量子网络理论及相关数学问题

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1 Tensor Networks

- A tensor network is simply a collection of tensors connected by contractions.

  Tensor network methods are employed in modern quantum information science, condensed matter physics, mathematics and computer science, representation theory, category theory, etc.

  Tensor networks come with an intuitive graphical language, which was dated back to the early 1970's by Roger Penrose:

  [1]. Roger Penrose. Applications of negative dimensional tensors. Combinatorial Mathematics and Its Applications, 1971

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- Let V be a finite-dimensional Hilbert space, V^* be the dual space of V. Then each basis $\{e^j\}_j$ of V, there is a dual basis $\{\eta_k\}_k$ of V^*, that is $\eta_j(e^i) = \delta^i_j$.

  For each finite-dimensional Hilbert space W_i, given a basis $\{e^{(i)k}\}_k$ of W_i and a dual basis $\{\eta^{(i)}_k\}_k$ of V_i^*, then each order-(p, q) tensor

$$T \in W_1 \otimes W_2 \otimes \cdots \otimes W_p \otimes V_1^* \otimes V_2^* \otimes \cdots \otimes V_q^*$$

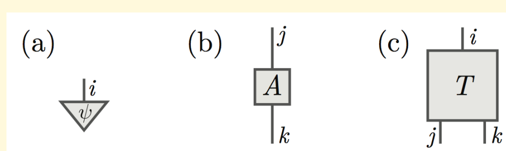
  can be represented by

$$T = \sum_{i_1 \cdots i_p; j_1 \cdots j_q} T_{i_1 \cdots i_p}{}^{j_1 \cdots j_q} e^{(1)^{i_1}} \otimes \cdots \otimes e^{(p)^{i_p}} \otimes \eta^{(1)}_{j_1} \otimes \cdots \otimes \eta^{(q)}_{j_q}.$$

- By the *Einstein summation convention*, we can denote

$$T = T_{i_1 \cdots i_p}{}^{j_1 \cdots j_q} e^{(1)^{i_1}} \otimes \cdots \otimes e^{(p)^{i_p}} \otimes \eta^{(1)}_{j_1} \otimes \cdots \otimes \eta^{(q)}_{j_q}.$$

Example. Diagram (a) represents a vector, diagram (b) is a matrix, and diagram (c) the tensor $T^i{}_{jk}$.



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- The map $W \otimes W^* \to \mathbb{K}, w \otimes \phi \mapsto \phi(w)$ decides a natural bilinear map.

  One can apply this map to *contraction* of the corresponding upper and lower indices.

  For example, if we happen to have $W_1 = V_1$ we may contract the corresponding indices on T:

$$\begin{aligned} C_{1,1}(T) &= T_{i_1\cdots i_p}{}^{j_1\cdots j_q} \eta^{(1)}_{j_1}(e^{(1)^{i_1}}) e^{(2)^{i_2}} \otimes \cdots \otimes e^{(p)^{i_p}} \otimes \eta^{(2)}_{j_2} \otimes \cdots \otimes \eta^{(q)}_{j_q} \\ &= T_{ki_2\cdots i_p}{}^{kj_2\cdots j_q} e^{(2)^{i_2}} \otimes \cdots \otimes e^{(p)^{i_p}} \otimes \eta^{(2)}_{j_2} \otimes \cdots \otimes \eta^{(q)}_{j_q}, \end{aligned}$$

  since the defining property of a dual basis is $\eta^{(1)}_{j_1}(e^{(1)i_1}) = \delta^{i_1}_{j_1}$.

❑ The order-(p,q) tensor T can be reinterpreted as multilinear map T' from vectors to vectors:

$$T' : V_1 \otimes \cdots \otimes V_q \to W_1 \otimes \cdots \otimes W_p,$$

$$T'(v^{(1)} \otimes \cdots \otimes v^{(q)}) = T_{i_1 \cdots i_p}{}^{j_1 \cdots j_q} e^{(1)^{i_1}} \otimes \cdots e^{(p)^{i_p}} \times \eta^{(1)}_{j_1}(v^{(1)}) \times \cdots \eta^{(q)}_{j_q}(v^{(q)}),$$

where the tensor T contract the corresponding indices.

