prod_{i=1}^k\epsilon_i^2 = \frac{(d-k)!}{2^d d!}\sum_{\epsilon_1,\dots,\epsilon_d\in\{\pm1\}}\prod_{i=1}^k 1$$

$$= \frac{(d-k)!}{d!}$$

since the last sum gives 2^d summands, which are copies of 1. □

Remark 6.2 (Hyperoctahedral series)**.** Clearly this argument works just as well for the so-called *hyperoctahedral series* of wreath products $H_d^s := \mathbf{Z}_s \wr S_d$ with $s \ge 3$, which in matrix terms are the groups of "signed" permutation matrices where the signs are s-th roots of unity. It also works for $H_d^\infty := \mathbb{T} \wr S_d$, or in other words $\widehat{\mathbf{Z}} \wr S_d$, with an integral over the d-torus $\mathbb{T}^d$ instead of a sum over d copies of $\mathbf{Z}_2$.

7. Convolution formulae and quadrature

Finally we finish the proof of Theorem 1.2, or rather a generalization:

Theorem 7.1 ([9, Theorems 2.10 & 2.12 & 2.14])**.** *Let $G \leq U_d$ be a compact subgroup with the quadrature property. Then*

(1) *for $0 \leq k \leq d$, we have*
$$\int_G \mathsf{e}_k(A + UBU^*)\, dU = \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \mathsf{e}_i(A)\mathsf{e}_j(B)$$
for self-adjoint $A, B \in M_d(\mathbb{C})$;

(2) *for $0 \leq k \leq d$, we have*
$$\int_G \mathsf{e}_k(AUBU^*)\, dU = \frac{k!(d-k)!}{d!} \mathsf{e}_k(A)\mathsf{e}_k(B)$$
for positive semidefinite $A, B \in M_d(\mathbb{C})$;

(3) *for $0 \leq k \leq d$, we have*
$$\iint_G \mathsf{e}_k((A + UBV)(A + UBV)^*)\, dU\, dV$$
$$= \sum_{i+j=k} \left(\frac{(d-i)!(d-j)!}{d!(d-k)!} \right)^2 \mathsf{e}_i(AA^*)\mathsf{e}_j(BB^*)$$
for $A, B \in M_d(\mathbb{C})$.

This clearly gives Theorem 1.2 with $G = U_d$.

7.1. Symmetric additive convolution.

We have already done a large portion of the proof for the symmetric additive convolution in Section 3.

Proof of (1) in Theorem 7.1. Let us pick back up from the computations at the beginning of Section 3. Notice that since we assume $A = \operatorname{diag}(a_1, \ldots, a_d)$ and $B = \operatorname{diag}(b_1, \ldots, b_d)$, we have
$$\mathsf{e}_i(A) = \frac{1}{i!} \sum_{\substack{\mathbf{p}:[i]\to[d] \\ \text{injective}}} a_{\mathbf{p}(1)} \cdots a_{\mathbf{p}(i)} = \sum_{\substack{\mathbf{p}:[i]\to[d] \\ \mathbf{p}(1)<\cdots<\mathbf{p}(i)}} a_{\mathbf{p}(1)} \cdots a_{\mathbf{p}(i)}$$
and similarly for $\mathsf{e}_j(B)$. So we have
$$\mathbb{E}_U \det(W(S,S))$$
$$= \sum_{R \subseteq S} \left(\prod_{i \in R} a_i \right) \sum_{\mathbf{p}:S\setminus R \to [d]} \left(\prod_{i \in S\setminus R} b_{\mathbf{p}(i)} \right)$$
$$\sum_{\sigma \in \mathrm{Sym}(S\setminus R)} \operatorname{sgn}(\sigma) \mathbb{E}_U \left(\prod_{i \in S\setminus R} u_{i\mathbf{p}(i)} \overline{u_{\sigma(i)\mathbf{p}(i)}} \right)$$
$$= \sum_{R \subseteq S} \frac{(d - |S \setminus R|)!}{d!} \left(\prod_{i \in R} a_i \right) \sum_{\substack{\mathbf{p}:S\setminus R \to [d] \\ \text{injective}}} \left(\prod_{i \in S\setminus R} b_{\mathbf{p}(i)} \right)$$

$$= \sum_{R \subseteq S} \frac{|S \setminus R|!(d - |S \setminus R|)!}{d!} \det(A(R,R)) \mathsf{e}_{|S \setminus R|}(B)$$

and then

$$\begin{aligned}
\mathbb{E}_U(\mathsf{e}_k(W)) &= \sum_{|S|=k} \mathbb{E}_U(\det(W(S,S))) \\
&= \sum_{|S|=k} \sum_{R \subseteq S} \frac{|S \setminus R|!(d - |S \setminus R|)!}{d!} \det(A(R,R)) \mathsf{e}_{|S \setminus R|}(B) \\
&= \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \mathsf{e}_j(B) \sum_{|R|=i} \det(A(R,R)) \\
&= \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \mathsf{e}_i(A) \mathsf{e}_j(B)
\end{aligned}$$

so we are done. □

7.2. **Symmetric multiplicative convolution.** The computations for the symmetric multiplicative convolution are somewhat simpler:

