Moser-Trudinger type inequality for the complex Monge-Ampère equations

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Outline

I. The classical Sobolev and Moser-Trudinger inequality

II. The Moser-Trudinger inequality of the complex Monge-Ampère equation

III. Applications to the regularity of the complex Monge-Ampère equation

IV. Futher question: The manifold case

(joint work with Jiaxiang Wang and Xu-jia Wang)

Part I. The classical Sobolev and Moser-Trudinger inequality

The classical Sobolev inequalities

Let Ω be a bounded domain in $\mathbb{R}^n$ and $W_0^{1,p}(\Omega)$ Sobolev space. Then

When $p < n$: for all $1 < q \leq n^* := \frac{np}{n-p}$, there is constant $C > 0$, such that

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W_0^{1,p}(\Omega)}, \quad u \in W_0^{1,p}(\Omega).$$

Moreover, the embedding map $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact when $q < n^*$.

Remark. Poincare inequality

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{W_0^{1,p}(\Omega)}, \quad u \in W_0^{1,p}(\Omega).$$

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The classical Moser-Trudinger inequality

When $p = n$:

$$\int_\Omega e^{\alpha\left(\frac{|u|}{\|u\|_{W_0^{1,n}(\Omega)}}\right)^\beta} \leq C, \; u \in W_0^{1,n}(\Omega)$$

for $0 < \alpha \leq \alpha_0$, $1 \leq \beta \leq \beta_0$, where

$$\alpha_0 = n\omega_{n-1}^{\frac{1}{n-1}}, \quad \beta_0 = \frac{n}{n-1}.$$

Remark. Moser's proof used Schwarz symmetrization. Other proof by blow-up.

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When $p > n$,

$$\|u\|_{C^{\alpha}(\Omega)} \leq C\|u\|_{W_0^{1,p}(\Omega)}, \ \alpha < \left[\frac{n}{p}\right] + 1 - \frac{n}{p}.$$

Generalizations to nonlinear equations

- k-Hessian equations.
- complex k-Hessian equations(including complex Monge-Ampère).

The real Hessian operators

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$.

- The k-Hessian operator is

$$S_k[u] = \sigma_k(\lambda(D^2u)),\ 1 \leq k \leq n$$

where $\lambda(D^2u) = (\lambda_1, ..., \lambda_n)$ are the eigenvalues of D^2u, and

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

- $S_1[u] = \triangle u$; $S_n[u] = \det D^2u$.

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Admissible functions

A smooth function u is k-admissible(k-convex, k-subharmonic) if $S_j[u] \geq 0$ for all $1 \leq j \leq k$, i.e., $\lambda(D^2 u) \in \bar{\Gamma}_k$ where

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_i(\lambda) > 0,\ 1 \leq i \leq k\}.$$

- when $k = 1$, subharmonic functions.
- when $k = n$, convex functions.

Let $\Phi^k(\Omega)$ be the set of all smooth k-admissible functions.

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Remark.

The k-admissibility can be extended to U. S. C. functions (denote by $\overline{\Phi}^k(\Omega)$) and

$$\overline{\Phi}^k(\Omega) \subset W^{1,q}_{loc}$$

for $q < \frac{nk}{n-k}$. When $k \leq \frac{n}{2}$,

$$\overline{\Phi}^k(\Omega) \subset L^p_{loc}, p < \frac{nk}{n-2k}.$$

The exponent is optimal.

A theory of geometric inequalities for real Hessian integrals (Sobolev inequlity, the isoperimetric inequalities for quermassintegrals, etc.) and the equation

$$S_k[u] = (-u)^p$$

was developed by K. S. Chou, X. J. Wang and N. Trudinger.

Let $\Phi_0^k(\Omega)$ be the subspace of functions in $\Phi^k(\Omega)$ vanishing on $\partial\Omega$. Assume that $\partial\Omega$ is $(k-1)$-convex.

