Stereotype dualities in Geometry

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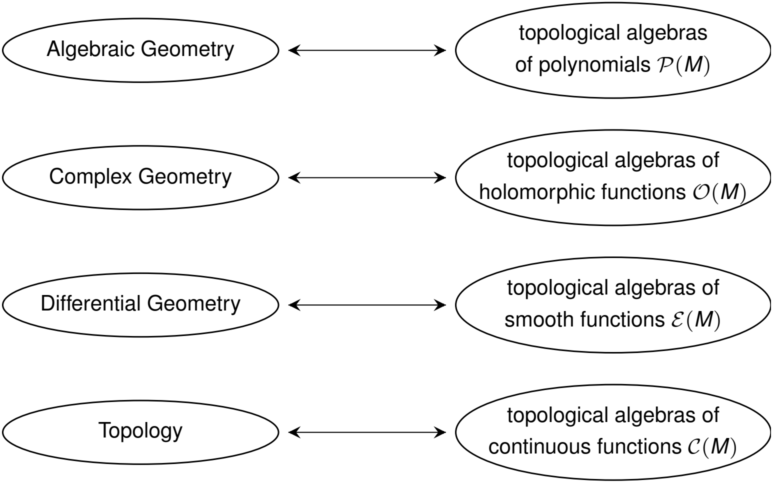
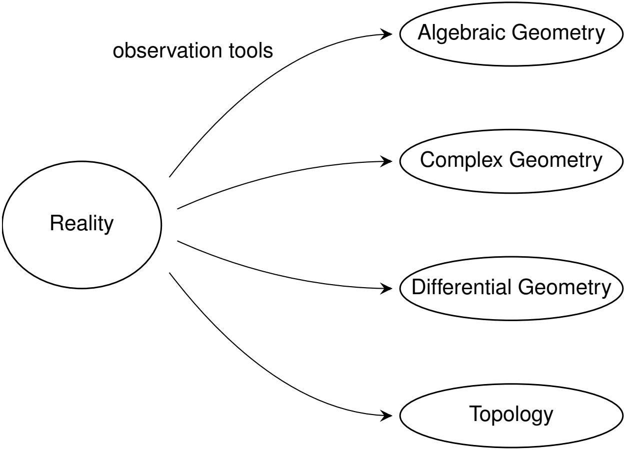
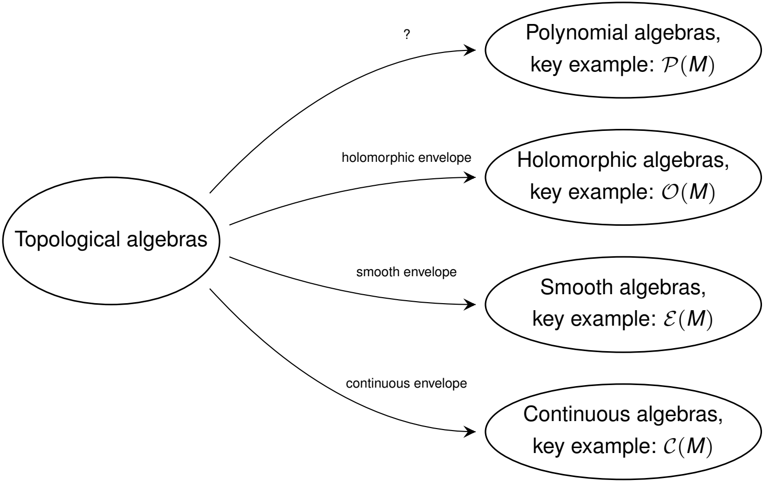
Geometric disciplines

Parallels between disciplines:




observation tools
Algebraic Geometry
Complex Geometry
Reality
Differential Geometry
Topology


Topological algebras
?
holomorphic envelope
smooth envelope
continuous envelope
Polynomial algebras,
key example: $\mathcal{P}(M)$
Holomorphic algebras,
key example: $\mathcal{O}(M)$
Smooth algebras,
key example: $\mathcal{E}(M)$
Continuous algebras,
key example: $\mathcal{C}(M)$

Monoidal category

Key example:

Category $\mathrm{Vect}_{\mathbb{C}}$ of vector spaces over $\mathbb{C}$.

1) a category $\mathtt{M}$,
2) a covariant furctor $\otimes : \mathtt{M} \times \mathtt{M} \to \mathtt{M}$ called the *tensor product*:

$$1_{X \otimes Y} = 1_X \otimes 1_Y, \qquad (\chi \otimes \chi') \circ (\varphi \otimes \varphi') = (\chi \circ \varphi) \otimes (\chi' \circ \varphi')$$

