A class of Monge-Ampère equations on the sphere

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Positive convex weak solution to (MAE):

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- when $q \neq n+1$: Ω satisfies $d\widetilde{C}_q(\Omega, \cdot) = u^p d\mu$;
- when $q = n+1$ and $p \geq 1$: Ω satisfies $d\mathcal{S}(\Omega, \cdot) = u^{p-1} d\mu$;
- when $q = n+1$ and $p < 1$: Ω satisfies $u^{1-p} d\mathcal{S}(\Omega, \cdot) = d\mu$.

- $\mathcal{S}(\Omega,\cdot)$ is the surface area measure: $\mathcal{S}(\Omega,\omega)=\mathcal{H}^n(\{z\in\partial\Omega:\ \nu_\Omega(z)\in\omega\})$.
- $\widetilde{C}_q(\Omega,\cdot)$ is the q-th dual curvature measure: $\widetilde{C}_q(\Omega,\omega)=\int_{\mathscr{A}^*_\Omega(\omega)} r^q(\xi)d\sigma_{\mathbb{S}^n}(\xi)$, introduced by Huang-Lutwak-Yang-Zhang [Acta Math. 2016].
- $\widetilde{C}_{p,q}(\Omega,\cdot)$ is the L_p dual curvature measure: $\widetilde{C}_{p,q}(\Omega,\omega)=\int_{\mathscr{A}^*_\Omega(\omega)} \frac{r^q(\xi)}{u^p(\mathscr{A}_\Omega(\xi))}d\sigma_{\mathbb{S}^n}(\xi)$, introduced by Lutwak-Yang-Zhang [Adv. Math. 2018].
- Support function: $u=u_\Omega:\mathbb{S}^n\to\mathbb{R}$, defined by

$$u(x)=\max\{x\cdot z:\ z\in\Omega\}.$$

- Radial function: $r=r_\Omega:\mathbb{S}^n\to\mathbb{R}$, defined by

$$r(\xi)=\max\{t:\ t\xi\in\Omega\}.$$

- Radial mapping: $\vec{r}=\vec{r}_\Omega:\mathbb{S}^n\to\partial\Omega$, defined by $\vec{r}(\xi)=r_\Omega(\xi)\xi$.
- Radial Gauss mapping: $\mathscr{A}=\mathscr{A}_\Omega$, set-valued mapping,

$$\mathscr{A}(\omega)=\{\nu_\Omega(\vec{r}_\Omega(\xi)):\ \xi\in\omega\},\ \text{for } \omega\subset\mathbb{S}^n.$$

- Reverse radial Gauss mapping: $\mathscr{A}^*=\mathscr{A}^*_\Omega$, set-valued mapping,

$$\mathscr{A}^*(\omega)=\{\xi\in\mathbb{S}^n\ :\ \nu_\Omega(\vec{r}_\Omega(\xi))\in\omega\},\ \text{for } \omega\subset\mathbb{S}^n.$$

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  - Chou-Wang [Adv Math 2006].
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Theorem 1 (H.Chen-Li, 2018)

Let $p > 0$ and $q \neq n+1$. Let $\mu \in NCH$ (finite, not concentrated on any closed hemisphere).

- *if $p > q$, then (MAE) admits a positive convex weak solution.*
- *if $p = q$, then (MAE) admits a non-negative convex weak solution with μ replaced by $\lambda\mu$ for some constant $\lambda > 0$.*
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Moreover if $d\mu = fd\sigma_{\mathbb{S}^n}$ with $f > 0, \in C^\infty(\mathbb{S}^n)$, and $p \geq q$, then (MAE) has a positive, smooth, uniformly convex classical solution u.

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- If $f > 0, \in C^\infty(\mathbb{S}^n)$, then the multiplier λ (when $p = q$) is unique. Solution u is unique ($p > q$), and is unique up to a dilation ($p = q$).

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$$\fint_{\mathbb{S}^n} r_{\widetilde{\Omega}_\varepsilon}^{q_\varepsilon} = 1.$$

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Let $\widetilde{u}_\varepsilon$ be the support function of $\widetilde{\Omega}_\varepsilon$.
We show that $(\widetilde{\Omega}_\varepsilon, \lambda_\varepsilon)$ converges to (u_∞, λ) as $\varepsilon \to 0$, and u_∞ satisfies (MAE) with μ replaced by $\lambda\mu$.