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The Hessian integrals(energies)

Denote

$$I_k(u) = \int_\Omega (-u) S_k[u]\, dx.$$

- $I_0 = \int_\Omega -u;$
- $I_1 = \int_\Omega |Du|^2;$
- $I_n = \int_\Omega (-u) \det D^2 u.$

Denote

$$\|u\|_{\Phi_0^k(\Omega)} = [I_k(u)]^{\frac{1}{k+1}}.$$

One can easily verify that $\|\cdot\|_{\Phi_0^k(\Omega)}$ is a norm in $\Phi_0^k(\Omega)$.

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The Hessian Sobolev inequality

Theorem(X. J. Wang, 1994). $u \in \Phi_0^k(\Omega)$.

- If $1 \le k < \frac{n}{2}$,

$$\|u\|_{L^{p+1}(\Omega)} \le C\|u\|_{\Phi_0^k(\Omega)}, \ \forall 1 \le p+1 \le \gamma(n,k),$$

  where $\gamma(n,k) = \frac{n(k+1)}{n-2k}$, C depends on p, k, n and Ω.

- If $k = \frac{n}{2}$,

$$\|u\|_{L^p(\Omega)} \le C\|u\|_{\Phi_0^k(\Omega)}, \ \forall p < \infty,$$

  where C depends on p, n and Ω.

- If $\frac{n}{2} < k \le n$,

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{\Phi_0^k(\Omega)}$$

  where C depends on k, n and Ω.

Moser-Trudinger type inequality

Theorem(G. J. Tian-X. J. Wang, 2010). Let $k = \frac{n}{2}$.

$$\int_\Omega e^{\alpha\left(\frac{-u}{\|u\|_{\Phi_0^k(\Omega)}}\right)^\beta} \leq C,\ u \in \Phi_0^k(\Omega)$$

for $0 < \alpha \leq \alpha_0$, $1 \leq \beta \leq \beta_0$, where

$$\alpha_0 = n\left[\frac{\omega_{n-1}}{k}\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)\right]^{\frac{2}{n}},\ \ \beta_0 = \frac{n+2}{n}.$$

Poincare type inequalities

Theorem(Trudinger-Wang, 1998). For $0 \le l < k \le n$, there exists $C > 0$, such that

$$\|u\|_{\Phi_0^l(\Omega)} \le C\|u\|_{\Phi_0^k(\Omega)}, \; u \in \Phi_0^k(\Omega).$$

The best constant C is attained by the solution of the Dirichlet problem

$$\begin{cases} \frac{S_k[u]}{S_l[u]} = 1, & \Omega, \\ u = 0, & \partial\Omega \end{cases}$$

Remark. In particular, $l = 0$, $k = 1$

$$\|u\|_{L^1(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}.$$

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- All these results were proved by using gradient flow.
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Part II. The Moser-Trudinger inequality of the complex Monge-Ampère equation

Complex Hessian operators

Let Ω be a bounded domain in $\mathbb{C}^n$, $u \in C^2(\Omega)$.

- Let $H_k[u]$ be the complex k-Hessian. $H_1[u] = \triangle u$; $H_n[u] = \det u_{i\bar{j}}$.
- A function u is k-plurisubharmonic if $H_j[u] \geq 0$ for all $1 \leq j \leq k$.
- Denote by $\mathcal{PSH}^k(\Omega)$ be the set of k-plurisubharmonic in $C^2(\Omega)$.

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Remark

The k-plurisubharmonicity can also be extended to U. S. C. functions(denote by $\overline{\mathcal{PSH}}^k(\Omega)$), and Blocki showed

$$\overline{\mathcal{PSH}}^k(\Omega) \subset L^p_{loc}, \;\; p < \frac{n}{n-k}.$$

He also conjectured it for $p < \frac{nk}{n-k}$.

complex Hessian Integral

- Let $\mathcal{PSH}_0^k(\Omega)$ the subset of functions in $\mathcal{PSH}^k(\Omega)$ which vanish on $\partial\Omega$.
- Denote the complex Hessian energy

$$I_k(u) = \int_\Omega (-u) H_k[u]$$

In particular, when $k = 0$, we define $I_0(u) = -\int_\Omega u$.

- For simplicity, we denote by

$$\| \cdot \| = \| \cdot \|_{\mathcal{PSH}_0^k(\Omega)} := [I_k(u)]^{\frac{1}{k+1}}.$$

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Poincare type inequality

Assume Ω is strong k-pseudoconvex.