3) an isomorphism of functors $\square : \Big((X, Y, Z) \mapsto (X \otimes Y) \otimes Z \Big) \rightarrowtail \Big((X, Y, Z) \mapsto X \otimes (Y \otimes Z) \Big)$, called the *associativity isomorphism*, such that $\forall A, B, C, D$

$$\begin{array}{ccc}
(A \otimes (B \otimes C)) \otimes D & \xrightarrow{\ \square_{A, B \otimes C, D}\ } & A \otimes ((B \otimes C) \otimes D) \\
\uparrow{\scriptstyle \square_{A,B,C} \otimes 1_D} & & \downarrow{\scriptstyle 1_A \otimes \square_{B,C,D}} \\
((A \otimes B) \otimes C) \otimes D & & A \otimes (B \otimes (C \otimes D)) \\
& \searrow{\scriptstyle \square_{A \otimes B, C, D}} \quad (A \otimes B) \otimes (C \otimes D) \quad \nearrow{\scriptstyle \square_{A, B, C \otimes D}} &
\end{array}$$

4) an object I called the *unit object* in the category $\mathtt{M}$, and two isomorphisms of functors $\lhd : \Big(X \mapsto I \otimes X\Big) \rightarrowtail \Big(X \mapsto X\Big)$, and $\rhd : \Big(X \mapsto X \otimes I\Big) \rightarrowtail \Big(X \mapsto X\Big)$, called the *left identity* and the *right identity*, such that

$$(\lhd_I : I \otimes I \to I) = (\rhd_I : I \otimes I \to I),$$

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow[\square_{X,I,Y}]{} & X \otimes (I \otimes Y) \\ & {\scriptstyle \rhd_X \otimes 1_Y} \searrow \quad \swarrow {\scriptstyle 1_X \otimes \lhd_Y} & \\ & X \otimes Y & \end{array}$$

5) an isomorphism of functors $\diamond : \big((X,Y)\mapsto X\otimes Y\big) \rightarrowtail \big((X,Y)\mapsto Y\otimes X\big)$ called the *symmetry*, such that

$$
\begin{array}{ccc}
I\otimes X & \xrightarrow{\ \diamond_{I,X}\ } & X\otimes I \\
 & {\scriptstyle \lhd_X}\searrow \quad \swarrow{\scriptstyle \rhd_X} & \\
 & X &
\end{array}
$$

$$
\begin{array}{ccc}
X\otimes Y & \xrightarrow{\ 1_{X\otimes Y}\ } & X\otimes Y \\
 & {\scriptstyle \diamond_{X,Y}}\searrow \quad \nearrow{\scriptstyle \diamond_{Y,X}} & \\
 & Y\otimes X &
\end{array}
$$

$$
\begin{array}{ccccc}
 & X\otimes(Y\otimes Z) & \xrightarrow[\diamond_{X,Y\otimes Z}]{} & (Y\otimes Z)\otimes X & \\
{\scriptstyle \square_{X,Y,Z}}\nearrow & & & & \searrow{\scriptstyle \square_{Y,Z,X}} \\
(X\otimes Y)\otimes Z & & & & Y\otimes(Z\otimes X) \\
{\scriptstyle \diamond_{X,Y}\otimes 1_Z}\searrow & & & & \nearrow{\scriptstyle 1_Y\otimes\diamond_{X,Z}} \\
 & (Y\otimes X)\otimes Z & \xrightarrow[\square_{Y,X,Z}]{} & Y\otimes(X\otimes Z) &
\end{array}
$$

6) a bifunctor $(X, Y) \mapsto \frac{Y}{X} : \mathtt{M} \times \mathtt{M} \to \mathtt{M}$, contravariant in the first variable and covariant in the second:

$$\frac{1_Y}{1_X} = 1_{\frac{Y}{X}}, \qquad \frac{\chi' \circ \varphi'}{\varphi \circ \chi} = \frac{\chi'}{\chi} \circ \frac{\varphi'}{\varphi}$$

7) an isomorphism of functors

$$\Big((X, Y, Z) \mapsto Z \boxslash (X \otimes Y)\Big) \overset{\#}{\rightarrowtail} \Big((X, Y, Z) \mapsto \frac{Z}{Y} \boxslash X\Big),$$

Remark

$$Y \boxslash X = \mathrm{Mor}(X, Y)$$

$$\frac{Y}{X} = \mathrm{Hom}(X, Y)$$

$$\mathrm{Mor}(X \otimes Y, Z) \cong \mathrm{Mor}(X, \mathrm{Hom}(Y, Z))$$

Pure algebra vs Topological algebra

Pure algebra:	Topological algebra:
The category $\mathrm{Vect}_{\mathbb{C}}$ of vector spaces over $\mathbb{C}$ is closed monoidal: $L(X \otimes Y, Z) \cong L(X, L(Y, Z))$	Only the category $\mathrm{Ban}_{\mathbb{C}}$ of Banach spaces over $\mathbb{C}$ is closed monoidal: $B(X \hat{\otimes} Y, Z) \cong B(X, B(Y, Z))$

$$\Downarrow$$

Topological algebra is a “non-categorical theory”: only its “Banach branch” is categorical, but the problem is that there are not so many Banach algebras, for example, the algebras $C^\infty(M)$, $\mathrm{Diff}(M)$ are not Banach.