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Firstly observe that (MAE) is (up to a rescale) the Euler equation of

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One attempts to use variational argument:

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Good news: Take a minimising sequence Ω_j, then

$$C \geq \int_{\mathbb{S}^n} u_{\Omega_j}^p f d\sigma_{\mathbb{S}^n} \geq \delta_{n,f} (\max_{\mathbb{S}^n} u_{\Omega_j})^p \quad (\text{using } p > 0).$$

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Hence Ω_j converges to a limit Ω, which minimises the extreme problem.
Difficulty: It is possible that $o \in \partial\Omega$. If this occurs, then we are in trouble to show the minimiser Ω is a solution to the problem.

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$$\widehat{F}_\varepsilon(z) = \begin{cases} \dfrac{1}{p} z^p, & \text{if } z \geq 2\varepsilon, \\ \dfrac{z^{p+\delta+\varepsilon}}{p+\delta+\varepsilon}, & \text{if } 0 \leq z < \varepsilon. \end{cases}$$

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Consider the flow

$$\partial_t X = \Big(\phi(t) u - f u F_\varepsilon(u) r^{n+1-q} K\Big)\nu \quad \textit{(flow)}$$

with $X(\cdot, 0)$ a C^∞, uniformly convex hypersurface, enclosing the origin, where

$$\phi(t) := \Big(\int_{\mathbb{S}^n} f u F_\varepsilon(u) d\sigma_{\mathbb{S}^n} \Big) \Big/ \Big(\int_{\mathbb{S}^n} r^q(\cdot, 0) d\sigma_{\mathbb{S}^n} \Big)$$

such that $\int_{\mathbb{S}^n} r^q(\cdot, t) d\sigma_{\mathbb{S}^n} = \text{const.}$ under the (flow).

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Let Ω_t be the convex body with support function $u(\cdot, t)$. Consider

$$\mathcal{J}_\varepsilon(\Omega_t) = \int_{\mathbb{S}^n} \widehat{F}_\varepsilon(u) f d\sigma_{\mathbb{S}^n} - \frac{1}{q} \int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}.$$

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It can be verified that $\frac{d}{dt}\mathcal{J}_\varepsilon \leq 0$ under the (flow), and equality holds iff

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$$\min_{\mathbb{S}^n} u(\cdot, t) \quad \geq \quad \min \left\{ \left[\frac{\phi(t)}{\max_{\mathbb{S}^n} f} \right]^{\frac{1}{\varepsilon}}, \varepsilon, \min_{\mathbb{S}^n} u(\cdot, 0) \right\}.$$

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By $\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n} = \text{const.}$, we derive a lower bound of $\max_{\mathbb{S}^n} u(\cdot, t)$, which gives a positive lower bound of $\phi(t)$. Hence

$$\min_{\mathbb{S}^n} u(\cdot, t) \quad \geq \quad C_\varepsilon, \quad \forall t.$$

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C^2-estimates: $1/C_\varepsilon \le \kappa_i(\cdot, t) \le C_\varepsilon, \ \forall t$. (i) $K(\cdot, t) \le C_\varepsilon, \ \forall t$; (ii) $\frac{1}{\kappa_i}(\cdot, t) \le C_\varepsilon, \ \forall t$.

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Approximation: Denote by Ω_ε the convex body whose support function is u_ε.

$$\int_{\mathscr{A}^*_{\Omega_\varepsilon}(\omega)} r_\varepsilon^q \, d\sigma_{\mathbb{S}^n} = \int_\omega u_\varepsilon F_\varepsilon(u_\varepsilon) f d\sigma_{\mathbb{S}^n}.$$

Since $u_\varepsilon \le C$ uniformly, $u_\varepsilon \to u$ (as $\varepsilon \to 0$) which solves (MAE) in weak sense.

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- For $p < 0$, we have the following result

Theorem 2 (H.Chen-Li, 2018)

Let $p < 0$.

- *For $p > q$, (MAE) admits a positive and convex weak solution u, if $\mu \in NCH$. Moreover if $d\mu = fd\sigma_{\mathbb{S}^n}$ with $f > 0, \in C^\infty(\mathbb{S}^n)$, then the solution $u \in C^\infty$ and is uniformly convex, and is unique.*
- *For $p = q$, if $d\mu = fd\sigma_{\mathbb{S}^n}$ with $f > 0, \in C^\infty(\mathbb{S}^n)$, then there is a unique $\lambda > 0$ such that (MAE) has a unique positive, smooth and uniformly convex solution u, with f replaced by λf.*

For origin-symmetric case, we can prove the existence below by variational argument.