Theorem(Z. L. Hou, 2008). For $0 \leq l < k \leq n$, there exists $C > 0$, such that

$$\|u\|_{\mathcal{PSH}_0^l(\Omega)} \leq C\|u\|_{\mathcal{PSH}_0^k(\Omega)}, \ u \in \mathcal{PSH}_0^k(\Omega).$$

Sobolev type inequality

Theorem(Zhou, 2013). Let $u \in \mathcal{PSH}_0^k(\Omega)$, $1 < k < n$. Then for all

$$0 \leq p+1 \leq \tilde{\gamma}(k,n) = \frac{n(k+1)}{n-2},$$

we have

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{\mathcal{PSH}_0^k(\Omega)},$$

where C depends on n, k, p and Ω. Moreover, the embedding map

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is compact when $p < \tilde{\gamma}(k,n)$.

- For real k-Hessian equation($k < \frac{n}{2}$), $\gamma(k,n) = \frac{n(k+1)}{n-2k}$.

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Radially symmetric functions

Let $\Omega = B_R$ and $u = u(r)$ is radially symmetric, where $r = |z|$. Assume $u(R) = 0$. Then

$$
\begin{aligned}
u_{i\bar{j}} &= u_r \delta_{ij} + u_{rr} \bar{z}_i z_j, \\
H_k[u] &= \frac{\binom{n-1}{k-1}}{k} \cdot (u_r^k r^n)_r r^{1-n}, \\
I_k[u] &= \frac{\omega_{2n-1} \binom{n-1}{k-1}}{2k(k+1)} \cdot \int_0^R u_r^{k+1} r^{n-1}\, dr.
\end{aligned}
$$

Let $\mathcal{R} = \{u \in C^1([0,R]) : u(R) = 0\}$. Then for all $0 \le p+1 \le \frac{n(k+1)}{n-k}$,

$$
\left(\int_0^R |u|^{1+p} r^{n-1}\, dr \right)^{\frac{1}{1+p}} \le C \left(\int_0^R u_r^{k+1} r^{n-1}\, dr \right)^{\frac{1}{1+k}}, \quad u \in \mathcal{R}.
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Critical exponent

Consider the equation

$$\begin{cases} H_k[u] = (-u)^p & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases}$$

Theorem(C. Li, 2013).

(i) When $p+1 \geq \frac{n(k+1)}{n-k}$, the above equation has no nontrivial nonpositive solution in $C^2(\bar{B}_R) \cap C^4(B_R)$;

(ii) When $1 < p+1 < \frac{n(k+1)}{n-k}$ and $p \neq k$, the above equation has a negative solution in $C^2(\bar{B}_R)$, which is radially symmetric.

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Now we focus on the complex Monge-Ampère equation($k = n$).

The classical case(In complex dim 1): Suppose $\Omega \subset \mathbb{R}^2$. Then

Moser-Trudinger inequality(M-T):

$$\int_\Omega e^{4\pi \frac{-u}{\|\nabla u\|_{L^2(\Omega)}}} dx \le C$$

for u with vanishing boundary value.

Brezis-Merle inequality(B-M):

$$\int_\Omega e^{(4\pi-\delta)\frac{-u}{\|\triangle u\|_{L^1(\Omega)}}} dx \le \frac{4\pi^2}{\delta}(diam(\Omega))^2.$$

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In general dimension: Let $\Omega \in \mathbb{C}^n$ and $u \in PSH(\Omega) \cap C_0^2(\bar{\Omega})$.

complex Monge-Ampere integral(energy):

$$\begin{aligned} \mathcal{E}(u) &= \frac{1}{n+1}\int_\Omega (-u)(dd^c u)^n \\ &= \frac{n!}{(n+1)\pi^n}\int_\Omega (-u)\det(u_{i\bar{j}}) \end{aligned}$$

complex Monge-Ampere mass:

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Aubin's conjecture

There exists $C > 0$, such that

$$\int_\Omega e^{n\left(\frac{-u}{\mathcal{E}(u)^{1/(n+1)}}\right)^{\frac{n+1}{n}}} \leq C$$

for $u \in PSH(\Omega) \cap C_0^2(\bar{\Omega})$, or

$$\int_\Omega e^{n(-u)^{\frac{n+1}{n}}} \leq C$$

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Quasi Moser-Trudinger inequality

Theorem(Berman-Berndtsson 2011). For any $\delta > 0$,

$$\int_\Omega e^{-(n+1-\delta)u} \leq C\delta^{-(n-1)} e^{n!(n+1-\delta)\mathcal{E}(u)}$$

for $u \in PSH(\Omega) \cap C_0^2(\Omega)$.