A *stereotype space* is a topological vector space X over $\mathbb{C}$ such that the natural map

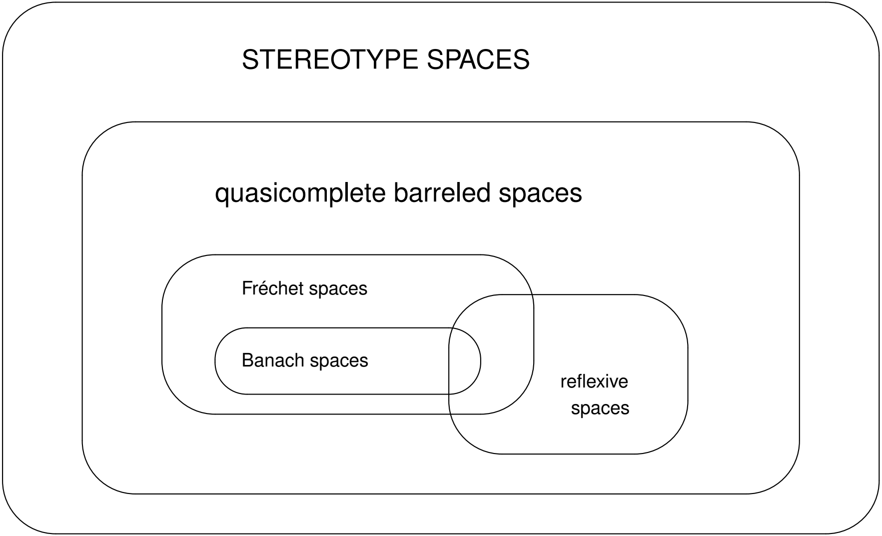
$$i_X : X \to X^{\star\star}, \qquad i_X(x)(f) = f(x), \qquad x \in X,\ f \in X^{\star}$$

is an isomorphism of topological vector spaces (i.e. a linear and a homeomorphic map). Here the dual space $X^{\star}$ is defined as the space of all linear continuous functionals $f : X \to \mathbb{C}$ endowed with the topology of *uniform convergence on totally bounded sets in* X, and the second dual space $X^{\star\star}$ is the space dual to $X^{\star}$ in the same sense.
A set $D \subseteq X$ is said to be *capacious* if for each totally bounded set $A \subseteq X$ there is a finite set $F \subseteq X$ such that $A \subseteq D + F$.

A topological vector space X is said to be

- *pseudocomplete*, if each totally bounded Cauchy net in X converges,
- *pseudosaturated*, if each closed convex balanced capacious set D in X is a neighborhood of zero in X.

Criterion: a locally convex space X is stereotype if and only if it is pseudocomplete and pseudosaturated.


STEREOTYPE SPACES
quasicomplete barreled spaces
Fréchet spaces
Banach spaces
reflexive spaces

Category "Ste" of stereotype spaces

The class Ste of stereotype spaces forms a category with linear continuous maps as morphisms and with the following properties:

- Ste is *pre-abelian*, i.e. additive with kernels and cokernels;
- Ste is *bicomplete*, i.e. has projective and injective limits;
- Ste has *nodal decomposition*:

$$\begin{array}{ccc} X & \xrightarrow{\quad\varphi\quad} & Y \\ {\scriptstyle \mathrm{StrongEpi}}\downarrow & & \uparrow{\scriptstyle \mathrm{StrongMono}} \\ X' & \underset{\mathrm{Epi}\,\cap\,\mathrm{Mono}}{\dashrightarrow} & Y' \end{array}$$

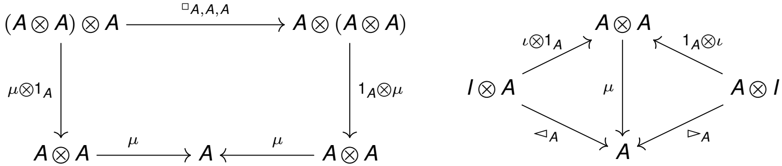
- Ste is a **-autonomous category*, i.e. symmetric closed monoidal with the duality functor $\star$, and

$$X^{\star\star} \cong X, \qquad X^{\star} \oslash (Y \circledast Z) \cong (X \circledast Y)^{\star} \oslash Z.$$

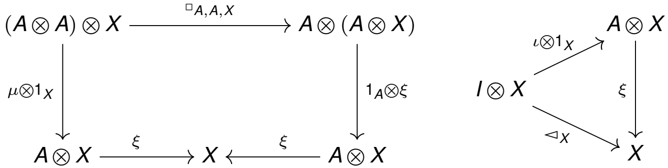
Key example:

Algebra $\mathcal{C}(M)$ of continuous functions on a (locally compact) topological space M.