Theorem 3 (H.Chen-S.Chen-Li, 2018)

Let $d\mu = f d\sigma_{\mathbb{S}^n}$, f be an even function on $\mathbb{S}^n$, and $1/C \leq f \leq C$. Suppose $q > 0$ and $-q^* < p < 0$, where $q^* > 0$ is defined as

$$q^* = \begin{cases} \dfrac{q}{q-n} & \text{if } \ q > n+1, \\ n+1 & \text{if } \ q = n+1, \\ \dfrac{nq}{q-1} & \text{if } \ 1 < q < n+1, \\ +\infty & \text{if } \ 0 < q \leq 1. \end{cases}$$

Then (MAE) has an even, positive, weak solution u, and $\Omega = \Omega_u$ is origin-symmetric and has a strictly convex and $C^{1,\alpha}$ boundary.
Moreover, if f is additionally smooth, then (MAE) has an even, positive, smooth and uniformly convex solution u.

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Solve the optimisation problem

$$\min_{\Omega \in \mathcal{K}_0^e} \left\{ \widehat{\Phi}_{p,f}(\Omega) : \fint_{\mathbb{S}^n} r_\Omega^q d\sigma_{\mathbb{S}^n} = 1 \right\}, \quad (\textit{Min Prob})$$

where $\widehat{\Phi}_{p,f}(\Omega) = \frac{1}{p} \int_{\mathbb{S}^n} f u_\Omega^p d\sigma_{\mathbb{S}^n}$.

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Given $q > 0$, and $\gamma \in (0, q^*]$, $\gamma \neq +\infty$, there is a $C_{n,q,\gamma} > 0$ such that

$$\left(\fint_{\mathbb{S}^n} r_\Omega^q d\sigma_{\mathbb{S}^n} \right)^{\frac{1}{q}} \left(\fint_{\mathbb{S}^n} u_\Omega^{-\gamma} d\sigma_{\mathbb{S}^n} \right)^{\frac{1}{\gamma}} \leq C_{n,q,\gamma}, \quad \forall\ \Omega \in \mathcal{K}_0^e.$$

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Let Ω_j be a minimising sequence. Let $\gamma=-p$. Then $0<\gamma<q^*$.

$$\begin{aligned}\widehat{\Phi}_{p,f}(\Omega_j) &= -\frac{1}{\gamma}\left[\int_{\{u_{\Omega_j}\text{very small}\}}+\int_{\{u_{\Omega_j}\ \text{ not small, not large}\}}+\int_{\{u_{\Omega_j}\ \text{ very large}\}}\right]u_{\Omega_j}^{-\gamma} f d\sigma_{\mathbb{S}^n}\\ &\to 0\ \text{ as }\ \max u_{\Omega_j}\to+\infty.\end{aligned}$$

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Note that $\widehat{\Phi}_{p,f}(\Omega_j) \leq \widehat{\Phi}_{p,f}(B_1) = -\frac{1}{\gamma} \int_{\mathbb{S}^n} f < 0$. Hence Ω_j is uniformly bounded and so converges to a limit Ω_0 solving (Min Prob).
Since Ω_0 contains the origin in its interior, we show that $\partial\Omega_0$ is strictly convex and $C^{1,\alpha}$.
We then verify that u_{Ω_0} solves (MAE).

A key ingredient for studying (MAE) is its variational property: (MAE) is the Euler equation (up to a rescale) of

$$\mathcal{J}_{p,q,f,1}(\Omega) = \Phi_{p,f}(u_\Omega) + \Psi_{q,1}(r_\Omega),$$

where

$$\Phi_{p,f}(u) = \begin{cases} \frac{1}{p} \log \fint_{\mathbb{S}^n} f u^p \, d\sigma_{\mathbb{S}^n}, & \text{if } p \neq 0, \\ \fint_{\mathbb{S}^n} f \log u \, d\sigma_{\mathbb{S}^n}, & \text{if } p = 0, \end{cases}$$

and

$$\Psi_{q,g}(r) = \begin{cases} -\frac{1}{q} \log \fint_{\mathbb{S}^n} g r^q \, d\sigma_{\mathbb{S}^n}, & \text{if } q \neq 0, \\ -\fint_{\mathbb{S}^n} g \log r \, d\sigma_{\mathbb{S}^n}, & \text{if } q = 0. \end{cases}$$

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Theorem 4 (H.Chen-S.Chen-Li, 2018)

If p and q satisfy one of the following (A1)-(A3), then

$$\det(\nabla^2 u + uI) = \frac{\sqrt{u^2 + |\nabla u|^2}^{\,n+1-q}}{u^{1-p}} \quad on\ \mathbb{S}^n$$

has at least two solutions: $u_1 \equiv 1$, $u_2 \not\equiv 1$.