Proof of (2) in Theorem 7.1. As for (1), assume without loss of generality that A and B are diagonal with $A = \operatorname{diag}(a_1, \ldots, a_d)$ and $B = \operatorname{diag}(b_1, \ldots, b_d)$. Write $W = AUBU^T$ and $U = (u_{ij})_{i,j}$, so the (i,j)-th entry of W is $a_i \sum_{p=1}^{d} u_{ip} b_p \overline{u_{jp}}$ and for a subset $S \subseteq [d]$ with $|S| = k$, we have

$$\begin{aligned}
\det(W(S,S)) &= \sum_{\sigma \in \mathrm{Sym}(S)} \operatorname{sgn}(\sigma) \prod_{i \in S} \left(a_i \sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right) \\
&= \det(A(S,S)) \sum_{\sigma \in \mathrm{Sym}(S)} \operatorname{sgn}(\sigma) \prod_{i \in S} \left(\sum_{p=1}^{d} u_{ip} b_p \overline{u_{\sigma(i)p}} \right).
\end{aligned}$$

Switching the product and sum, we have

$$\begin{aligned}
\mathbb{E}_U(\det(W(S,S))) &= \det(A(S,S)) \sum_{\mathbf{p}: S \to [d]} \left(\prod_{i \in S} b_{\mathbf{p}(i)} \right) \\
&\qquad \sum_{\sigma \in \mathrm{Sym}(S)} \operatorname{sgn}(\sigma) \mathbb{E}_U \left(\prod_{i \in S} u_{ip} \overline{u_{\sigma(i)\mathbf{p}(i)}} \right) \\
&= \det(A(S,S)) \sum_{\substack{\mathbf{p}: S \to [d] \\ \text{injective}}} \left(\prod_{i \in S} b_{\mathbf{p}(i)} \right) \frac{(d-|S|)!}{d!} \\
&= \det(A(S,S)) \frac{k!(d-k)!}{d!} \mathsf{e}_k(B),
\end{aligned}$$

thus

$$\mathbb{E}_U \mathsf{e}_k(AUBU^*) = \frac{k!(d-k)!}{d!} \sum_{|S|=k} \det(A(S,S)) \mathsf{e}_k(B) = \frac{k!(d-k)!}{d!} \mathsf{e}_k(A) \mathsf{e}_k(B)$$

and we are done. □

7.3. **Asymmetric additive convolution.** The computations here, for the asymmetric additive convolution, are rather more involved than for the symmetric convolutions; we refer to [8, Section 2.3.2] at some points for details which do not concern our techniques. Let us first make some simplifying notation:

Notation 7.2. For $A, B \in M_d(\mathbb{C})$, write

$$h[A,B](x) := \mathbb{E}_{U,V} c_x((A+UBV)(A+UBV)^*)$$

and

$$\mathrm{dil}(A) := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}.$$

Proof of (3) in Theorem 7.1. Assume without loss of generality – passing to singular value decompositions if necessary – that A and B are diagonal with $A = \mathrm{diag}(a_1, \dots, a_d)$ and $B = \mathrm{diag}(b_1, \dots, b_d)$. We have

$$h[A,B](x^2) = \mathbb{E}_{U,V} c_x(\mathrm{dil}(AU+VB))$$

so with $M := AU + VB$, we have

$$\mathbb{E}_{U,V} c_x(\mathrm{dil}(M)) = \sum_{k=0}^{2d} x^{2d-k} \sum_{\substack{W \subseteq [2d] \\ |W|=k}} \sum_{\sigma \in \mathrm{Sym}(W)} \mathrm{sgn}(\sigma) \mathbb{E}_{U,V} \left(\prod_{i \in W} \mathrm{dil}(M)_{i\sigma(i)} \right).$$