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$$\int_\Omega e^{(1-\delta)n\left(\frac{-u}{\mathcal{E}(u)^{1/(n+1)}}\right)^{\frac{n+1}{n}}} \leq C$$

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Optimal constant

- The optimal constant is open.
- The optimal constant holds for S^1-invariant functions. $(f(e^{i\theta}z_1, \cdots, e^{i\theta}z_n) = f(z_1, \cdots, z_n))$
- The Schwarz symmetrization works for S^1-invariant functions.

Remark

The proof also gives a Brezis-Merle type inequality

$$\int_\Omega e^{-nu} \leq A(1-\mathcal{M}(u))^{-1},\ \mathcal{M}(u) < 1.$$

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Compare with Tian's α-invariant

Let (M, ω_g) be a Kähler manifold. Define

$$P(M,g) = \{\phi \in C^2(M,\mathbb{R}) \mid \omega_\phi := \omega_g + \sqrt{-1}\partial\bar{\partial}\phi > 0, \sup_M \phi = 0\}.$$

Theorem (Tian, 87')

There exists $\alpha > 0$ and $C > 0$ depending on (M, ω_g) such that

$$\int_M e^{-\alpha\phi}\omega_g^n \leq C,\ \forall \phi \in P(M,g). \tag{1}$$

- In a fixed Kähler class, $\int_M \omega_\phi^n = [\omega_g]^n$ is a constant.

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A PDE proof of the weak Moser-Trudinger inequality

Theorem

Let Ω be a hyperconvex domain. There exist $\alpha > 0$, and a constant $C > 0$ depending on n, α, and diam(Ω), such that

$$\int_\Omega e^{\alpha \frac{-u}{\|u\|_{\mathcal{PSH}_0(\Omega)}}} \leq C, \quad u \in \mathcal{PSH}_0(\Omega) \cap C^2(\bar{\Omega}),\ u \not\equiv 0.$$

- We use the gradient flow method.
- The constant C depends on the *diam*(Ω).

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The Sobolev inequality

Theorem

Let $u \in \mathcal{PSH}_0(\Omega)$. Then for all $p > 0$,

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{\mathcal{PSH}_0(\Omega)}, \; u \in \mathcal{PSH}_0(\Omega) \cap C^2(\bar{\Omega})$$

where C depends on n, p and Ω.

Denote

$$T_{p,\Omega} =: \inf_{u \in \mathcal{PSH}_0(\Omega)} \frac{\mathcal{E}(u)}{\|u\|_{L^{p+1}(\Omega)}^{n+1}}.$$

It suffices to prove

$$T_{p,\Omega} \geq \lambda$$

for some small constant $\lambda > 0$.

Proof of Sobolev inequality

Step 1: Assume the Sobolev inequality

$$\|u\|_{L^p(B)} \leq C_0 \|u\|_{\mathcal{PSH}_0(B)}, \ u \in \mathcal{PSH}_0(B) \cap C^\infty(\bar{B})$$

holds for $p > 0$ on any ball $B \subset \mathbb{C}^{n-1}$. Then the following inequality

$$\left(\int_\Omega |u|^p \right)^{\frac{1}{p}} \leq C \cdot C_0 \left(\int_\Omega (dd^c u)^n \right)^{\frac{1}{n}}, \ u \in \mathcal{PSH}_0(\Omega) \cap C^\infty(\bar{\Omega}).$$

holds on any ball $\Omega \subset \mathbb{C}^n$ with the same radius as B. Here C depends on the radius and is independent of p.

A relation between MA energy and MA mass

Assume the balls are all centered at the origin.

