Some Liouville Theorem and Bernstein Theorem

Sheng Li

(Joint work with An-Min Li)

Sichuan University

几何分析与非线性偏微分方程研讨会,

哈尔滨工业大学数学研究院, 2018

Liouville Theorem

Classical Liouville Theorem. Let f be a holomorphic function (i.e. $\bar{\partial} f = 0, \quad in \quad \mathbb{C},$) with

$$|f| \leq M \leq \infty, \qquad z \in \mathbb{C}.$$

Then f is a constant.

Some notations

Let $\Omega = \mathbb{C}^n$, denote

$$Psh^\infty(\Omega) := \{f \in C^\infty(\Omega) \mid f \text{ is a real function and } (f_{i\bar{j}}) > 0\},$$

where $(f_{i\bar{j}}) = \left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)$. For $f \in Psh^\infty(\Omega)$, (Ω, ω_f) is a Kähler manifold.

We consider the complex Monge-Ampere equations

$$\det(f_{i\bar{j}}) = 1. \tag{1}$$

We know that $f = z_1\bar{z}_1 + ... + z_n\bar{z}_n$ is a special global solution of (1). When ω_f is complete, ω_f is a complete Calabi-Yau metric.

Real Monge-Ampere

A celebrated result of Jörgens,($n = 2$) Calabi ($n \leq 5$) and Pogorelov ($n \geq 2$) stated that every strictly convex solutions u to real Monge-Ampere equations

$$\det(u_{ij}) = 1, \quad x \in \mathbb{R}^m.$$

must be a quadratical polynomial.

Unlike the real Monge-Ampere equation, global solutions of (1) cannot be classified without any restriction on solutions growth at infinity.

LeBrun constructs, for all positive real numbers $m \geq 0$, a family of Kähler metrics $g_{(m)}$ on $\mathbb{C}^2$, whose associated Kähler form is given by $\omega_{(m)} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}G_{(m)}$, where

$$G_{(m)}(u, v) = u^2 + v^2 + m(u^4 + v^4),$$

u and v are implicitly defined by

$$x_1 = |z_1| = e^{m(u^2 - v^2)}u, \qquad x_2 = |z_2| = e^{m(v^2 - u^2)}v,$$

For $m = 0$, one gets the flat metric $g_{(0)}$, thus

$$\omega_{(0)} = \sqrt{-1}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2),$$

while for $m > 0$ each of the $g_{(m)}$s represents the first example of complete Ricci flat and non-flat metric on $\mathbb{C}^2$. $g_{(m)}$ is isometric (up to dilation and rescaling) to the Taub-NUT metric.

One can check that they have the same volume form of the flat metric $g_{(0)}$, thus

$$\omega_{(m)} \wedge \omega_{(m)} = \omega_{(0)} \wedge \omega_{(0)},$$

but the Taub-NUT metric has cubic volume growth and the flat metric has Euclidean volume growth.

On $\mathbb{C}^2$, Tian showed that every Calabi-Yau metric of Euclidean volume growth has to be flat.

Tian also conjectured that the same should hold true on $\mathbb{C}^n$ for all $n \geq 3$.

Does Calabi-Yau manifolds with Euclidean volume growth that have a unique tangent cone at infinity? In particular, Does Calabi-Yau manifolds with Euclidean volume growth have a nontrivial tangent cone at infinity?

Recently, Li($n = 3$) construct Calabi-Yau manifolds with Euclidean volume growth, whose tangent cone at infinity is the singular cone $\mathbb{C}^2/\mathbb{Z}^2 \times \mathbb{C}$.

Conlon-Rochon, Szekelyhidi (inspired by the work of Hein and Naber) found a counterexample to this conjecture for all ($n \geq 3$) independently, which have a tangent cone at infinity with singular cross-section.

To obtain the Liouville theorem of (1) people need to strengthen the assumption near ∞. Szekelyhidi asked the following question:

Question. Let f be plurisubharmonic solution to $det(f_{i\bar{j}}) = 1$ on $\mathbb{C}^n$ satisfying

$$C^{-1}(|z|^2 + 1) \leq f \leq C(|z|^2 + 1)$$

then f is quadratic.

