Symmetrization with respect to Mixed Volumes

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- Let Ω be an open bounded set of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be a measurable function.
- Symmetric rearrangement $\Omega^\sharp$ of Ω is the centered open ball having the same volume as Ω, i.e.

$$\Omega^\sharp = B_R(0), \text{ where } \omega_n R^n = |\Omega|.$$

- Schwarz Symmetrization (symmetric decreasing rearrangement) $u^\sharp$ of u is

$$\begin{aligned} u^\sharp : \quad & \Omega^\sharp \to \mathbb{R}, \\ & u^\sharp(x) = u^\sharp(|x|) = \sup\{t < 0 : \{-|u| < t\} \leq \omega_n |x|^n\}. \end{aligned}$$

- By definition,

$$\mathrm{Vol}(\{u^\sharp < t\}) = \mathrm{Vol}(\{-|u| < t\}).$$

Schwarz symmetrization

Properties of Schwarz symmetrization:

- (Cavalieri's principle)

$$\int_\Omega |u|^p dx = \int_{\Omega^\sharp} |u^\sharp|^p dx.$$

- (Polya-Szego's principle)

$$\int_\Omega |\nabla u|^p dx \geq \int_{\Omega^\sharp} |\nabla u^\sharp|^p dx.$$

- (Hardy-Littlewood's inequality)

$$\int_\Omega uv dx \leq \int_{\Omega^\sharp} u^\sharp v^\sharp dx.$$

Schwarz symmetrization

Applications of Schwarz symmetrization I:

- (Rayleigh-Faber-Krahn's inequality for first Dirichlet eigenvalue)
$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\sharp).$$
- (Saint-Venant's principle for torsional rigidity)
$$\tau(\Omega) \leq \tau(\Omega^\sharp).$$
- Because of the variational property
$$\lambda_1(\Omega) = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}, \quad \tau(\Omega) = \sup_{u \neq 0} \frac{\left(\int_\Omega u\right)^2}{\int_\Omega |\nabla u|^2}$$

Schwarz symmetrization

Applications of Schwarz symmetrization II:

- (Talenti '76) Let u be the solution to

  $$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

  and

  $$\Delta v = f^\sharp \text{ in } \Omega^\sharp, \quad v = 0 \text{ on } \partial\Omega^\sharp,$$

  Then

  $$|u^\sharp| \leq v \text{ in } \Omega^\sharp.$$

- Because $\sup|u| = \sup|u^\sharp|$, this gives a sharp estimate for $|u|$.

Tatenti-Tso symmetrization

Talenti-Tso symmetrization (Symmetrization w.r.t. Quermassintegral)
(Talenti '81 $n = k = 2$ and Tso '89 any n and k)

- Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain with C^2 boundary.
- Steiner's formula:

$$\mathrm{Vol}\,(\Omega + tB) = \sum_{k=0}^{n} \binom{n}{k} t^k W_k(\Omega),$$

where $W_k(\Omega)$ is k-th quermassintegral given by

$$W_k(\Omega) = \frac{1}{n} \int_{\partial\Omega} \frac{1}{\binom{n-1}{k-1}} \sigma_{k-1}(\kappa) d\mathcal{H}^{n-1}.$$

Let $\zeta_k(\Omega)$ be k-mean radius given by

$$\zeta_k(\Omega) := \left(\frac{W_k(\Omega)}{\omega_n} \right)^{\frac{1}{n-k}}.$$

Tatenti-Tso symmetrization

- Define the following class of admissible functions

$$\Phi_0(\Omega) := \left\{ u \in C^2(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega, u \text{ strictly convex} \right\}.$$

- Let $\Omega^\sharp_{k-1}$ be the centered open ball having the same W_k as Ω, i.e.

$$\Omega^\sharp_{k-1} = B_R(0), \text{ where } R = \zeta_k(\Omega).$$

For $u \in \Phi_0(\Omega)$,

$$u^\sharp_{k-1}(x) = \sup\left\{ t \leq 0 : \zeta_{k-1}(\{u \leq t\}) \leq |x| \right\}.$$

Namely,

$$\zeta_{k-1}(\{u \leq t\}) = \zeta_{k-1}(\{u^\sharp_{k-1} \leq t\}).$$

- The case $k = 1$ is Schwarz symmetrization.