Write $z = (w, \xi) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Let D be the disk in $\mathbb{C}$ with the same radius as B. For any $\xi = t + \sqrt{-1}s \in D$, denote $D_\xi := \{w \in \mathbb{C}^{n-1} \mid |w|^2 \leq 1 - |\xi|^2\}$.

For $u(z) \in \mathcal{PSH}_0(\Omega) \cap C_0^\infty(\bar{\Omega})$, Then denote

$$v(\xi) = \int_{D_\xi} (-u)(d_w d_w^c u)^{n-1}.$$

Then it holds

$$\int_D |-\triangle_\xi v(\xi)|\, dt\, ds \leq 2 \int_\Omega (dd^c u)^n.$$

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By the Sobolev inequality in dimension $n-1$,

$$\begin{aligned}
\left(\int_\Omega |u|^p\right)^{\frac{1}{p}} &= \left(\int_{|\xi|^2\le 1}\int_{D_\xi} |u|^p \, d\mu_w \, d\mu_\xi\right)^{\frac{1}{p}} \\
&\le C_0 \left(\int_{|\xi|^2\le 1}\left(\int_B (-u)\det(u_{w^i\bar{w}^j})\right)^{\frac{p}{n}} d\mu_\xi\right)^{\frac{1}{p}} \\
&= C_0 \left(\int_{|\xi|^2\le 1} [v(\xi)]^{\frac{p}{n}} \, d\mu_\xi\right)^{\frac{1}{p}} \\
&\le C\cdot C_0 \left(\int_{|\xi|^2\le 1} |-\triangle_\xi v(\xi)|\right)^{\frac{1}{n}} \le C\cdot C_0 \left(\int_\Omega (dd^c u)^n\right)
\end{aligned}$$

The Brezis-Merle inequality in real dimension 2 is used in the last inequality.

Proof of Sobolev inequality

Step 2: We show the Sobolev inequality holds for any smooth pseudo-convex domain $\Omega \subset \mathbb{C}^n$ under the assumption

$$\left(\int_\Omega |u|^p\right)^{\frac{1}{p}} \leq C\left(\int_\Omega (dd^c u)^n\right)^{\frac{1}{n}}, \; u \in \mathcal{PSH}_0(\Omega) \cap C^2(\bar{\Omega})$$

holds.

Proof of Sobolev inequality

We denote

$$f(t) = \begin{cases} |t|^p & |t| \le M, \\ e^{-M} t^{-2} & |t| \ge M + e^{-M}, \end{cases}$$

where $M > 1$ is a large constant. Denote

$$J(u) = \int_\Omega (-u) \det(u_{i\bar{j}}) \, dV - \lambda \Big[(p+1) \int_\Omega F[u] \Big]^{\frac{n+1}{p+1}}.$$

Here $F(t) = \int_0^t f(s) \, ds$.

If the Sobolev inequality is not true, then for a small $\lambda > 0$ and large M, we have

$$\inf_{u \in \mathcal{PSH}_0(\Omega) \cap C^2(\bar{\Omega})} J(u) < -1.$$

Proof of Sobolev inequality

Introduce a descent gradient flow for the functional J.

$$\begin{cases} u_t - \log\det(u_{i\bar{j}}) = -\log\lambda\beta(u)f(u) & \text{in } Q = \Omega\times(0,\infty), \\ u(x,0) = w_\epsilon, \quad \text{and} \quad u = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases}$$

where w_ϵ is chosen such that

$$J(w_\epsilon) \leq \inf_{u\in\mathcal{PSH}_0(\Omega)\cap C^2(\bar{\Omega})} J(u) + \epsilon < -1,$$

and

$$\beta(u) = \left[(p+1)\int_\Omega F(u)\right]^{\frac{n-p}{p+1}}.$$

The solution to the flow converges to $u = u_\epsilon$, which solves

$$\begin{aligned} \det(u_{i\bar{j}}) &= \lambda\beta(u)f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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Proof of Sobolev inequality

Claim: We have

$$\begin{aligned} f(u) &= (1+o(1))|u|^p, \\ \beta(u) &= (1+o(1))\Big[\int_\Omega |u|^{p+1}\Big]^{(n-p)/(p+1)} \approx \|u\|_{L^{p+1}}^{n-p} \end{aligned}$$

when M goes to infinity.