The order-(p,q) tensor T can be also reinterpreted as multilinear map T'' from dual vectors to dual vectors:

$$T'' : W_1^* \otimes \cdots \otimes W_p^* \to V_1^* \otimes \cdots \otimes V_q^*,$$

$$T''(\varphi_{(1)} \otimes \cdots \otimes \varphi_{(p)}) = T_{i_1 \cdots i_p}{}^{j_1 \cdots j_q} \varphi_{(1)}(e^{(1)^{i_1}}) \times \cdots \varphi_{(p)}(e^{(p)^{i_p}}) \times \eta^{(1)}_{j_1} \otimes \cdots \eta^{(q)}_{j_q}.$$

Thus, we may move any of the vector spaces to the other side of the arrow by taking their dual:

$$W \otimes V^* \;\cong\; \mathbb{K} \to W \otimes V^* \;\cong\; V \to W \;\cong\; V \otimes W^* \to \mathbb{K} \;\cong\; W^* \to V^*.$$

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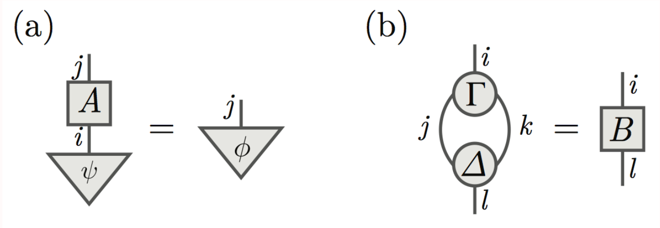
- Connecting two tensor legs with a wire means that the corresponding indices are contracted, that is, summed over.



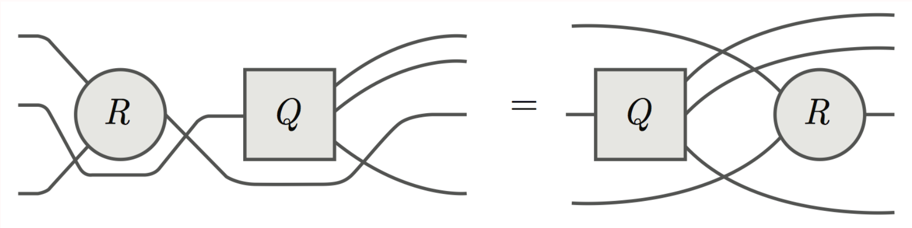
In diagram (a):

$$A^j{}_i \psi^i = \phi^j.$$

In diagram (b):

$$\Gamma^i{}_{jk} \Delta^{jk}{}_l = B^i{}_l.$$

- The wires are allowed to cross tensor symbols and other wires:



- Two or more tensors in a diagram form a tensor network.

Recently, there are many important papers:

[2]. Jacob Biamonte. Charged String Tensor Networks. Proceedings of the National Academy of Sciences of U. S. A, 2017

[3]. Glen Evenbly. Hyperinvariant Tensor Networks and Holography. Physical Review Letter. 119, 141602, 2017

[4]. F. Motzoi, M. P. Kaicher, F. K. Wilhelm. Linear and Logarithmic Time Compositions of Quantum Many-Body Operators. Physical Review Letter. 119, 160503, 2017

- An important milestone was David Deutsch's pioneering use of the diagrammatic notation in quantum computing, developing the so called quantum circuit model.

  The quantum circuits model is widely used to describe experimental implementations of quantum algorithms, to classify the entangling properties and computational power of quantum gates.

  [5]. D. Deutsch. Quantum computational networks. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 1989.

An Example, Quantum circuits

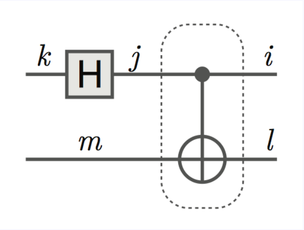
Quantum circuit diagrams is as follows:

(1). Time goes from left to right.

(2). Horizontal lines represent qubits.

(3). Quantum gates and measurements are represented by various symbols.

We consider a quantum circuit that consists of two tensors, a controlled NOT gate C-NOT, and a Hadamard gate H:



- The CNOT and Hadamard gates are:

$$
\begin{aligned}
\mathsf{CNOT} &= \sum_{ab} |a, a \oplus b\rangle\langle a, b|, \\
\mathsf{H} &= \frac{1}{\sqrt{2}} \sum_{ab} (-1)^{ab} |a\rangle\langle b|,
\end{aligned}
$$

where the addition in the CNOT gate is modulo 2, that is $1 \oplus 1 = 0 \oplus 0 = 0, 1 \oplus 0 = 0 \oplus 1 = 1, 1 \oplus a = \neg a$ where $\neg a$ is the Boolean negation of a.