- The k-Hessian integral

$$I_k[u,\Omega] = \int_\Omega (-u)\sigma_k(\nabla^2 u)dx$$

Theorem (Talenti, Tso)

For $u \in \Phi_0(\Omega)$,

$$I_k[u,\Omega] \geq I_k[u^\sharp_{k-1}, \Omega^\sharp_{k-1}].$$

Equality holds if and only if Ω is a ball and u is radial.

- The proof used crucially Alexandrov-Fenchel's inequality between quermassintegrals for convex domains and Reilly's work on Hessian operators.
  Trudinger '97 generalized this to k-convex domains and u having k-convex level sets, however, used his Alexandrov-Fenchel's inequality between quermassintegrals for k-convex domains, whose proof has gap.

Convex symmetrization

- Let F be a norm on $\mathbb{R}^n$, i.e., positive, convex and 1-homogenous.
- Let F^0 be its dual norm, i.e.,

$$F^0(\xi) = \sup_{x \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}.$$

- The set $\mathcal{W}_F := \{F^0(\xi) \leq 1\}$ is called unit Wulff ball and $\partial \mathcal{W}_F$ is unit Wulff shape. Denote $\kappa_n = |\mathcal{W}|$ and $\mathcal{W}_R$ the centered Wulff ball with radius R.
- Anisotropic Dirichlet integral $\int_\Omega F(\nabla u)^2$
- Anisotropic Laplacian

$$\Delta_F u = \operatorname{div}(\nabla_\xi (\frac{1}{2} F^2)(\nabla u))$$

Convex symmetrization

Alvino-Ferone-Lions-Trombetti '97 introduces the convex symmetrization which diminishes the anisotropic Dirichlet integral

- Convex symmetrization Ω^* of Ω

$$\Omega^* = \mathcal{W}_R, \text{ where } |\mathcal{W}_R| = \kappa_n R^n = |\Omega|.$$

The convex symmetrization of

$$u^* : \Omega^* \to \mathbb{R},$$
$$u^*(x) = u^*(F^0(x)) = \sup\{t < 0 : \{-|u| < t\} \leq \kappa_n (F^0(x))^n\}.$$

Convex symmetrization

Theorem (Alvino-Ferone-Lions-Trombetti '97)

$$\int_\Omega F(\nabla u)^2 \geq \int_{\Omega^*} F(\nabla u^*)^2.$$

Equality holds if and only if Ω *is a Wulff ball and* u *is radial w.r.t.* F*, namely,* $u(x) = u(F^0(x))$*.*

Corollary

Let u *be the solution to*

$$\Delta_F u = f \;\; in\; \Omega, \quad u = 0 \;\; on\; \partial\Omega,$$

and

$$\Delta_F v = f^* \;\; in\; \Omega^*, \quad v = 0 \;\; on\; \partial\Omega^*,$$

Then

$$|u^*| \leq v \;\; in\; \Omega^*.$$

Symmetrization w.r.t. mixed volumes

- Our aim is to study the Talenti-Tso symmetrization in the anisotropic case.
- Motivation: Alexandrov-Fenchel's inequality holds for mixed volumes for two convex bodies.
  To be precise, Let F be a given norm whose Wulff ball is given by $\mathcal{W}_F$, Let Ω be a convex domain, then

$$\mathrm{Vol}((1-t)\Omega + t\mathcal{W}_F) = \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k W_k(\Omega, \mathcal{W}_F).$$

  Denote

$$\zeta_{k,F}(\Omega) = \left(\frac{W_k(\Omega, \mathcal{W}_F)}{\kappa_n} \right)^{\frac{1}{n-k}}$$

  Then

$$\zeta_{k,F}(\Omega) \geq \zeta_{l,F}(\Omega), \text{ for } 0 \leq l < k \leq n-1.$$