If k is odd, then for any $W \subseteq [2d]$ with $|W| = k$ and for any $\sigma \in \mathrm{Sym}(W)$, there is some $i_0 \in W$ such that $\mathrm{dil}(M)_{i_0\sigma(i_0)} = 0$. So we may assume k is even, say $k = 2l$. The coefficient of x^{2d-2l} is

$$\sum_{\substack{S \text{ set of } l \text{ rows} \\ T \text{ set of } l \text{ columns}}} \sum_{\substack{\rho: S \to T \\ \text{bijection}}} \sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \mathbb{E}_{U,V} \left(\prod_{i \in S} M_{i\rho(i)} \overline{M_{\sigma(i)\rho(i)}} \right)$$

and we have

$$\begin{aligned}
&\mathbb{E}_{U,V} \left(\prod_{i \in S} M_{i\rho(i)} \overline{M_{\sigma(i)\rho(i)}} \right) \\
&= \mathbb{E}_{U,V} \left(\prod_{i \in S} (a_i u_{i\rho(i)} + v_{i\rho(i)} b_{\rho(i)}) \overline{(a_{\sigma(i)} u_{\sigma(i)\rho(i)} + v_{\sigma(i)\rho(i)} b_{\rho(i)})} \right) \\
&= \sum_{R \subset S} \mathbb{E}_U \left(\prod_{i \in R} a_i \overline{a_{\sigma(i)}} u_{i\rho(i)} \overline{u_{\sigma(i)\rho(i)}} \right) \mathbb{E}_V \left(\prod_{i \in S \setminus R} b_{\rho(i)} \overline{b_{\rho(i)}} v_{i\rho(i)} \overline{v_{\sigma(i)\rho(i)}} \right) \\
&= \sum_{R \subset S} \left(\prod_{i \in R} a_i a_{\sigma(i)} \right) \left(\prod_{i \in S \setminus R} b_{\rho(i)}^2 \right) \int_G \prod_{i \in R} u_{i\rho(i)} \overline{u_{\sigma(i)\rho(i)}} \, dU \\
&\qquad \int_G \prod_{i \in S \setminus R} v_{i\rho(i)} \overline{v_{\sigma(i)\rho(i)}} \, dV
\end{aligned}$$

since the cross terms vanish in the second-last line.

Putting this back into the larger sum, we get

$$\sum_{\substack{S \text{ set of } l \text{ rows} \\ T \text{ set of } l \text{ columns}}} \sum_{\substack{\rho: S\to T \\ \text{bijection}}} \sum_{\sigma\in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma)\mathbb{E}_{U,V}\left(\prod_{i\in S} M_{i\rho(i)}\overline{M_{\sigma(i)\rho(i)}}\right)$$

$$= \sum_{\substack{S \text{ set of } l \text{ rows} \\ T \text{ set of } l \text{ columns}}} \sum_{\substack{\rho: S\to T \\ \text{bijection}}} \sum_{\sigma\in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \sum_{R\subseteq S}\left(\prod_{i\in R} a_i a_{\sigma(i)}\right)\left(\prod_{i\in S\setminus R} b^2_{\rho(i)}\right)$$

$$\int_G \prod_{i\in R} u_{i\rho(i)}\overline{u_{\sigma(i)\rho(i)}}\, dU \int_G \prod_{i\in S\setminus R} v_{i\rho(i)}\overline{v_{\sigma(i)\rho(i)}}\, dV$$

$$= \sum_{\substack{S \text{ set of } l \text{ rows} \\ T \text{ set of } l \text{ columns}}} \sum_{\substack{R\subseteq S \\ Z\subseteq T \\ |R|+|Z|=l}} \sum_{\substack{\rho: S\to T \\ \text{bijection} \\ \rho(S\setminus R)=Z}} \left(\prod_{i\in R} a_i^2\right)\left(\prod_{i\in S\setminus R} b^2_{\rho(i)}\right)$$

$$\sum_{\substack{\sigma_1\in \mathrm{Sym}(R) \\ \sigma_2\in \mathrm{Sym}(S\setminus R)}} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)\int_G \prod_{i\in R} u_{i\rho(i)}\overline{u_{\sigma_1(i)\rho(i)}}\, dU$$

$$\int_G \prod_{i\in S\setminus R} v_{i\rho(i)}\overline{v_{\sigma_2(i)\rho(i)}}\, dV$$

since only the terms with $\sigma \in \mathrm{Sym}(R)$ or $\sigma \in \mathrm{Sym}(S \setminus R)$ survive. Now by the quadrature property, this is equal to

$$\sum_{\substack{S \text{ set of } l \text{ rows} \\ T \text{ set of } l \text{ columns}}} \sum_{\substack{R\subseteq S \\ Z\subseteq T \\ |R|+|Z|=l}} \sum_{\substack{\rho: S\to T \\ \text{bijection} \\ \rho(S\setminus R)=Z}} \left(\prod_{i\in R} a_i^2\right)\left(\prod_{i\in S\setminus R} b^2_{\rho(i)}\right)\frac{(d-|R|)!}{d!}\frac{(d-|S\setminus R|)!}{d!}$$

$$= \sum_{r+z=l}\left(\frac{(d-r)!(d-z)!}{d!(d-l)!}\right)^2 \mathsf{e}_r(AA^T)\mathsf{e}_z(BB^T)$$

where the last equality follows as in [8, Section 2.3.2]. □

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