(A1) $q - 2n - 2 > p \geq 0$,

(A2) $p + 2n + 2 < q \leq 0$,

(A3) $q > 0$ *and* $-q^* < p < \min\{0, q - 2n - 2\}$, *where*

$$q^* = \begin{cases} \dfrac{q}{q-n} & \text{if } q > n+1, \\ n+1 & \text{if } q = n+1, \\ \dfrac{nq}{q-1} & \text{if } 1 < q < n+1, \\ +\infty & \text{if } 0 < q \leq 1. \end{cases}$$

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For any even function $\eta \in C^\infty(\mathbb{S}^n)$, $\eta \neq$ const.,

$$\Omega_t^\eta := \{z \in \mathbb{R}^{n+1} : x \cdot z \leq 1 + t\eta(x),\ x \in \mathbb{S}^n\} \in \mathcal{K}_0^e.$$

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When $|t|$ is sufficiently small, Ω_t^η has support function $u(x,t) = 1 + t\eta(x)$.
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$$\frac{d}{dt}\bigg|_{t=0} \mathcal{J}_{p,q,1,1}(\Omega_t^\eta) = 0$$

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The Poincaré inequality on the sphere said

$$\inf\left\{\frac{\fint_{\mathbb{S}^n} |\nabla\eta|^2 d\sigma_{\mathbb{S}^n}}{\fint_{\mathbb{S}^n} \eta^2 d\sigma_{\mathbb{S}^n}} :\ \eta \in C^\infty(\mathbb{S}^n) \text{ is even}, \fint_{\mathbb{S}^n} \eta d\sigma_{\mathbb{S}^n} = 0, \eta \not\equiv 0\right\} = 2n+2;$$

Proof of Theorem 4:

Therefore if $q > p + 2n + 2$ then there is an η_0 such that

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{J}_{p,q,1,1}(\Omega_t^{\eta_0}) < 0.$$

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Consequently

$$\begin{aligned} \mathcal{J}_{p,q,1,1}(\Omega_t^{\eta_0}) &= \mathcal{J}_{p,q,1,1}(B_1) + t\frac{d}{dt}\Big|_{t=0}\mathcal{J}_{p,q,1,1}(\Omega_t) + \frac{1}{2}t^2\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{J}_{p,q,1,1}(\Omega_t) + o(t^2) \\ &< \mathcal{J}_{p,q,1,1}(B_1), \quad \text{for } t \in (0, \varepsilon_0'), \end{aligned}$$

This shows that B_1 is not a minimiser.

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This shows that B_1 is not a minimiser.

But under the condition (A1)-(A3), we can show that there is a $\Omega \in \mathcal{K}_0^e$ minimising $\mathcal{J}_{p,q,1,1}$, and u_Ω is a solution to (MAE).

So $u_\Omega \not\equiv 1$, completing the proof.

$\square$

Remark. Poincaré inequality on the sphere

$$\inf\left\{\frac{\fint_{\mathbb{S}^n}|\nabla\eta|^2 d\sigma_{\mathbb{S}^n}}{\fint_{\mathbb{S}^n}\eta^2 d\sigma_{\mathbb{S}^n}}:\ \eta\in C^\infty(\mathbb{S}^n)\text{ is even},\ \fint_{\mathbb{S}^n}\eta d\sigma_{\mathbb{S}^n}=0,\eta\not\equiv 0\right\}=2n+2;$$

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Compute the second variation of the functionals

$$\widetilde{\mathcal{J}}_{p,n+1,1,1}(\Omega)=\frac{1}{p}\log\fint_{\mathbb{S}^n}u^p d\sigma_{\mathbb{S}^n}-\frac{1}{n+1}\log\fint r^{n+1}d\sigma_{\mathbb{S}^n},$$

for even case; and for non-even case,

$$\widetilde{\mathcal{J}}_p(\Omega,z)=\frac{1}{p}\log\fint_{\mathbb{S}^n}u_z^p d\sigma_{\mathbb{S}^n}-\frac{1}{n+1}\log\fint r_z^{n+1}d\sigma_{\mathbb{S}^n}$$

The functional is decreasing under flow

$$\partial_t X=-K^\alpha\nu,\alpha>\frac{1}{n+2},$$

which deforms hypersurfaces into a round point (Andrews-Guan-Ni [Adv Math 2016]; Brendle-Daskalopoulos-Choi [Acta Math 2017]).

Thank you for your attention.