Symmetrization w.r.t. mixed volumes

- Let $F \in C^3(\mathbb{R}^n \setminus \{0\})$ be a strongly convex norm on $\mathbb{R}^n$, strongly convex means $\mathrm{Hess}(\frac{1}{2}F^2)$ is positive definite.
- Denote by $A_F[u] = ((A_F)_{ij}[u])$ the matrix

$$\begin{aligned}(A_F)_{ij}[u] &:= \partial_{x_j}\left[\partial_{\xi_i}\left(\frac{1}{2}F^2\right)(\nabla u)\right] \\ &= \sum_l \left(\frac{1}{2}F^2\right)_{il}(\nabla u)u_{lj}, \text{ when } \nabla u \neq 0.\end{aligned}$$

  We regard $A_F[u] = 0$ when $\nabla u = 0$, in the case that F is not the Euclidean norm.
- In case F is the Euclidean norm, $A_F[u] = \nabla^2 u$.

Symmetrization w.r.t. mixed volumes

- The anisotropic k-Hessian operator of u is defined as

$$S_{k,F}[u] := S_k(A_F[u]).$$

- The anisotropic k-Hessian integral of u is defined by

$$\begin{aligned} I_{k,F}[u,\Omega] &= \int_\Omega (-u) S_{k,F}[u]\,dx = \int_\Omega (-u) S_k(A_F[u])\,dx \\ &= \int_\Omega S^{ij}_{k,F}[u] F F_i u_j\,dx. \end{aligned}$$

  The second line follows from $\partial_j S^{ij}_{k,F} = 0$.

- In case F is the Euclidean norm,

$$A_F[u] = \nabla^2 u, \quad S_{k,F}[u] = S_k(\nabla^2 u), \quad I_{k,F}[u,\Omega] = I_k[u,\Omega].$$

Symmetrization w.r.t. mixed volumes

- Let Ω^*_{k-1} be the centered open ball having the same $\zeta_{k,F}$ as Ω, i.e.

$$\Omega^*_{k-1} = \mathcal{W}_R, \text{ where } R = \zeta_{k,F}(\Omega).$$

For $u \in \Phi_0(\Omega)$,

$$u^*_{k-1}(x) = \sup\left\{t \leq 0 : \zeta_{k-1,F}(\{u \leq t\}) \leq F^0(x)\right\}.$$

Namely,

$$\zeta_{k-1,F}(\{u \leq t\}) = \zeta_{k-1,F}(\{u^*_{k-1} \leq t\}).$$

Symmetrization w.r.t. mixed volumes

Theorem (Della Pietra-Gavitone '15, Della Pietra-Gavitone-X. '19)

For $u \in \Phi_0(\Omega)$,

$$\int_\Omega |u|^p dx \leq \int_{\Omega^*_{k-1}} |u^*_{k-1}|^p dx,$$

$$I_{k,F}[u,\Omega] \geq I_{k,F}[u^*_{k-1}, \Omega^*_{k-1}].$$

Equality holds if and only if Ω is a Wulff ball and u is radial w.r.t. F, namely, $u(x) = u(F^0(x))$.

- Della Pietra-Gavitone '15 proved the case $n = k = 2$ by direct computation.
- Difficulty for general case, compare to the case of Euclidean norm, is the study of S_k on non-symmetric matrix $A_F[u]$.

- Define anisotropic L^p $k-$Hessian integral

$$I_{k,p,F}[u,\Omega] = \int_\Omega S_k^{ij}[u] F^{p-k} F_i u_j \, dx.$$

In particular,

$$I_{k,k+1,F} = k \, I_{k,F}, \quad I_{1,p,F} = \int_\Omega F^p(\nabla u) \, dx.$$

Theorem (Della Pietra-Gavitone-X. '19)

For $u \in \Phi_0(\Omega)$, $p \geq 1$,

$$I_{k,p,F}[u,\Omega] \geq I_{k,p,F}[u^*_{k-1}, \Omega^*_{k-1}].$$

Equality holds if and only if Ω is a Wulff ball and u is radial w.r.t. F, namely, $u(x) = u(F^0(x))$.