The diagram translates into the following equation:

$$
\mathsf{CNOT}^{il}{}_{jm} \mathsf{H}^{j}{}_{k} = \frac{1}{\sqrt{2}} \sum_{jkm} (-1)^{jk} |j, j \oplus m\rangle\langle k, m|.
$$

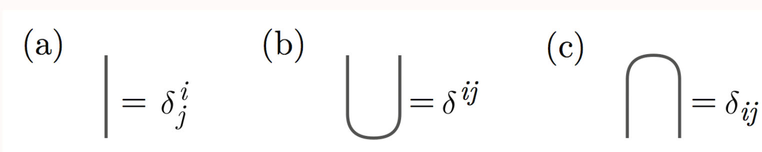
- In quantum information science, one often introduces a computational basis $\{|k\rangle\}_k$ for each space V, $\{\langle j\,|\}_j$ for its dual basis, and

$$T = \sum_{ijk} T^i{}_{jk} |i\rangle\langle jk|$$

is a tensor in $V \otimes V^* \otimes V^*$.

- 1. Cup and cap tensors

In the previous section, wires are used to denote the contraction of pairs of tensor indices. However, it is often useful to interpret certain wire structures as independent tensors of their own. We start with three of these special wire tensors that allow one to rearrange the arms and legs of another tensor:



The identity tensor (a) is used for index contraction, cup (b) and the cap (c) raise and lower tensor indices by bending the corresponding tensor legs.

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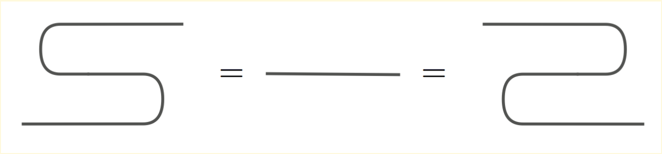
❑ Expanding them in the basis are:

$$
\begin{aligned}
|I\rangle &= \delta^i{}_j |i\rangle\langle j| = \sum_k |k\rangle\langle k|, \\
|\cup\rangle &= \delta^{ij} |ij\rangle = \sum_k |kk\rangle, \\
\langle\cap| &= \delta_{ij} \langle ij| = \sum_k \langle kk|.
\end{aligned}
$$

2. Snake tensors

One can raise and then lower an index or vice versa, which amounts to doing nothing at all. This idea is captured diagrammatically by the so called snake:

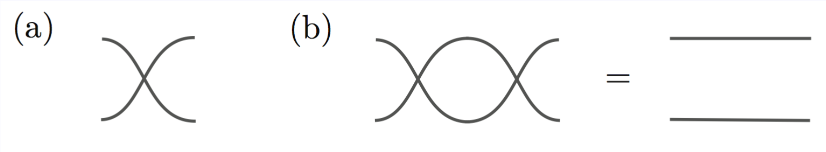


In abstract index notation it is expressed succinctly as

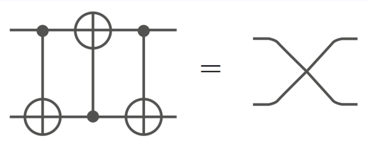
$$\delta^{ij}\delta_{jk} = \delta^{i}{}_{k} = \delta_{kj}\delta^{ji}.$$

❑ 3. Swap tensor

Crossing two wires can be thought of as swapping the relative order of two vector spaces. If both wires represent the same vector space, it represents swapping the states of the two subsystems:

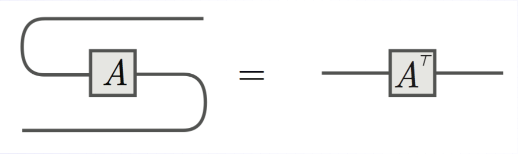


It is an important quantum gate in quantum computing, with a well-known implementation in terms of three controlled-NOT gates:



4. Transpose of Matrices

Given $A^i{}_j$, we may reverse the positions of its indices using a cup and a cap. This is equivalent to transposing the corresponding linear map in the basis:



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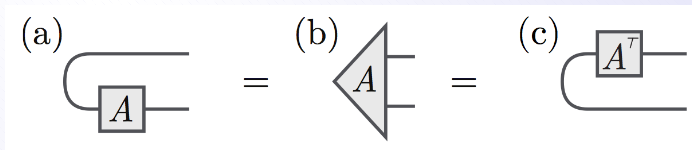
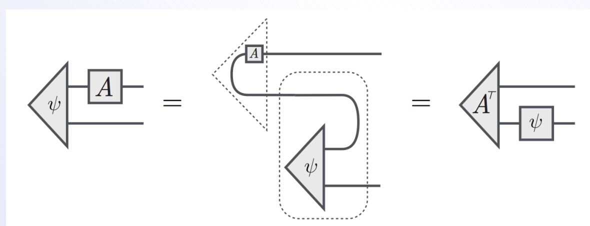
❑ 5. Map-state duality