An *algebra* in a monoidal category $\mathtt{M}$ is a triple (A, μ, ι) such that



A *left module* over (A, μ, ι) is a pair (X, ξ) such that



As a symmetric monoidal category Ste generates the notions of stereotype algebra A and of stereotype module over A. Analytical definitions:

- A is a *stereotype algebra* if the multiplication $(a, b) \mapsto a \cdot b$ is continuous as a bilinear mapping,
- M is a *left stereotype module* over A if the multiplication $(a, x) \mapsto a \cdot x$ is continuous as a bilinear mapping,
- M is a *right stereotype module* over A if the multiplication $(x, a) \mapsto x \cdot a$ is continuous as a bilinear mapping,

Theorem. The categories ${}_A\mathrm{Ste}$ and Ste_A of left and right stereotype modules over A are enriched categories over Ste.

Hopf algebras in monoidal categories

Key example:

Algebra $\mathcal{C}(G)$ of continuous functions on a (locally compact) topological group G.

A *Hopf algebra* in a symmetric monoidal category $\mathtt{M}$ is a sextuple $(H, \mu, \iota, \varkappa, \varepsilon, \sigma)$:

$$\mu : H \otimes H \to H \quad \text{(multiplication)},$$
$$\iota : I \to H \quad \text{(unit)},$$
$$\varkappa : H \to H \otimes H \quad \text{(comultiplication)},$$
$$\varepsilon : H \to I \quad \text{(counit)},$$
$$\sigma : H \to H \quad \text{(antipode)}$$

1) the triple (H, μ, ι) is a monoid in $\mathtt{M}$,

$$\begin{array}{ccccc}
(H\otimes H)\otimes H & & \xrightarrow{\quad \square_{H,H,H} \quad} & & H\otimes(H\otimes H) \\
{\scriptstyle \mu\otimes 1_H}\downarrow & & & & \downarrow{\scriptstyle 1_H\otimes\mu} \\
H\otimes H & \xrightarrow{\ \mu\ } & H & \xleftarrow{\ \mu\ } & H\otimes H
\end{array}
\qquad
\begin{array}{ccccc}
 & \overset{\iota\otimes 1_H}{\nearrow} & H\otimes H & \overset{1_H\otimes\iota}{\nwarrow} & \\
I\otimes H & & \downarrow{\scriptstyle\mu} & & H\otimes I \\
 & \underset{\lhd_H}{\searrow} & H & \underset{\rhd_H}{\swarrow} &
\end{array}$$

2) the triple $(H, \varkappa, \varepsilon)$ is a comonoid in $\mathtt{M}$,

$$\begin{array}{ccccc}
(H\otimes H)\otimes H & & \xrightarrow{\quad \square_{H,H,H} \quad} & & H\otimes(H\otimes H) \\
{\scriptstyle \varkappa\otimes 1_H}\uparrow & & & & \uparrow{\scriptstyle 1_H\otimes\varkappa} \\
H\otimes H & \xleftarrow{\ \varkappa\ } & H & \xrightarrow{\ \varkappa\ } & H\otimes H
\end{array}
\qquad
\begin{array}{ccccc}
 & \overset{\varepsilon\otimes 1_H}{\swarrow} & H\otimes H & \overset{1_H\otimes\varepsilon}{\searrow} & \\
I\otimes H & & \uparrow{\scriptstyle\varkappa} & & H\otimes I \\
 & \underset{\lhd_H^{-1}}{\nwarrow} & H & \underset{\rhd_H^{-1}}{\nearrow} &
\end{array}$$

3) the morphisms $\varkappa : H \to H \circledast H$ and $\varepsilon : H \to I$ are homomorphisms of monoids, and the morphisms $\mu : H \circledast H \to H$ and $\iota : I \to H$ are homomorphisms of comonoids:

$$\begin{array}{ccccc}
 & (H \circledast H) \circledast (H \circledast H) & \xrightarrow{\between_{H,H,H,H}} & (H \circledast H) \circledast (H \circledast H) & \\
\nearrow^{\varkappa\circledast\varkappa} & & & & \searrow^{\mu\circledast\mu} \\
H \circledast H & & \xrightarrow{\mu} \ H \ \xrightarrow{\varkappa} & & H \circledast H
\end{array}$$

$$\begin{array}{ccc}
I & \xrightarrow{\lhd_I^{-1}} & I \circledast I \\
\iota \downarrow & & \downarrow \iota\circledast\iota \\
H & \xrightarrow{\varkappa} & H \circledast H
\end{array} \quad ; \quad
\begin{array}{ccc}
H \circledast H & \xrightarrow{\mu} & H \\
\varepsilon\circledast\varepsilon \downarrow & & \downarrow \varepsilon \\
I \circledast I & \xrightarrow{\lhd_I} & I
\end{array} \quad ; \quad
\begin{array}{ccc}
 & H & \\
\nearrow^{\iota} & & \searrow^{\varepsilon} \\
I & \xrightarrow{1_I} & I
\end{array}$$

4) axiom of antipode:

$$\begin{array}{ccccc}
 & H \circledast H & \xrightarrow{\sigma\circledast 1_H} & H \circledast H & \\
\nearrow^{\varkappa} & & & & \searrow^{\mu} \\
H & \xrightarrow{\varepsilon} & I & \xrightarrow{\iota} & H \\
\searrow_{\varkappa} & & & & \nearrow_{\mu} \\
 & H \circledast H & \xrightarrow{1_H\circledast\sigma} & H \circledast H &
\end{array}$$