Known Results

Riebesehl and Schulz proved that if the solution of (1) satisfies

$$\left|\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right| \leq C < \infty, \quad \forall z \in \mathbb{C}^n,$$

then the second derivatives of f of mixed type are constants.
They also proved that if

$$\left|\nabla^2 f\right| \leq C < \infty, \quad \forall z \in \mathbb{C}^n,$$

then f is quadratic.

By using the small perturbation result of Savin, Wang prove the Liouville theorem for complex Monge-Ampere equations under the assumption $f = |z|^2 + o(|z|^2)$, as $|z| \to \infty$.

Hein proved a Liouville theorem for the complex Mong-Ampere equation on product manifolds

Li, Li and Zhang obtained a Liouville type theorem for the complex Monge- Ampere equation on product manifolds

Theorem(Li and S.) Let $f \in Psh^{\infty}(\Omega)$ satisfying (1). Suppose that

(1) There is a constant $\varepsilon > 0$ such that $f \geq \varepsilon(\sum_{i=1}^{n} z_i \bar{z}_i)$ as $|z| \to \infty$.

(2) ω_f is complete.

Then the second derivatives of f of mixed type are constants.

Corollary Let $f \in Psh^{\infty}(\Omega)$ satisfying (1). Suppose that

(1) There is a constant $C > 0$ such that

$$C^{-1}|z|^2 \leq f \leq C|z|^2, \quad as\ |z| \to \infty$$

(2) ω_f is complete.

Then f must be a quadratical polynomial.

Observation

Denote $T = \sum_{i=1}^{n} f^{i\bar{i}}$. We take a linear transformation

$$\tilde{f} = \lambda f, \quad \tilde{z} = \sqrt{\lambda} z.$$

Denotes $\tilde{f}_{i\bar{j}} = \frac{\partial^2 \tilde{f}}{\partial \tilde{z}_i \partial \bar{\tilde{z}}_j}$ and $(\tilde{f}^{i\bar{j}})$ the inverse matrix of $(\tilde{f}_{i\bar{j}})$. Then

- $\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 \tilde{f}}{\partial \tilde{z}_i \partial \bar{\tilde{z}}_j}, \quad \tilde{T} := \sum \tilde{f}^{i\bar{i}} = T, \quad \det(\tilde{f}_{i\bar{j}}) = 1.$
- $C^{-1}|\tilde{z}|^2 \leq \tilde{f} \leq C|\tilde{z}|^2$,
- $\omega_{\tilde{f}}$ is complete,
- Laplacian comparison Theorem tell us $r\Delta r \leq C(n)$. This inequality is invariant under the re-scaling. So it may be easy to estimate geometric quantity in the geodesic.

For example, if you want to estimate

$$L = (a^2 - r^2)^2 F$$

at some maximal point p^*. Then at p^*,

$$-\frac{(2r^2)_i}{a^2 - r^2} + F_i/F = 0,$$

$$-\frac{Cr\Delta r}{a^2 - r^2} - \frac{C}{(a^2 - r^2)^2} + \frac{\Delta F}{F} - (F_i)^2/F^2 = 0.$$

Then all the term is similar as the calculation on compact manifold.

Sketch of the Proof.

Chen-Li-S. proved that the following estimates of the gradient of f.