Using the snake tensor, we have:



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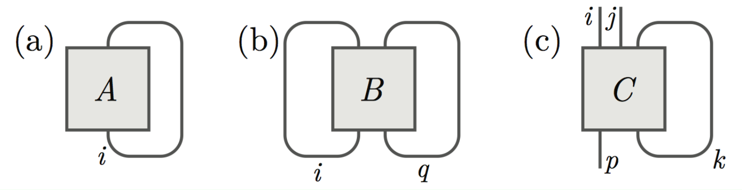
6. Trace of matrices

Diagram (a) below represents the trace ${A^i}_i$. Diagram (b) represents the trace ${B^{iq}}_{iq}$.



Partial trace means contracting only party of tensors, such as the tensor ${C^{ijk}}_{pk}$ shown in diagram (c).

7. Singular value decomposition

Singular value decomposition of matrices is at the heart of many numerical simulation algorithms. If we can factor (1,1) tensor into blocks with simple properties:

(i) (1,1) diagonal tensor storing the singular values,

(ii) two (1,1) unitary tensors,

The we can factor all (p,q) tensors.

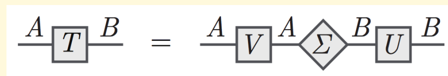
(1,1) tensor can be considered as linear maps $T : A \to B$, where A and B be two vector spaces, thus,

$$T^b{}_a = U^b{}_j \Sigma^j{}_i V^i{}_a,$$

where U and V are unitary, and Σ is real, non-negative, diagonal and has the singular values $\{\sigma_k\}_k$ of T on its diagonal. It can be expanded as

$$\Sigma = \sum_k \sigma_k |k\rangle_B \langle k|_A.$$

❑ Diagrammatically this is represented as



Now using the wire bending techniques, we have the famous Schmidt decomposition theorem:

Given a vector $|\psi\rangle \in A \otimes B$, we may use the snake equation to convert it into a linear map $\psi : A \to B$ and apply the above process on ψ, we have

$$|\psi\rangle_{A\otimes B} = \sum_i \sigma_i |\psi_i\rangle_A |\phi_i\rangle_B.$$

The singular values $\{\sigma_k\}_k$ now correspond to the Schmidt coefficients.

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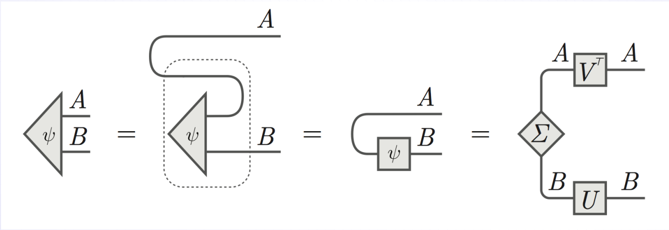
❑ Schmidt decomposition theorem has important applications in quantum information theory.

Diagrammatically Schmidt decomposition theorem is represented as



8. Matrix product states

Given a n qubits quantum state $|\psi\rangle$, fully describing this state requires an amount of information that grows exponentially with n, that is

$$|\psi\rangle = \sum_{ij\cdots k} \psi^{ij\cdots k}|ij\cdots k\rangle,$$

has 2^n coefficients $\psi^{ij\cdots k}$, this is too difficult !!

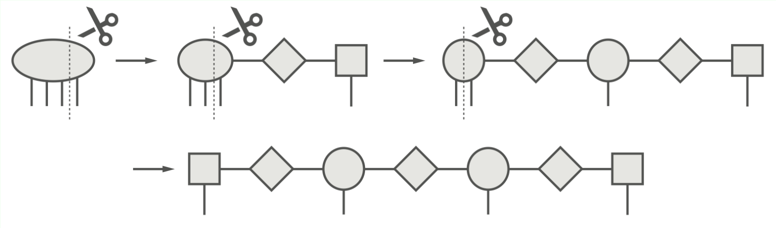
We need to find new representation of $|\psi\rangle$ such that the data is less intensive. We wish to write $|\psi\rangle$ as

$$|\psi\rangle = \sum_{ij\cdots k} Tr(A_i^{[1]}A_j^{[2]}\cdots A_k^{[n]})|ij\cdots k\rangle,$$

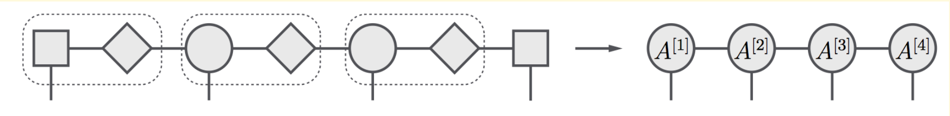
where $A_i^{[1]}, A_j^{[2]}, \cdots, A_k^{[n]}$ are indexed sets of matrices. Calculating the components of $|\psi\rangle$ becomes to calculate the the products of matrices, hence the name matrix product state.