We have

$$\|u\|_{L^{p+1}} \le C\left(\int_\Omega (dd^c u)^n\right)^{\frac{1}{n}} = C\left(\int_\Omega \lambda\beta(u)f(u)\right)^{\frac{1}{n}} \le C\lambda^{\frac{1}{n}}\beta^{\frac{1}{n}}\|u\|_{L^{p+1}}^{\frac{p}{n}}.$$

We get $\lambda \ge C$. This is a contradiction to that λ is small.

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Proof of Sobolev inequality

Step 3: For any pseudoconvex domains Ω_1, Ω_2 with $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$, We have

$$T_{p,\Omega_1} \geq T_{p,\Omega_2}.$$

Similar to the Hessian equation.

Proof of Sobolev inequality

Step 4: Induction arguments:

- By the Sobolev inequality in real dimension 2, i.e., complex dimension 1,

$$\left(\int_\Omega |u|^p\right)^{\frac{1}{p}} \leq C\left(\int_\Omega (dd^c u)^n\right)^{\frac{1}{n}},\ u \in \mathcal{PSH}_0(\Omega) \cap C^2(\bar{\Omega})$$

  holds for any ball in $\mathbb{C}^2$.

- By Step 1, we have Sobolev inequality for any ball in C^2.

- By Step 2, the Sobolev inequality for any hyperconvex domain $\Omega \subset \mathbb{C}^2$ follows.

- all dimensions.

Proof of Moser-Trudinger inequality

Let $C_{n,p+1}$ be the Sobolev constant in dimension n, i.e.,

$$\|u\|_{L^p(\Omega)} \leq C_{n,p} \cdot \|u\|_{\mathcal{PSH}_0(\Omega)}.$$

Equivalently, it holds

$$\int_\Omega \left(\frac{|u|}{\|u\|_{\mathcal{PSH}_0(\Omega)}} \right)^p d\mu \leq C^p_{n,p+1}.$$

By checking the proof of Sobolev inequality, we have $C_{n,p} \leq C \cdot C_{n-1,p}$ for some constant indpendent of p.

Hence, by the Moser-Trudinger inequality when $n = 1$(real dimension 2), there exists $\alpha > 0$,

$$\int_\Omega e^{\alpha \frac{-u}{\|u\|_{\mathcal{PSH}_0(\Omega)}}} d\mu \quad = \quad \int_\Omega \sum_{j=1}^{\infty} \frac{1}{j!} \left(\alpha \frac{-u}{\|u\|_{\mathcal{PSH}_0(\Omega)}} \right)^j d\mu \leq C.$$

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About Brezis-Merle type inequality

The proof above also implies a PDE proof to Brezis-Merle type inequality:

Suppose Ω is a hyper-convex domain. There exists a constant $\alpha > 0$ such that

$$\int_\Omega e^{\alpha(-u)} \leq C, \quad \mathcal{M}(u) = 1.$$

Part III. Applications in regularity of the complex Monge-Ampère equation

Let $\Omega \in \mathbb{C}^n$ and $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ be a given function. We consider

$$\begin{cases} \det(u_{i\bar{j}}) = (dd^c u)^n = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{2}$$

- Caffarelli-Kohn-Nirenberg-Spruck: smooth data on f, φ, Ω.
- $f \in L^2(\Omega)$: L^∞-estimate by Cheng-Yau, Cegrell-Persson, Bedford, Blocki, etc.

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Theorem (Kolodziej)

Suppose $f \in L^p(\Omega)$, $p > 1$ and $\varphi \in L^\infty(\Omega)$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a plurisubharmonic solution to (2). Then there is a constant $C > 0$ depending on n, p and Ω such that

$$|\inf_{\Omega} u| \leq |\inf_{\Omega} \varphi| + C\|f\|_{L^p(\Omega)}^{\frac{1}{n}}. \tag{3}$$

Remark

- Kolodziej's proof used capacity theory(Bedford-Taylor)

$$cap(K,\Omega) := \sup\left\{\int_K (dd^c u)^n : \ u \in PSH(\Omega),\ -1 \le u < 0\right\}.$$

- The L^∞-estimate holds when $L^1 \log L^{n+\epsilon}$(Lorenz-Zygmumd space)

$$L^1(\log L)^q(\Omega) := \left\{f \mid \int_\Omega |f|(\log(e+|f|))^q \, dx < \infty\right\}.$$

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Question(Blocki-Kolodziej): Find a PDE proof for the L^∞ estimate.