Lemma Let $\tilde{f} \in Psh^{\infty}(\Omega)$ with $\tilde{f}(0) = \inf_{\Omega} \tilde{f} = 0$. Suppose that

$$Ric(\omega_{\tilde{f}}) \geq -\mathrm{N}_1 \omega_{\tilde{f}}, \ \ in \ \ B_{\tilde{f}}(0, 2),$$

where $R_{i\bar{j}}(\omega_{\tilde{f}})$ is the Ricci curvature of the metric $\omega_{\tilde{f}}$. Then in $B_{\tilde{f}}(0, 1)$

$$\frac{\|\nabla \tilde{f}\|^2_{\tilde{f}}}{(1+\tilde{f})^2} \leq \mathrm{C}_1$$

where $\mathrm{C}_1 > 0$ is a constant depending only on n and N_0. Then, for any $q \in B_{\tilde{f}}(0, 1)$,

$$\tilde{f}(q) - \tilde{f}(0) \leq C.$$

Using $C^{-1}|\tilde{z}|^2 \leq \tilde{f}$ and the Lemma we have

$$|\tilde{z}|^2 \leq C, \qquad in\ B(0,1).$$

We can obtain the estimates of eigenvalue of the hession $(f_{i\bar{j}})$ from the following lemma (Chen-Li-S.)

Lemma Let $\tilde{f} \in Psh^{\infty}(\Omega)$ and $B_{\tilde{f}}(0,1) \subset \Omega$. Suppose

$$det(\tilde{f}_{i\bar{j}}) \leq \mathrm{N}_1, \quad Ric(\omega_{\tilde{f}}) \geq -\mathrm{N}_1\omega_{\tilde{f}}, \quad |\tilde{z}| \leq \mathrm{N}_1.$$

in $B_{\tilde{f}}(0,1)$, for some constant $\mathrm{N}_1 > 0$. Then there exists a constant $\mathrm{C}_2 > 1$ such that

$$\mathrm{C}_2^{-1} \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \mathrm{C}_2, \ \ \forall\ q \in B_{\tilde{f}}(0,1/2).$$

where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of the matrix $(\tilde{f}_{i\bar{j}})$, C_2 is a positive constant depending on n and N_1.

Then we conclude that $f_{i\bar{j}}, 1 \leq i, j \leq n$ are constants.

Set $v = \sum_{i,j} z_i f_{i\bar{j}}(0)\bar{z}_j$. Then $f - v$ satisfying

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j}(f - v) = 0, \qquad \forall 1 \leq i, j \leq n,$$

$$|f - v| \leq C(1 + |z|^2).$$

Then, $f - v$ is a harmonic function. By the estimates of harmonic function we have, for any $R > 1$ and any multi-index ν with $|\nu| = 2$,

$$|\nabla^\nu (f - v)(p)| \leq \frac{C(n)}{R^2} \max_{D(p,R)} |f - v|,$$

Choose R big enough, we have $|\nabla^\nu (f - v)(p)| \leq C$. Then by Liouville Theorem we have $f - v$ is quadratic.

7. Complex affine technique

Set

$$W = \det(f_{s\bar{t}}), \quad \Psi = \|\nabla \log \det(f_{s\bar{t}})\|_f^2,$$
$$P = \exp\left(\kappa W^{\alpha}\right)\sqrt{W}\Psi.$$

Note that Ψ is a complex version of Φ in Calabi geometry. Denote

$$\|V_{,i\bar{j}}\|_f^2 = \sum f^{i\bar{j}} f^{k\bar{l}} V_{i\bar{l}} V_{k\bar{j}}, \quad \|V_{,ij}\|_f^2 = \sum f^{i\bar{j}} f^{k\bar{l}} V_{,ik} V_{,\overline{lj}}.$$

Denote by $\square = \sum f^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$ the Laplacian operator.

Denote by $\Box = \sum f^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$. We have

$$\frac{\Box P}{P} \geq \frac{\|R_{,i\bar{j}}\|_f^2}{2\Psi} + \alpha^2 \kappa (1 - 2\kappa W^\alpha) W^\alpha \Psi$$
$$- \frac{2|\langle \nabla \mathcal{S}, \nabla V \rangle|}{\Psi} - \left(\alpha\kappa W^\alpha + \tfrac{1}{2}\right) \mathcal{S},$$

where $\langle , \rangle$ denotes the inner product with respect to the metric ω_f.