- If the matrices are bounded in size, then the amount of information required to describe them is only linear in n. For instance, if the matrices are at most χ by χ, the size of the representation scales as $n\chi^2$.

  We can represent quantum states into quantum networks by recursive application of the singular value decomposition process, for example for 4 qubit state $|\psi\rangle = \sum_{ijkm} \psi^{ijkm}|ijkm\rangle$, we have:



- If the tensors are grouped, we have



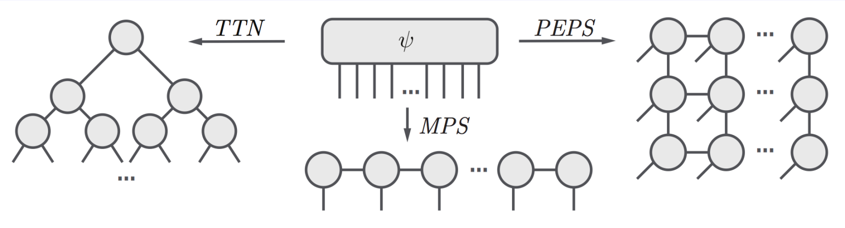
and

$$|\psi\rangle = \sum_{ijkm} A_i^{[1]} A_j^{[2]} A_k^{[3]} A_m^{[4]} |ijkm\rangle.$$

- There is a lot of excitement about tensor network algorithms, for example:

  Matrix Product States (MPS), Tree Tensor Networks (TTN), Projected Entangled Pair States (PEPS), etc.

  Diagrammatically these methods are represented as



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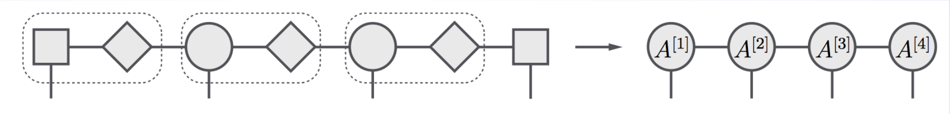

and

$$|\psi\rangle = \sum_{ijkm} A_i^{[1]} A_j^{[2]} A_k^{[3]} A_m^{[4]} |ijkm\rangle.$$

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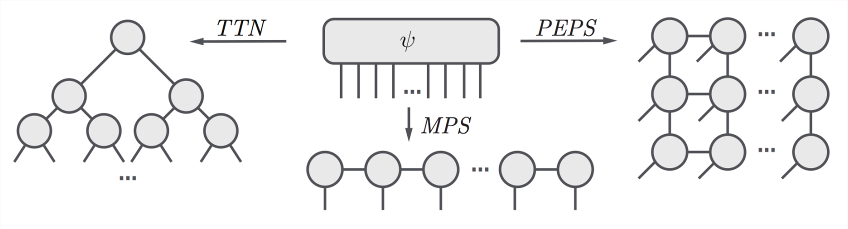


2 Quantum Networks

❑ 为了研究量子问题而建立的量子网络理论是近 20 年来发展起来的一个重要研究方向。代表性论文是：

[1]. J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi. Quantum State Transfer and Entanglement Distribution among Distant Nodes in a Quantum Network. Physical Review Letters 78, 3221, 1997.

[2]. R. Milo, S. Itzkovitz, N. Kashtan. Response to comment on network motifs: Simple building blocks of complex networks and superfamilies of evolved and designed networks. Sciences, 305, 2004

[3]. S. Perseguers, M. Lewenstein, A. Acin, J. I. Cirac. Quantum complex networks. Nature Physics 6, 539 - 543, 2010

❑ [4]. G. Chiribella, G. M. D. Ariano, P. Perinotti. Optimal Cloning of Unitary Transformation. Physical Review Letters 101, 180504, 2008

[5]. G. Chiribella, G. M. D. Ariano, P. Perinotti. Quantum Circuit Architecture. Physical Review Letters 101, 060401, 2008