Symmetrization w.r.t. mixed volumes

Corollary (Anisotropic Sobolev type inequality)

For $u \in \Phi_0(\Omega)$*,*

- *if* $p < n - k + 1$*, then*

$$\|u\|^p_{L^{\frac{np}{n-k+1-p}}(\Omega)} \leq C(n,k,p,F) I_{k,p,F}[u,\Omega],$$

- *if* $p > n - k + 1$*, then*

$$\|u\|^p_{L^\infty(\Omega)} \leq C(n,k,p,F) I_{k,p,F}[u,\Omega],$$

- *if* $p = n - k + 1$*, then*

$$\|u\|^p_{L^\Psi(\Omega)} \leq C I_{p,k,F}[u;\Omega].$$

where $L^\Psi(\Omega)$ *is the Orlicz space associated to the function* $\Psi(t) = e^{|t|^{\frac{p}{p-1}}} - 1$*.*

Symmetrization w.r.t. mixed volumes

Theorem (A priori estimate for anisotropic Hessian equation)

Let $u \in \Phi_0(\Omega)$ be a solution of the following Dirichlet problem

$$\begin{cases} S_{k,F}[u] = f(x) & a.e.\ in\ \Omega \\ u = 0 & on\ \partial\Omega. \end{cases}$$

Then

$$u^*_{k-1}(x) \geq v(x)\ in\ \Omega^*_{k-1},$$

where v is the unique anisotropic radially symmetric solution of the following symmetrized problem:

$$\begin{cases} S_{k,F}[v] = f^*_0(x) & in\ \Omega^*_{k-1,F} \\ v = 0 & on\ \Omega^*_{k-1,F}. \end{cases}$$

- A study of S_k for non-symmetric matrix

$$S_k(A) = \frac{1}{k!} \sum_{1 \le i_1, \cdots, i_k, j_1, \cdots, j_k \le n} \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k} A_{i_1 j_1} \cdots A_{i_k j_k},$$

$$\begin{aligned} S_k^{ij}(A) &= \frac{\partial S_k(A)}{\partial A_{ij}} \\ &= \frac{1}{(k-1)!} \sum_{1 \le i_1, \cdots, i_k, j_1, \cdots, j_k \le n} \delta_{i_1 \cdots i_{k-1} i}^{j_1 \cdots j_{k-1} j} A_{i_1 j_1} \cdots A_{i_{k-1} j_{k-1}}. \end{aligned}$$

Important ingredients

Proposition

For an $n \times n$ *matrix* $A = (A_{ij})$, *we have*

$$S_k^{ij}(A) = S_{k-1}(A)\delta_{ij} - \sum_l S_{k-1}^{il}(A) A_{jl}.$$

Proposition

For an $n \times n$ *matrix* $A = (A_{ij})$, *we have*

$$\sum_j \partial_j S_{k,F}^{ij}[u] = 0.$$

Important ingredients

- A study of anisotropic curvatures of level sets
- Let M be a smooth closed hypersurface in $\mathbb{R}^n$ and ν be the unit Euclidean outer normal of M. The anisotropic outer normal of M is defined by

$$\nu_F = \nabla F(\nu).$$

The anisotropic principal curvatures $\kappa_F = (\kappa_1^F, \ldots, \kappa_{n-1}^F) \in \mathbb{R}^{n-1}$ are defined as the eigenvalues of the map

$$d\nu_F \colon T_pM \to T_{\nu_F(p)}\mathcal{W}_F.$$

For $k = 1, \ldots, n$ the anisotropic k-th mean curvature of M is $S_k(\kappa_F)$.