Estimates for Ψ

LemmaLet $f \in \mathcal{P}sh^{\infty}(\mathbb{C}^n)$. Suppose that there are constants $\mathrm{N}_1, \mathrm{N}_2 > 0$ such that

$$Ric(\omega_f) \geq -\mathrm{N}_1 \omega_f, \qquad \det(f_{i\bar{j}}) \leq \mathrm{N}_2,$$

in $B_f(0, a)$. Then in $B_f(0, a/2)$

$$[\det(f_{i\bar{j}})]^{\frac{1}{2}} \Psi \quad \leq \quad \mathrm{C}_3 \left[\max_{B_f(0,a)} \left(|\mathcal{S}| + \|\nabla \mathcal{S}\|_f^{\frac{2}{3}} \right) + a^{-1} + a^{-2} \right].$$

where C_3 is a constant depending only on n and N_1.

Note that

$$[\det(f_{i\bar{j}})]^{\frac{1}{2}}\Psi = 16\|\nabla[\det(f_{i\bar{j}})]^{\frac{1}{4}}\|_f^2.$$

Substituting $\mathcal{S} = 0$ into the inequality of Ψ we have

$$\|\nabla[\det(f_{i\bar{j}})]^{\frac{1}{4}}\|_f^2 \leq C_1(a^{-1} + a^{-2}), \quad \text{in } B_f(0, a/2).$$

Using the geodesic completeness and by taking $a \to \infty$ we have

$$\nabla[\det(f_{i\bar{j}})^{\frac{1}{4}}] = 0, \quad \text{in } \mathbb{C}^n.$$

It follows that

$$\det(f_{i\bar{j}}) = const$$

Then we have

Corollary. Let $f \in \mathcal{P}sh^{\infty}(\mathbb{C}^n)$ satisfying $\mathcal{S}(f) \equiv 0$. Suppose that

(1) There is a constant $\mathrm{N}_0 > 0$ such that

$$\mathrm{N}_0^{-1}(\sum_{i=1}^{n} z_i \bar{z}_i) \le f \le \mathrm{N}_0(\sum_{i=1}^{n} z_i \bar{z}_i), \quad as\ |z| \to \infty$$

(2)ω_f is complete,

(3) there are constants $\mathrm{N}_1, \mathrm{N}_2 > 0$ such that

$$Ric(\omega_f) \ge -\mathrm{N}_1 \omega_f,$$

$$\det(f_{i\bar{j}}) \le \mathrm{N}_2, \ \text{ in } \mathbb{C}^n,$$

Then f is a quadratical polynomial.

Bernstein Theorem

In differential geometry, Bernstein's theorem is as follows:

Theorem. If Σ is an entire minimal graph in $\mathbb{R}^3$, then u is a linear function.

Let $r(x, y) = (x, y, f(x, y))$. Then the equations of minimal surface can be written as

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0.$$

The Bernstein theorem can be seen as a rigidity theorem of minimal surface.

Consider the Levi-transformation, the the Bernstein theorem can be obtained from the Liouville theorem.

By some transformation the equation is equivalent to the equation

$$\det(D^2\phi) = 1.$$

defined on $\mathbb{R}^2$.

Then Berstein theorem follows from the Theorem of Jörgens.

Above famous Jörgens, Calabi and Pogorelov result on real Monge-Ampere equation can be seen as a rigidity theorem.

There are a lot of extension of this result, such as

- Caffarelli and Li established a quantitative version of the theorem of Jörgens, Calabi and Pogorelov, and showed that this result holds for viscosity solutions.
- Gutierrez and Huang extended to the parabolic Monge-Ampre equation.
- Bao and Xiong proved this type theorem for parabolic MongeAmpre equations with the isolated singularities.
- ......

We consider the extension of the another direction. Let

$$\omega_f = \sqrt{-1}\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$$

be the Kahler metric defined on $(\mathbb{C}^*)^n$ with T^n-action, thus

$$(e^{\sqrt{-1}\theta_1}, \cdots, e^{\sqrt{-1}\theta_n}) \cdot (z_1, \cdots, z_n) = (e^{\sqrt{-1}\theta_1} z_1, \cdots, e^{\sqrt{-1}\theta_n} z_n).$$

Suppose that ω_f is T^n-invariant. Then the plurisubharmonic function f can be reduced to the convex function defined on $\mathbb{R}^n$.