The identity

$$\mathcal{C}(G \times H) \cong \mathcal{C}(G) \odot \mathcal{C}(H)$$

implies that for each locally compact group G

(i) $\mathcal{C}(G)$ is a Hopf algebra in $(\mathtt{Ste}, \odot)$ with

$$\begin{array}{lll}
\mu : \mathcal{C}(G \times G) \to \mathcal{C}(G), & \mu(w)(t) = w(t,t) & \text{(multiplication)} \\
\iota : \mathbb{C} \to \mathcal{C}(G), & \iota(\lambda)(t) = \lambda & \text{(unit)} \\
\varkappa : \mathcal{C}(G) \to \mathcal{C}(G \times G), & \varkappa(u)(s,t) = u(s \cdot t) & \text{(comultiplication)} \\
\varepsilon : \mathcal{C}(G) \to \mathbb{C}, & \varepsilon(u) = u(1_G) & \text{(counit)} \\
\sigma : \mathcal{C}(G) \to \mathcal{C}(G), & \sigma(u)(t) = u(t^{-1}) & \text{(antipode)}
\end{array}$$

(ii) $\mathcal{C}^\star(G)$ is a Hopf algebra in $(\mathtt{Ste}, \circledast)$ with

$$\begin{array}{lll}
\mu^\star : \mathcal{C}^\star(G) \to \mathcal{C}^\star(G \times G), & \mu^\star(\alpha)(w) = \displaystyle\int_G w(t,t)\, \alpha(dt) & \text{(comultiplication)} \\
\iota^\star : \mathcal{C}^\star(G) \to \mathbb{C}, & \iota^\star(\alpha) = \displaystyle\int_G 1\, \alpha(dt) & \text{(counit)} \\
\varkappa^\star : \mathcal{C}^\star(G \times G) \to \mathcal{C}^\star(G), & \varkappa^\star(\gamma) = \displaystyle\int_{G\times G} u(s \cdot t)\, \gamma(ds, dt) & \text{(multiplication)} \\
\varepsilon^\star : \mathbb{C} \to \mathcal{C}^\star(G), & \varepsilon^\star(\lambda) = \lambda \cdot \delta^{1_G} & \text{(unit)} \\
\sigma^\star : \mathcal{C}^\star(G) \to \mathcal{C}^\star(G), & \sigma^\star(\alpha) = \alpha \circ \sigma & \text{(antipode)}
\end{array}$$

$\mathcal{C}^\star(G)$ as a group algebra

A *continuous representation* of a locally compact group G in a stereotype algebra A is an arbitrary continuous multiplicative map $\pi : G \to A$:

$$\pi(1_G) = 1_A, \qquad \pi(s \cdot t) = \pi(s) \cdot \pi(t), \qquad s, t \in G.$$

Example

The delta-function $\delta : G \to \mathcal{C}^\star(G)$.

Theorem

For each locally compact group G and for each stereotype algebra A the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\delta\quad} & \mathcal{C}^\star(G) \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\varphi} & \\ & A & \end{array}$$

establishes a bijection between

— continuous representations π of G in A, and

— homomorphisms φ of $\mathcal{C}^\star(G)$ into A.

Hopf algebras $\mathcal{E}(G)$ and $\mathcal{E}^\star(G)$

The identities

$$\mathcal{E}(G \times H) \cong \mathcal{E}(G) \odot \mathcal{E}(H) \cong \mathcal{E}(G) \circledast \mathcal{E}(H)$$

imply that for each real Lie group G

(i) $\mathcal{E}(G)$ is a Hopf algebra in $(\texttt{Ste}, \odot)$ and in $(\texttt{Ste}, \circledast)$ with

$$\begin{array}{lll}
\mu : \mathcal{E}(G \times G) \to \mathcal{E}(G), & \mu(w)(t) = w(t,t) & \text{(multiplication)} \\
\iota : \mathbb{C} \to \mathcal{E}(G), & \iota(\lambda)(t) = \lambda & \text{(unit)} \\
\varkappa : \mathcal{E}(G) \to \mathcal{E}(G \times G), & \varkappa(u)(s,t) = u(s \cdot t) & \text{(comultiplication)} \\
\varepsilon : \mathcal{E}(G) \to \mathbb{C}, & \varepsilon(u) = u(1_G) & \text{(counit)} \\
\sigma : \mathcal{E}(G) \to \mathcal{E}(G), & \sigma(u)(t) = u(t^{-1}) & \text{(antipode)}
\end{array}$$