Important ingredients

Proposition

Assume Σ_t is a non-degenerate level set of u, i.e., $\nabla u \neq 0$ along Σ_t. Then the anisotropic k-th mean curvature $\sigma_k(\kappa_F)$ of Σ_t

$$\sigma_k(\kappa_F) = S_k\left(\sum_l F_{il} u_{lj}\right) = \frac{1}{F^{k+1}} \sum_{i,j} S^{ij}_{k+1,F}[u] u_j F_i,$$

- In the case F is the Euclidean norm, it reduces to (Reilly '70s)

$$\sigma_k(\kappa) = \sum_{i,j} \frac{S^{ij}_{k+1}(\nabla^2 u) u_i u_j}{|\nabla u|^{k+2}},$$

- In the case $k = 1$, it reduces to (Wang-X. '11)

$$H_F = \sum_{i,j} F_{ij} u_{ij} = \frac{1}{F}\left(\Delta_F u - \sum_{i,j} F_i F_j u_{ij}\right).$$

Important ingredients

Proposition (Reilly '73, '76)

$$\frac{d}{dt}W_{k,F}(\overline{\Omega_t}) = \frac{1}{\binom{n}{k}}\int_{\Sigma_t}\frac{S_k(\kappa_F)F(\nu)}{F(\nabla u)}\,d\mathcal{H}^{n-1}.$$

$$\frac{d}{dt}\zeta_{k,F}(\overline{\Omega_t}) = \frac{1}{(n-k)\kappa_n\binom{n}{k}}\frac{1}{[\zeta_{k,F}(\overline{\Omega_t})]^{n-k-1}}\int_{\Sigma_t}\frac{S_k(\kappa_F)F(\nu)}{F(\nabla u)}\,d\mathcal{H}^{n-1}.$$

Important ingredients

- A study of anisotropic Hessian integral on anisotropic radial function

Proposition

Let $u(x) = v(r)$, where $r = F^0(x)$. Then

$$S_{k,F}[u] = \binom{n-1}{k-1} \frac{v''(r)}{r} \left(\frac{v'(r)}{r}\right)^{k-1} + \binom{n-1}{k} \left(\frac{v'(r)}{r}\right)^{k}$$
$$= \binom{n-1}{k-1} r^{-(n-1)} \left(\frac{r^{n-k}}{k}(v'(r))^k\right)'.$$

$$I_{k,F}[u, \mathcal{W}_R] = \kappa_n \binom{n}{k} \int_0^R r^{n-k} v'(r)^{k+1}\, dr.$$

Sketch of proof

$$
\begin{aligned}
& C_{n,k}[\zeta_{k-1}(\overline{\Omega_t})]^{(n-k)(k+1)} \leq C_{n,k}[\zeta_k(\overline{\Omega_t})]^{(n-k)(k+1)} \\
= & \left(\int_{\Sigma_t} S_{k-1}(\kappa_F)F(\nu)d\mathcal{H}^{n-1}\right)^{k+1} \\
\leq & \left(\int_{\Sigma_t} \frac{S_{k-1}(\kappa_F)}{F(\nabla u)}F(\nu)d\mathcal{H}^{n-1}\right)^k \int_{\Sigma_t} S_{k-1}(\kappa_F)F(\nabla u)^k F(\nu)d\mathcal{H}^{n-1} \\
= & \left\{\tilde{C}_{n,k}[\zeta_{k-1}(\overline{\Omega_t})]^{n-k}\frac{d}{dt}\zeta_{k-1}(\overline{\Omega_t})\right\}^k \int_{\Sigma_t} S_{k-1}(\kappa_F)F(\nabla u)^k F(\nu)d\mathcal{H}^{n-}
\end{aligned}
$$

$$
\begin{aligned}
I_k[u,\Omega] &= \frac{1}{k}\int_m^0 \int_{\Sigma_t} S_{k-1}(\kappa_F)F^k(\nabla u)F(\nu)\, d\mathcal{H}^{n-1}dt \\
&\geq \kappa_n \binom{n}{k} \int_m^0 \frac{[\zeta_{k-1}(\overline{\Omega_t})]^{n-k}}{\left[\frac{d}{dt}\zeta_{k-1}(\overline{\Omega_t})\right]^k}\, dt \\
&= \kappa_n \binom{n}{k} \int_0^R r^{n-k}(\rho'_{k-1}(r))^{k+1}\, dr = I_{k,F}[u^*_{k-1}, \mathcal{W}_R].
\end{aligned}
$$

Thank you for your attention!