In particular, the Ricci curvature of the metrics ω_f

$$R_{i\bar{j}} = \frac{\partial^2 \log \det(f_{kl})}{\partial x_i \partial x_j}$$

We consider a simple case $R_{i\bar{j}} \equiv 0$. Then we have the PDE:

$$\det\left(f_{kl}\right) = \exp\{\sum a_i x_i + b\},$$

it can be written as

$$\det\left(u_{ij}\right) = \exp\{-\sum a_i \frac{\partial u}{\partial \xi_i} - b\}.$$

Note that the equation of f on $\mathbb{R}^n$ have nontrivial solution when $a_i \neq 0$. For example, $a_1 = 1,\ a_i = 0,\ i \geq 2, b = 0$, we jave

$$f = e^{x_1} + \frac{1}{2}\sum_{i=2}^{n} x_i^2.$$

Theorem (Li-Xu) *Let* $u(\xi_1, ..., \xi_n)$ *be a* C^∞ *strictly convex function defined on whole* $\mathbb{R}^n$. *If* $u(\xi)$ *satisfies the equation above, then* u *must be a quadratic polynomial, and* $a_i = b = 0, i = 1, \cdots, n$.

This Theorem is also a generalization of the Jörgens, Calabi and Pogorelov's theorem.

We want extend the similar results to some linearized Monge-Ampere equations.

Some linearized Monge-Ampere equations

Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Consider the following equation

$$\sum_{i,j=1}^{n} U^{ij} w_{ij} = -L, \quad w = \left[\det\left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}\right)\right]^a \tag{2}$$

where L is some given C^∞ function, $u(\xi)$ is a smooth and strictly convex function defined in Ω, (U^{ij}) denotes the cofactor matrix of the Hessian matrix $\left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}\right)$ and $a \neq 0$ is a constant.

This PDE appears in different geometry problems.

When $a = -\frac{n+1}{n+2}$ and $L = 0$, the PDE (2) is the equation for affine maximal hypersurfaces.

When $a = -1$ the PDE (2) is called the Abreu equation, which appears in the study of the differential geometry of toric varieties, where L is the scalar curvature of the Kähler metric.

Affine Bernstein Problem

About complete affine maximal hypersurfaces there are two famous conjectures, Chern's conjecture and Calabi's conjecture.

Chern's Conjecture: Let $\xi_{n+1} = u(\xi_1, ..., \xi_n)$ be a strictly convex function defined for all $(\xi_1, ..., \xi_n) \in A^n$. if $M = \{(\xi, u(\xi)|\xi \in A^n\}$ is an affine maximal hypersurface, then M must be an elliptic paraboloid.

Calabi's Conjecture: A locally strongly convex, affine complete hypersurface $\xi : M \to A^{n+1}$ with affine mean curvature $L_1 \equiv 0$ is an elliptic paraboloid.

The two conjectures differ in the assumption on the completeness of the affine maximal surface considered. While Chern assumed that the hypersurface is Euclidean complete. Calabi assumed that the hypersurface is complete with respect to the Blaschk metric.

Generally, the affine completeness and the Euclidean completeness are not equivalent. Both problems are called the affine Bernstein problem. This is a long standing problem.

When $n = 2$ both conjectures are solved (Trudinger-Wang, Li-Jia). The higher dimensional affine Bernstein problem is much difficult. So far it remains open.

For a strictly smooth convex function $u(\xi)$, it is natural to consider the metric

$$G_u = \sum \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} d\xi_i d\xi_j,$$

One can also consider Legendre transformation of $(\xi, u(\xi))$.

$$x_i = \frac{\partial u}{\partial \xi_i}, \qquad f(x) = \sum \xi_i \frac{\partial u}{\partial \xi_i} - u(\xi),$$

It is to see that the metric

$$G^f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j,$$

is isometry to G_u.