(ii) $\mathcal{E}^\star(G)$ is a Hopf algebra in $(\texttt{Ste}, \odot)$ and in $(\texttt{Ste}, \circledast)$ with

$$\begin{array}{lll}
\mu^\star : \mathcal{E}^\star(G) \to \mathcal{E}^\star(G \times G), & \mu^\star(\alpha)(w) = \int_G w(t,t)\ \alpha(dt) & \text{(comultiplication)} \\
\iota^\star : \mathcal{E}^\star(G) \to \mathbb{C}, & \iota^\star(\alpha) = \int_G 1\ \alpha(dt) & \text{(counit)} \\
\varkappa^\star : \mathcal{E}^\star(G \times G) \to \mathcal{E}^\star(G), & \varkappa^\star(\gamma) = \int_{G \times G} u(s \cdot t)\ \gamma(ds, dt) & \text{(multiplication)} \\
\varepsilon^\star : \mathbb{C} \to \mathcal{E}^\star(G), & \varepsilon^\star(\lambda) = \lambda \cdot \delta^{1_G} & \text{(unit)} \\
\sigma^\star : \mathcal{E}^\star(G) \to \mathcal{E}^\star(G), & \sigma^\star(\alpha) = \alpha \circ \sigma & \text{(antipode)}
\end{array}$$

$\mathcal{E}^\star(G)$ as a group algebra

A *smooth representation* of a real Lie group G in a stereotype algebra A is an arbitrary continuous multiplicative map $\pi : G \to A$

$$\pi(1_G) = 1_A, \qquad \pi(s \cdot t) = \pi(s) \cdot \pi(t), \qquad s, t \in G.$$

that defines a continuous map

$$f \in A^\star \mapsto f \circ \pi \in \mathcal{E}(G).$$

Example

The delta-function $\delta : G \to \mathcal{O}^\star(G)$.

Theorem

For each real Lie group G and for each stereotype algebra A the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \delta\ } & \mathcal{E}^\star(G) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & A & \end{array}$$

establishes a bijection between

— smooth representations π of G in A, and

— homomorphisms φ of $\mathcal{E}^\star(G)$ into A.

Hopf algebras $\mathcal{O}(G)$ and $\mathcal{O}^\star(G)$

The identities

$$\mathcal{O}(G\times H)\cong \mathcal{O}(G)\odot\mathcal{O}(H)\cong \mathcal{O}(G)\circledast\mathcal{O}(H)$$

imply that for each Stein group G

(i) $\mathcal{O}(G)$ is a Hopf algebra in $(\mathtt{Ste},\odot)$ and in $(\mathtt{Ste},\circledast)$ with

$$\begin{array}{lll}
\mu:\mathcal{O}(G\times G)\to\mathcal{O}(G), & \mu(w)(t)=w(t,t) & \text{(multiplication)}\\
\iota:\mathbb{C}\to\mathcal{O}(G), & \iota(\lambda)(t)=\lambda & \text{(unit)}\\
\varkappa:\mathcal{O}(G)\to\mathcal{O}(G\times G), & \varkappa(u)(s,t)=u(s\cdot t) & \text{(comultiplication)}\\
\varepsilon:\mathcal{O}(G)\to\mathbb{C}, & \varepsilon(u)=u(1_G) & \text{(counit)}\\
\sigma:\mathcal{O}(G)\to\mathcal{O}(G), & \sigma(u)(t)=u(t^{-1}) & \text{(antipode)}
\end{array}$$

(ii) $\mathcal{O}^\star(G)$ is a Hopf algebra in $(\mathtt{Ste},\odot)$ and in $(\mathtt{Ste},\circledast)$ with

$$\begin{array}{lll}
\mu^\star:\mathcal{O}^\star(G)\to\mathcal{O}^\star(G\times G), & \mu^\star(\alpha)(w)=\int_G w(t,t)\,\alpha(dt) & \text{(comultiplication)}\\
\iota^\star:\mathcal{O}^\star(G)\to\mathbb{C}, & \iota^\star(\alpha)=\int_G 1\,\alpha(dt) & \text{(counit)}\\
\varkappa^\star:\mathcal{O}^\star(G\times G)\to\mathcal{O}^\star(G), & \varkappa^\star(\gamma)=\int_{G\times G} u(s\cdot t)\,\gamma(ds,dt) & \text{(multiplication)}\\
\varepsilon^\star:\mathbb{C}\to\mathcal{O}^\star(G), & \varepsilon^\star(\lambda)=\lambda\cdot\delta^{1_G} & \text{(unit)}\\
\sigma^\star:\mathcal{O}^\star(G)\to\mathcal{O}^\star(G), & \sigma^\star(\alpha)=\alpha\circ\sigma & \text{(antipode)}
\end{array}$$

$\mathcal{O}^\star(G)$ as a group algebra

A *holomorphic representation* of a Stein group G in a stereotype algebra A is an arbitrary continuous multiplicative map $\pi : G \to A$

$$\pi(1_G) = 1_A, \qquad \pi(s \cdot t) = \pi(s) \cdot \pi(t), \qquad s, t \in G.$$

that defines a continuous map

$$f \in A^\star \mapsto f \circ \pi \in \mathcal{O}(G).$$

Example

The delta-function $\delta : G \to \mathcal{O}^\star(G)$.