[6]. A. Bisio, G. Chiribella, G. M. D. Ariano, S. Facchini, and P. PerinottiG. Optimal Quantum Tomography of States, Measurements, and Transformations. Physical Review Letters 102, 010404, 2009

文献 [4], [5], [6] 基于著名的Choi-Jamiolkowski 同构来建立量子网络理论，我们将简单介绍该理论。

3 Linear Maps

❑ In this talk, all complex Hilbert space H is finite dimension. We have

$$\mathcal{L}(H_a \otimes H_b \otimes \cdots \times H_c) = \mathcal{L}(H_a) \otimes \mathcal{L}(H_b) \cdots \times \mathcal{L}(H_c).$$

This shows that if $A \in \mathcal{L}(H_a \otimes H_b) \otimes \cdots \times H_c)$, then there exists

$$A_i \in \mathcal{L}(H_a), \ \ B_i \in \mathcal{L}(H_b), \ \ \cdots, \ \ C_i \in \mathcal{L}(H_c)$$

such that

$$A = \sum_i A_i \otimes B_i \cdots \otimes C_i.$$

We adopt these convention:

- $H_{ab\ldots n} = H_a \otimes H_b \otimes \cdots \otimes H_n$.
- $A_{ab\ldots n}$ means $A \in \mathcal{L}(H_{ab\ldots n})$;
- $A_{b|a}$ means $A \in \mathcal{L}(H_a, H_b)$;
- $A_{ab}B_{bc}$ denotes $(A_{ab} \otimes I_c)(I_a \otimes B_{bc})$;
- Tr_a denotes partial trace over H_a;
- ^{T_a} denotes partial transposition over H_a.

4 Choi-Jamiolkowski Isomorphism

❑ **Theorem.** The map

$$\mathfrak{C} : \mathcal{L}(\mathcal{L}(H_0), \mathcal{L}(H_1)) \to \mathcal{L}(H_1 \otimes H_0)$$

is defined by

$$\mathfrak{C} : \mathcal{M} \mapsto M_{10}, \qquad M_{10} = \sum_{ij} \mathcal{M}(E_{ij}) \otimes E_{ij}$$

is an isomorphism between $\mathcal{L}(\mathcal{L}(H_0), \mathcal{L}(H_1))$ and $\mathcal{L}(H_1 \otimes H_0)$,

where E_{ij} is the matrix of i-th row and j-column is 1, otherwise is 0.

The operator $M = \mathfrak{C}(\mathcal{M})$ is called the Choi operator of $\mathcal{M}$.

- The inverse map

$$\mathfrak{C}^{-1}: \mathcal{L}(H_1 \otimes H_0) \to \mathcal{L}(\mathcal{L}(H_0), \mathcal{L}(H_1))$$

is decided by

$$[\mathfrak{C}^{-1}(M_{10})](X) = Tr_0[(I_1 \otimes X^T)M_{10}],$$

where $M_{10} \in \mathcal{L}(H_1 \otimes H_0), X \in \mathcal{L}(H_0)$.

5 The Link Product

- Let

$$\mathcal{M} \in \mathcal{L}(\mathcal{L}(H_0), \mathcal{L}(H_1)), \quad \mathcal{N} \in \mathcal{L}(\mathcal{L}(H_1), \mathcal{L}(H_2)).$$

Consider the composition

$$\mathcal{F} := \mathcal{N} \circ \mathcal{M} \in \mathcal{L}(\mathcal{L}(H_0), \mathcal{L}(H_2)),$$

we get

$$\mathfrak{C}(\mathcal{F}) = Tr_1[(I_2 \otimes M_{10}^{T_1})(N_{21} \otimes I_0)].$$

We denote

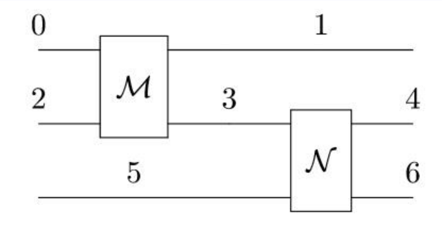
$$N * M = Tr_1[(I_2 \otimes M_{10}^{T_1})(N_{21} \otimes I_0)].$$

Where $M = M_{10}, \quad N = N_{21}.$

- Consider

$$\mathcal{M} \in \mathcal{L}(\mathcal{L}(H_0 \otimes H_2), \mathcal{L}(H_1 \otimes H_3)),$$