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References:
Dinew-Guedj-Zeriahi, Open problems in pluripotential theory, 2016.

AIM problem lists, available at http://aimath.org/pastworkshops/mongeampereproblems.pdf.

We establish a PDE approach based on the Sobolev type inequality

(1). L^∞-estimate

(2). Stability theorem

(3). Hölder regularity

(1). L^∞-estimate

Recall: L^∞-estimate in elliptic PDE theory

- Linear elliptic equation: De Giorgi, Moser, Stampaccia, etc.
- p-Laplacian: Boccardo-Murat-Puel.
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PDE Proof of the L^∞-estimate: preparation

- By quasi M-T,

$$\|u\|_{L^{p+1}(\Omega)} \le C[\mathcal{E}(u)]^{\frac{1}{n+1}}, \quad u \in \mathcal{PSH}_0(\Omega) \cap C_0^2(\Omega).$$

  Key: The constant C depends on *diam*(Ω).

- Assume $\|f\|_{L^p(\Omega)} = 1$.

- Replacing the boundary function by $\inf_\Omega \varphi$, it suffices to prove the estimate for $\varphi = 0$ by the comparison principle.

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Proof of the L^∞-estimate

Claim: For any $s > 0$, let $\Omega_s = \{u \in \Omega \mid u < -s\}$,

$$|\Omega_s| \leq C\frac{1}{s}|\Omega|^{1+\delta}, \tag{4}$$

where $\delta = \frac{1}{np^*} - \frac{1}{\beta}(1 + \frac{1}{np^*}) > 0$ when choosing $\beta > 1 + p^* k$.

Proof of the claim

$$
\begin{aligned}
\mathcal{E}(u) &= \frac{n!}{(n+1)\pi^n} \int_\Omega (-u) f \\
&\leq \frac{n!}{(n+1)\pi^n} \|f\|_{L^p(\Omega)} \|u\|_{L^{p^*}(\Omega)} \\
&\leq C|\Omega|^{\frac{1}{p^*}(1-\frac{1}{\beta})} \|u\|_{L^{\beta p^*}(\Omega)} \\
&\leq C|\Omega|^{\frac{1}{p^*}(1-\frac{1}{\beta})} [\mathcal{E}(u)]^{\frac{1}{n+1}},
\end{aligned}
$$

where p^* is conjugate to p and $\beta > 1$. It follows that

$$
[\mathcal{E}(u)]^{\frac{1}{n+1}} \leq C|\Omega|^{\frac{1}{np^*}(1-\frac{1}{\beta})}.
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Using Sobolev inequality again, we have

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\|u\|_{L^1(\Omega)} \leq |\Omega|^{1-\frac{1}{\beta}} \|u\|_{L^\beta(\Omega)} \leq C|\Omega|^{1+\delta}.
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Iteration argument

Choose s_0 sufficiently large such that for $\Omega_0 = \Omega_{s_0}$, $|\Omega_0| \leq \frac{1}{2}|\Omega|$.

For any $k \in \mathbb{Z}_+$, define

$$s_k = s_0 + \sum_{j=1}^{k} 2^{-\delta j}, \quad \Omega_k := \Omega_{s_k}, \quad u^k = u + s_k.$$

Then u^k satisfies

$$\begin{cases} \det(u_{i\bar{j}}) = f & \text{in } \Omega_k, \\ u = 0 & \text{on } \partial\Omega_k. \end{cases}$$

Then

$$\|u\|_{L^1(\Omega_k)} \leq C|\Omega_k|^{1+\delta}.$$

Hence the constants depend on the diameters of the domains, are uniform for k.