Theorem

For each Stein group G and for each stereotype algebra A the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \delta\ } & \mathcal{O}^\star(G) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & A & \end{array}$$

establishes a bijection between

— holomorphic representations π of G in A, and

— homomorphisms φ of $\mathcal{O}^\star(G)$ into A.

The identities

$$\mathcal{P}(G\times H)\cong\mathcal{P}(G)\odot\mathcal{P}(H)\cong\mathcal{P}(G)\circledast\mathcal{P}(H)$$

imply that for each complex affine algebraic group G

(i) $\mathcal{P}(G)$ is a Hopf algebra in $(\texttt{Ste},\odot)$ and in $(\texttt{Ste},\circledast)$ with

$$\begin{array}{lll}
\mu:\mathcal{P}(G\times G)\to\mathcal{P}(G), & \mu(w)(t)=w(t,t) & \text{(multiplication)}\\
\iota:\mathbb{C}\to\mathcal{P}(G), & \iota(\lambda)(t)=\lambda & \text{(unit)}\\
\varkappa:\mathcal{P}(G)\to\mathcal{P}(G\times G), & \varkappa(u)(s,t)=u(s\cdot t) & \text{(comultiplication)}\\
\varepsilon:\mathcal{P}(G)\to\mathbb{C}, & \varepsilon(u)=u(1_G) & \text{(counit)}\\
\sigma:\mathcal{P}(G)\to\mathcal{P}(G), & \sigma(u)(t)=u(t^{-1}) & \text{(antipode)}
\end{array}$$

(ii) $\mathcal{P}^\star(G)$ is a Hopf algebra in $(\texttt{Ste},\odot)$ and in $(\texttt{Ste},\circledast)$ with

$$\begin{array}{lll}
\mu^\star:\mathcal{P}^\star(G)\to\mathcal{P}^\star(G\times G), & \mu^\star(\alpha)(w)=\int_G w(t,t)\,\alpha(dt) & \text{(comultiplication)}\\
\iota^\star:\mathcal{P}^\star(G)\to\mathbb{C}, & \iota^\star(\alpha)=\int_G 1\,\alpha(dt) & \text{(counit)}\\
\varkappa^\star:\mathcal{P}^\star(G\times G)\to\mathcal{P}^\star(G), & \varkappa^\star(\gamma)=\int_{G\times G} u(s\cdot t)\,\gamma(ds,dt) & \text{(multiplication)}\\
\varepsilon^\star:\mathbb{C}\to\mathcal{P}^\star(G), & \varepsilon^\star(\lambda)=\lambda\cdot\delta^{1_G} & \text{(unit)}\\
\sigma^\star:\mathcal{P}^\star(G)\to\mathcal{P}^\star(G), & \sigma^\star(\alpha)=\alpha\circ\sigma & \text{(antipode)}
\end{array}$$

$\mathcal{P}^\star(G)$ as a group algebra

A *polynomial representation* of a complex affine algebraic group G in a stereotype algebra A is an arbitrary continuous multiplicative map $\pi : G \to A$

$$\pi(1_G) = 1_A, \qquad \pi(s \cdot t) = \pi(s) \cdot \pi(t), \qquad s, t \in G.$$

that defines a continuous map

$$f \in A^\star \mapsto f \circ \pi \in \mathcal{P}(G).$$

Example

The delta-function $\delta : G \to \mathcal{P}^\star(G)$.

Theorem

For each complex affine algebraic group G and for each stereotype algebra A the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\delta\quad} & \mathcal{P}^\star(G) \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\varphi} & \\ & A & \end{array}$$

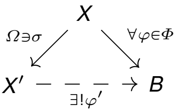
establishes a bijection between

— polynomial representations π of G in A, and

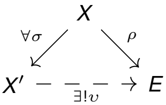
— homomorphisms φ of $\mathcal{P}^\star(G)$ into A.

Envelopes

- A morphism $\sigma : X \to X'$ is called an *extension of the object X in the class of morphisms Ω with respect to the class of morphisms Φ*, if $\sigma \in \Omega$, and for any morphism $\varphi : X \to B$ from the class Φ there exists a unique morphism $\varphi' : X' \to B$ such that



- An extension $\rho : X \to E$ of an object X in the class of morphisms Ω with respect to the class of morphisms Φ is called an *envelope of X in Ω with respect to Φ*, if for any other extension $\sigma : X \to X'$ (of X in Ω with respect to Φ) there is a unique morphism $\upsilon : X' \to E$ such that



Notations:

$$\rho = \mathrm{env}^{\Omega}_{\Phi} X, \qquad E = \mathrm{Env}^{\Omega}_{\Phi} X.$$