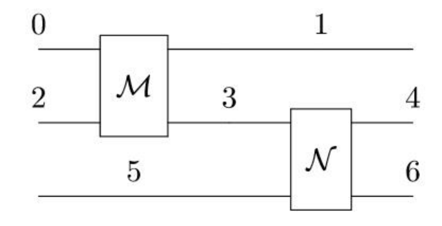
$$\mathcal{N} \in \mathcal{L}(\mathcal{L}(H_3 \otimes H_5), \mathcal{L}(H_4 \otimes H_6)),$$



We have

$$\mathcal{M} \leftrightarrow M \in \mathcal{L}(H_1 \otimes H_3 \otimes H_0 \otimes H_2),$$

$$\mathcal{N} \leftrightarrow N \in \mathcal{L}(H_4 \otimes H_6 \otimes H_3 \otimes H_5).$$



Define the composition

$$\mathcal{N} \star \mathcal{M} := (\mathcal{N} \otimes \mathcal{I}_1) \circ (\mathcal{M} \otimes \mathcal{I}_5),$$

and

$$\mathcal{N} \star \mathcal{M} \leftrightarrow N * M = Tr_3[(I_{456} \otimes M_{1302}^{T_3})(I_{012} \otimes N_{4635})].$$

Definition. Let

$$M \in \mathcal{L}(\bigotimes_{i\in\Lambda} H_i), \quad N \in \mathcal{L}(\bigotimes_{j\in J} H_j),$$

where Λ and J are two finite set of indexes.

Then the link product $N * M$ is an operator in

$$\mathcal{L}((\bigotimes_{i\in\Lambda\setminus J} H_i) \otimes (\bigotimes_{j\in J\setminus\Lambda} H_j))$$

is defined by

$$N * M := Tr_{\Lambda\cap J}[(I_{J\setminus\Lambda} \otimes M^{T_{\Lambda\cap J}})(I_{\Lambda\setminus J} \otimes N)].$$