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We claim that $|\Omega_{k+1}| \leq \frac{1}{2}|\Omega_k|$ for any k.
(proof: By induction, we assume the inequality holds for $k \leq l$.

$$|\Omega_{l+1}| \leq C2^{\delta(l+1)}|\Omega_l|^{1+\delta} \leq C2^{\delta(l+1)}\left(\frac{|\Omega_0|}{2^l}\right)^{\delta} \cdot |\Omega_l|$$
$$\leq C\frac{1}{s_0^{\delta}}|\Omega|^{\delta(1+\delta)}|\Omega_l| \leq \frac{1}{2}|\Omega_l|$$

provided s_0 is sufficiently large.) This implies that the set

$$\{u \in \Omega \mid u < -s_0 - \sum_{j=1}^{\infty}(\frac{1}{2^{\delta}})^j\}$$

has measure zero. Hence,

$$|\inf_{\Omega} u| \leq s_0 + \sum_{j=1}^{\infty}(\frac{1}{2^{\delta}})^j = s_0 + \frac{1}{2^{\delta}-1} \leq C.$$

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(2). The stability theorem

Theorem

Assume $\psi \in C^0(\partial\Omega)$. Let $v \in L^\infty(\Omega)$ be a PSH solution to

$$\begin{cases} (dd^c v)^n = g\mu & \text{in } \Omega, \\ v = \psi & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Then there exists a constant C depending on $\|f\|_{L^p(\Omega)}$, $\|g\|_{L^p(\Omega)}$ and n, and the upper bound of the diameter of Ω, such that

$$\|u - v\|_{L^\infty(\Omega)} \leq C\left\{\|f - g\|_{L^1(\Omega)}^{\frac{1}{n}\frac{\delta}{1+\delta}} + \|\varphi - \psi\|_{L^\infty(\partial\Omega)}^{\frac{\delta}{1+\delta}}\right\}$$

where δ is defined as before.

(3). Hölder regularity

The Hölder continuity

Theorem (Guedj-Kolodziej-Zeriahi 08')

Suppose Ω is strictly pseudo-convex. Assume $0 \le f \in L^p(\Omega)$, $p > 1$ and $\varphi \in C^{0,2\alpha}(\partial\Omega)$. Let $\hat{u}$ be the solution to the Dirichlet problem with $(dd^c\hat{u})^n = 0$ and boundary data φ, if $\triangle\hat{u}$ has finite mass in Ω, then

$$u \in C^{0,\alpha'}, \quad \text{for all } \alpha' < \min(\alpha, \frac{2}{p^* n + 1}).$$

Remark:

- The Hölder continuity was first proved by Bedford-Taylor under the assumption that $f^{\frac{1}{n}} \in C^{\alpha}(\Omega)$ and $\phi \in C^{2\alpha}(\partial\Omega)$.
- The technical condition of $\hat{u}$ is satisfied when $\varphi \in C^{1,1}(\partial\Omega)$.
- We give a PDE proof without using capacity theory.

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Part IV. Futher question: The manifold case

Manifold case

Assume (M, ω_g) is a Kähler manifold.

Question. Are there Sobolev and Moser-Trudinger typed inequalities for Kahler potentials $\phi \in [\omega_g]$ in terms of the Monge-Ampère energy

$$\mathcal{E}(\phi) = -\frac{1}{(n+1)!} \sum_i \int_M \phi \omega_\phi^i \wedge \omega_g^{n-i}?$$

On the two-sphere the inequality was first shown by Moser with sharp constant. Subsequently, the general Riemann surface case was settled by Fontana with the same sharp constant.

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On general Kähler manifold

Theorem(Berman-Berndtsson). Assume $[\omega_g]$ is an integral class. Then there exsit $c, C > 0$, such that

$$\int_M e^{c\left(\frac{-\phi}{\mathcal{E}^{1/(n+1)}(\phi)}\right)^{\frac{n+1}{n}}} \leq C$$

for $\phi \in [\omega_g]$.

- When $[\omega_g] \in H^2(M, \mathbb{Z})$(integral class), the metric can be identified with the curvature of a metric on an ample line bundle $L \to M$.
- The proof used convexity properties of certain functionals along geodesics.

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Thank you for your attention!