The theory of Caffarelli and Gutierrez

Caffarelli and Gutierrez obtained the regularity theory of the linearized Monge-Ampre equations of the form

$$\sum_{i,j} U^{ij} v_{ij} = g$$

where U^{ij} is the cofactor matrix of the Hessian matrix $D^2 u$ of a locally uniformly convex function u solving the Monge-Ampre equation

$$0 < \lambda \leq \det(u_{ij}) \leq \Lambda < +\infty$$

where λ and Λ are two positive constants. Caffarelli and Gutierrez's theory play a important role when one studies the equation (2).

3. Generalizations

Consider the following fourth order partial differential equation

$$\text{(2)} \qquad \sum_{i,j=1}^{n} U^{ij}(\psi(H))_{ij} = 0,$$

on a bounded convex domain $\Omega \subset R^n$, where (U^{ij}) is the cofactor matrix of the Hessian matrix D^2u of the strictly convex smooth function u , $H = \det(D^2u)$ and ψ is any smooth function on the half-line $(0, \infty)$ such that $\psi^{'}(t) \neq 0$.

We will discuss some Bernstein properties on the study of (1).

Equation (2) was first put forward by Donaldson. He derived the equation by calculating the Euler-Lagrange equation of the functional

$$\mathcal{F} = \int_{\Omega} \phi(H) d\xi_1 \cdots d\xi_n$$

where $\phi'(t) = \psi(t)$.

Donaldson showed that the construction of Dominic Joyce could be extended to this equation in dimension two assuming that $\phi(t)$ is any smooth, strictly convex function on the half-line $(0, +\infty)$

Equivalent differential equations

Denote the Calabi metric $G_u = \sum u_{ij} d\xi_i \otimes d\xi_j$. Denote $\rho = H^{-\frac{1}{n+2}}$. By a direct calculation we have

$$\Delta_C H = -c(H)\frac{||gradH||_C^2}{H},$$

$$\Delta_C \rho = -\beta\frac{||grad\ \rho||_C^2}{\rho},$$

$$\sum U^{ij} w_{ij} = \sum u^{ij}\frac{w_i w_j}{\omega^2}(2 + \frac{\psi''(H)H}{\psi'(H)}).$$

where $c(t) = \frac{1}{2} + \frac{t\psi''(t)}{\psi'(t)}$, $\beta = -\frac{3n+8}{2} - (n+2)\frac{\psi''(H)H}{\psi'(H)}$ and $w = \frac{1}{H}$,

here Δ_C denote the Laplacian operator w.r.t. G_u.

Equations under Affine Transformations

To using affine blow-up analysis we need obtain the change of the equation under the affine transformations. we consider the following affine transformations

$$\hat{\xi} = A\xi, \quad \hat{u} = \lambda u(A^{-1}\hat{\xi}), \quad \hat{\xi} \in A(\Omega).$$

Denote $\hat{H} = \det\left(\frac{\partial^2 \hat{u}}{\partial\hat{\xi}_i \partial\hat{\xi}_j}\right)$, $\hat{\rho} = \hat{H}^{-\frac{1}{n+2}}$ and $a = \lambda^{-n}|A|^2$. Then

$$\Delta_{\hat{C}} \hat{H} = -c(a\hat{H}) \frac{||grad \hat{H}||^2_{\hat{C}}}{\hat{H}}.$$

Then equation can be re-written as

$$\Delta_{\hat{C}}\hat{\rho} = -\hat{\beta}\frac{||grad\ \hat{\rho}||^2_{\hat{C}}}{\hat{\rho}}.$$

where $\hat{\beta} = -\frac{3n+8}{2} - (n+2)\frac{\psi''(a\hat{H})a\hat{H}}{\psi'(a\hat{H})}$.

$$\sum \hat{U}^{ij}\hat{w}_{ij} = \sum \hat{u}^{ij}\frac{\hat{w}_i\hat{w}_j}{\hat{w}^2}(2 + \frac{\psi''(a\hat{H})a\hat{H}}{\psi'(a\hat{H})}).$$