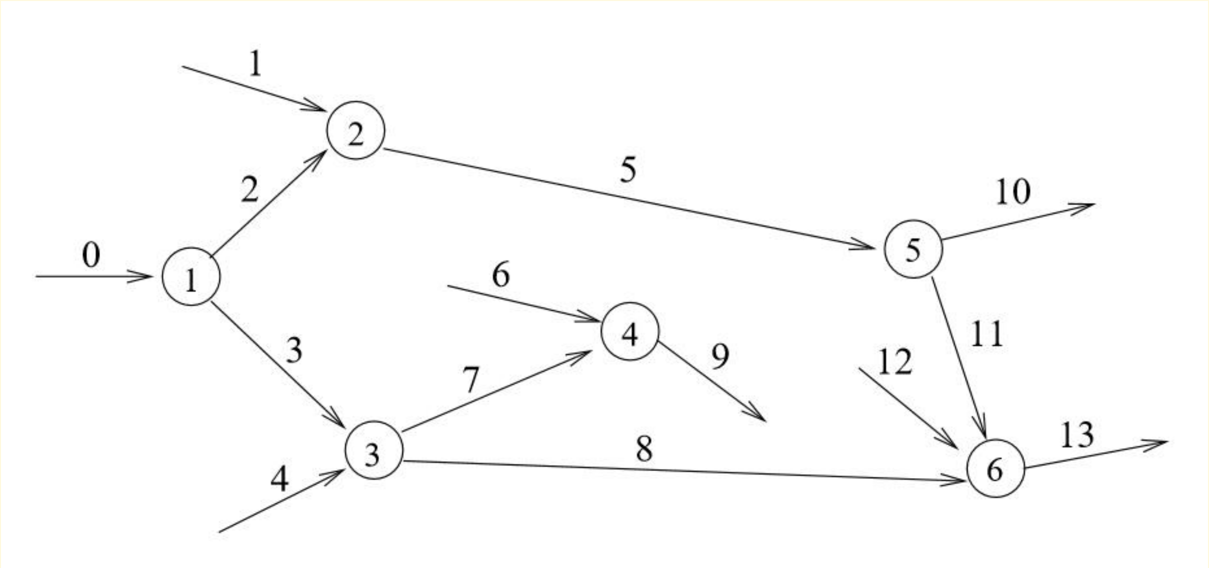
❑ If $\Lambda \cap J = \emptyset$, then

$$N * M = N \otimes M,$$

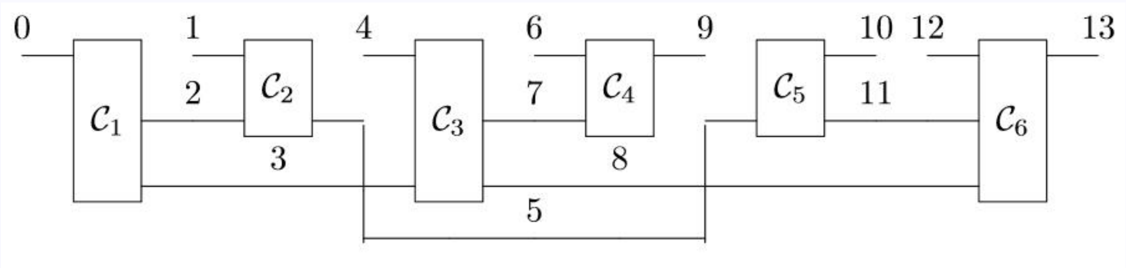
if $\Lambda = J$, then

$$N * M = Tr[M^T N].$$

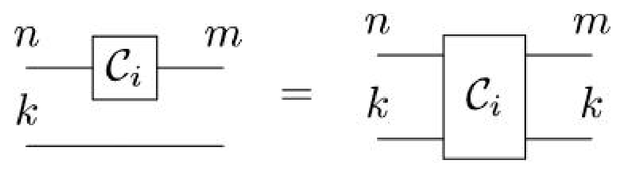
❑ **Definition.** A quantum network is a directed acyclic graph:



❑ Equivalent following network:

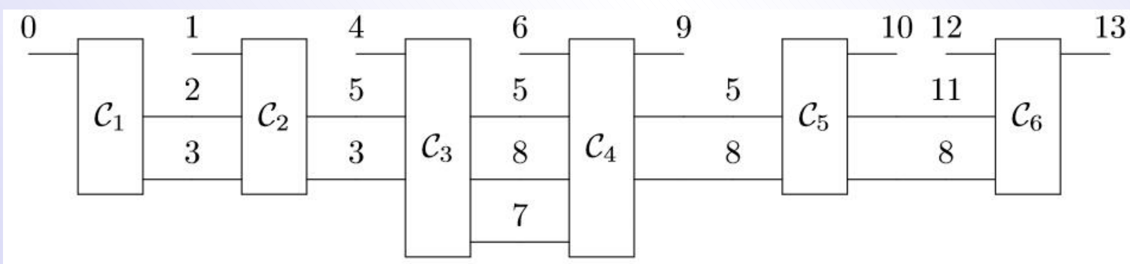


- If we let

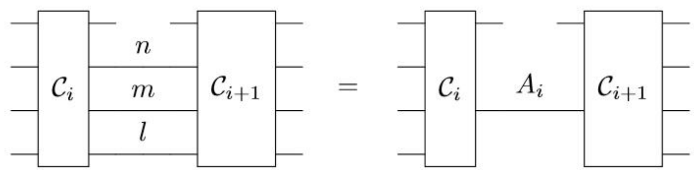


$$\mathcal{C}_i \to \mathcal{C}_i \otimes \mathcal{I}_k$$

The quantum network is:

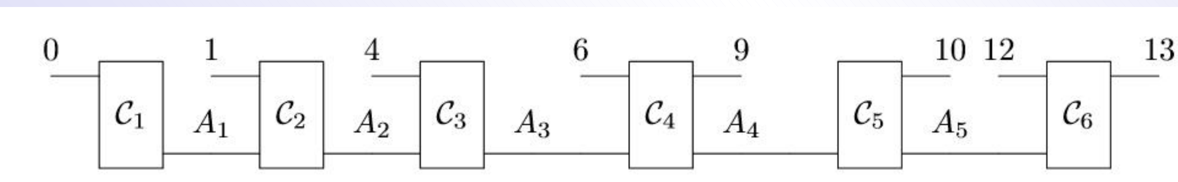


- If we let



$$\mathcal{H}_{A_i} = \mathcal{H}_n \otimes \mathcal{H}_m \otimes \mathcal{H}_l.$$

The quantum network is:



6 Conclusion

- For above network, note that the input space is:

$$H_{in} = H_0 \otimes H_1 \otimes H_4 \otimes H_6 \otimes H_{12},$$

output space is

$$H_{out} = H_9 \otimes H_{10} \otimes H_{13}.$$

- Theorem 1. Each quantum network is decide by a positive operator in $\mathcal{L}(H_{in} \otimes H_{out})$.

  Converse, under a constrain condition, each positive operator in $\mathcal{L}(H_{in} \otimes H_{out})$ can be realized by a quantum network which is made of a series of Isometries Quantum Channel.

- Link product told us that tensor networks are special quantum networks.

7 References

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Thank your attention !