In particular, equation (2) changes into

$$\sum \hat{U}^{ij}(\psi(a\hat{H}))_{ij} = 0.$$

Calculation of $\Delta\Phi$

The calculation of $\Delta\Phi$ is important to obtain the Berstein Theorem for the equations (2). Following from Li -Jia 's work we can obtain that

Proposition. Let u be a function defined as above. The following estimate holds

$$\begin{aligned}\Delta\Phi \geq & \frac{2\delta}{\rho^2}\sum(\rho_{,ij})^2 + \frac{n(1-\delta)}{2(n-1)}\frac{||grad\ \Phi||^2}{\Phi} \\ & -[\frac{2\beta(n-2+\delta)}{n-1} + 4 - \frac{2n(1-\delta)}{n-1}]\langle grad\ \Phi, grad\ \ln\rho\rangle \\ & +[\frac{(2\beta^2+4\beta)(1-\delta)+2-2n\delta}{n-1} - \frac{(n+2)^2(n-1)}{8n} - 2\beta'\rho]\Phi^2\end{aligned}$$

Estimates for determinant

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let E be the ellipsoid of minimum volume. Then there exists a constant α_n such that

$$\alpha_n E \subset \Omega \subset E,$$

Let T be an affine transformation such that $T(E) = D_1(0)$, the unit disk. Put $\tilde{\Omega} = T(\Omega)$, then

$$\alpha_n D_1(0) \subset \tilde{\Omega} \subset D_1(0)$$

We call T the normalizing transformation of Ω and $\tilde{\Omega} \subset \mathbb{R}^2$ the normalized convex domain.

Let Ω be a normalized convex domain. Denote by $\mathcal{F}(\Omega, C)$ the class of convex functions defined on Ω such that

$$\inf_{\Omega} u = 0, \quad u = C \ \ on \ \ \partial\Omega.$$

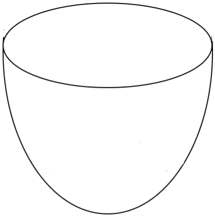

Figure:

As in Chen-Li-Sheng we can obtain the following estimates

Lemma. Let $u \in \mathcal{F}(\Omega, C)$ be a smooth and strictly convex function defined in Ω which satisfies

$$\sum U^{ij} w_{ij} = \sum u^{ij} \frac{w_i w_j}{w} (2 + \frac{\psi''(aH)aH}{\psi'(aH)}).$$

Suppose that

$$\frac{\psi''(t)t}{\psi'(t)} \leq -2, \quad \forall \ t > 0.$$

Then there is a constant $d_1 > 0$ depending only on n,b, d and $\frac{1}{C}$, such that

$$\exp\left\{-\frac{4C}{C-u}\right\} \frac{\det(u_{ij})}{(d+f)^4} \leq d_1$$

on $S_u(p, C)$.

By the method of Li-Jia word by word, we have

Lemma. Let u be a smooth and strictly convex function defined on a bounded convex domain $\Omega \in R^2$. Assume that u satisfies the equation

$$\Delta\rho = -\beta\frac{||grad\rho||^2}{\rho}$$

where $\beta \leq r$ is a continuous function for some postive constant r. Then the following estimate holds:

$$\det(u_{ij}) \geq d_2, \quad for \;\; \xi \in \nabla^f(\Omega^{'*}),$$

where Ω^* denotes the Lengendre transformation domain of u, $\Omega^{'*} \subset \Omega^*$ with $dist(\Omega^{'*}, \partial\Omega^*) > 0$.

Convergence Theorem.

The following kind of convergence theorem first given by Li-Jia, which is important for Bernstein Theorem when we use reduction to absurdity.

Theorem. Let $\Omega_k \in R^2$ be a sequence of normalized convex domain and $u_k \in \mathcal{F}(\Omega_k, C)$ be a sequence of functions and p_k^0 be the minimal point of u_k. Suppose that Ω_k converges to a normalized domain Ω. Then there exists a subsequence of functions, still denoted by u_k, locally uniformly converge to a function u_∞ in Ω and p_k^0 converges to p_∞^0 satisfying:

(1) there exists a constant s and C_2 such that $d_E(p_k^0, \partial\Omega) > 2s$, and in $D_s(p_\infty^0)$

$$||u_k||_{C^{3,\alpha}} \leq C_2$$

for any $\alpha \in (0,1)$; in particular u_k $C^{3,\alpha}$-converges to u_∞ in $D_s(p_\infty^0)$;
(2)there exists a constant $\delta \in (0,1)$, such that $S_{u_k}(p_k^0, \delta) \subset D_s(p_\infty^0)$;
(3)u_k $C^{k+3,\alpha}$-converges to u_∞ in $D_s(p_\infty^0)$.

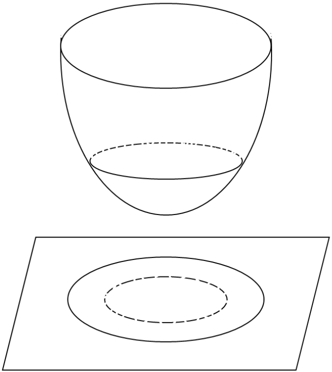

Figure:

Berstein Theorem

By the affine blow-down argument we have

Theorem. Let $u(\xi_1, \xi_2)$ be a C^∞ strictly convex function defined on R^2 and u satisfies the equation (1.1) with

$$r \leq \frac{\psi''(t)t}{\psi'(t)} \leq -2, \quad \forall \ \ t > 0,$$

where r is a negative constant and $\psi(t)$ is a smooth function on $(0, +\infty)$ with $\psi^{'}(t) \neq 0$. Then, u must be a quadratic polynomial.

A geometric interpretation

The differential equation (2) has a natural meaning in relative affine differential geometry.

Let $u(\xi_1, \xi_2, \cdots, \xi_n)$ be a C^∞ strictly convex function defined on a domain $\Omega \subset R^n$. Denote

$$M := \{(\xi, u(\xi)) | \xi_{n+1} = u(\xi_1, \cdots, \xi_n), \ \ (\xi_1, \cdots, \xi_n) \in \Omega\}.$$

We choose the following moving frame field along M:

$$e_i = (0, 0, \cdots, 1, \cdots, 0, u_i),$$

$$e_{n+1} = (0, 0, \cdots, 0, 1).$$

We consider the relative affine normal $Y = e_{n+1}$, then the conormal field U is given by

$$U = (-u_1, \cdots, -u_n, 1).$$

We consider the conformal transformation. Choose a new conormal field $\bar{U} = F(\rho)U$.

We can choose F such that the solution of equation (2) can be seen as a relative affine maximal hypersurface under some relative metric $ds^2 = F(\rho) \sum u_{ij} d\xi_i d\xi_j$.

Example of Rotation invariant solution

We can construct the rotation invariant solution of the equation (2).

Suppose that the solution u of the equation (2) is given by:

$$u(\xi) = \int^{r} v(s)ds, \quad r = |\xi|.$$

Then

$$u_{ij} = \frac{v}{r}\delta_{ij} + (\frac{v'}{r^2} - \frac{v}{r^3})\xi_i\xi_j$$

and

$$H = \det u_{ij} = v'(\frac{v}{r})^{n-1}.$$

Denote $g(v, v', r) = \psi(H)$.

Then $v(r)$ is a solution of the second order differential equation

$$g^{'}v^{n-1} = C.$$

The function

$$u = [(\xi_1)^2 + (\xi_2)^2 + \cdots + (\xi_n)^2]^{\beta}$$

solves the equation (2) on $\mathbb{R}^n/\{0\}$ when $\psi(H) = H^{\frac{\alpha}{n+2}-1}$, if $\beta = \frac{n(2\alpha-n-2)}{2(\alpha n-n-2)}$ and $\alpha \neq \frac{n^2+n-2}{n}, \frac{n+2}{n}$.

Thanks for your attention!