Stone—Čech compactification

In the category of Tikhonov spaces the Stone—Čech compactification $\beta : X \to \beta X$ is an envelope in the class of compact spaces with respect to the same class of spaces:

$$\beta X = \mathrm{Env}^{\mathrm{Com}}_{\mathrm{Com}} \mathrm{X}$$

Completion

In the category of locally convex spaces the completion $\blacktriangledown : X \to X^{\blacktriangledown}$ is an envelope in the class of all locally convex spaces with respect to the class of Banach spaces:

$$X^{\blacktriangledown} = \mathrm{Env}^{\mathrm{LCS}}_{\mathrm{Ban}} \mathrm{X}$$

Continuous envelope

A *continuous envelope* $\mathrm{env}_{\mathcal{C}} A : A \to \mathrm{Env}_{\mathcal{C}} A$ of an involutive stereotype algebra A is its envelope in the class DEpi of dense epimorphisms in the category InvSteAlg of involutive stereotype algebras with respect to the class of all homomorphisms into C^*-algebras:

$$\mathrm{Env}_{\mathcal{C}} A = \mathrm{Env}^{\mathrm{DEpi}}_{\mathrm{C}^*} A$$

Theorem

Let A be an involutive subalgebra in $\mathcal{C}(M)$ and the mapping $M \to \mathrm{Spec}(A)$ is an exact covering. Then

$$\mathrm{Env}_{\mathcal{C}} A = \mathcal{C}(M)$$

Example: $\mathrm{Env}_{\mathcal{C}} \mathcal{E}(M) = \mathcal{C}(M)$.

Differential morphisms

Let B be an involutive stereotype algebra, $d \in \mathbb{N}$ and $m \in \mathbb{N}^d$, set

$$I_m = \{x \in B[[d]] : \ \forall k \in \mathbb{N}^d \quad k \leqslant m \quad \Longrightarrow \quad x_k = 0\}.$$

(the ideal in the algebra $B[[d]]$ of power series with coefficients in B). The quotient algebra

$$B[m] := B[[d]]/I_m$$

is called the *algebra B with joined self-adjoint nilpotent elements (of order m)*. Take

$$\mathbb{N}[m] = \{k \in \mathbb{N}^d : \ k \leqslant m\}.$$

For each homomorphism $D : A \to B[m]$ of involutive stereotype algebras its *partial derivatives* are the operators

$$D_k : A \to B, \quad D_k(a) = D(a)^{(k)}, \quad k \in \mathbb{N}[m], \quad a \in A.$$

A homomorphism $D : A \to B[m]$ is *differential*, if its partial derivatives $\{D_k; \ k \in \mathbb{N}[m]\}$ are differential operators from A into B with respect to the homomorphism $D_0 : A \to B$ with the orders, not greater than $|k|$:

$$D_k \in \mathrm{Diff}^{|k|}(D_0),$$

i.e.

$$[...[D_k, a_0], ...a_{|k|}] = 0, \quad a_0, ..., a_{|k|} \in A,$$

with

$$[\Phi, a](x) = \Phi(a \cdot x) - D_0(a) \cdot \Phi(x), \qquad x \in X.$$

Smooth envelope

A *smooth envelope* $\mathrm{env}_{\mathcal{E}} A : A \to \mathrm{Env}_{\mathcal{E}} A$ of an involutive stereotype algebra A is its envelope in the class DEpi of dense epimorphisms in the category InvSteAlg of involutive stereotype algebras with respect to the class DiffMor of all differential homomorphisms into C^*-algebras $B[m]$ with the joined self-adjoint nilpotent elements:

$$\mathrm{Env}_{\mathcal{E}} A = \mathrm{Env}^{\mathrm{DEpi}}_{\mathrm{DiffMor}} A$$

Theorem

Let A be an involutive subalgebra in $\mathcal{E}(M)$ and the mapping $M \to \mathrm{Spec}(A)$ is an exact covering, and for each $s \in M$ the mapping $T_s(M) \to T_s[A]$ is an isomorphism. Then

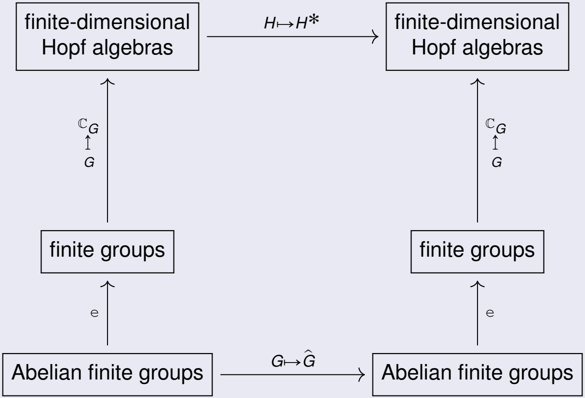
$$\mathrm{Env}_{\mathcal{E}} A = \mathcal{E}(M)$$

A *holomorphic envelope* $\mathsf{env}_{\mathcal{O}} A : A \to \mathsf{Env}_{\mathcal{O}} A$ of an involutive stereotype algebra A is its envelope in the class DEpi of dense epimorphisms in the category SteAlg of stereotype algebras with respect to the class Ban of all homomorphisms into Banach algebras:

$$\mathsf{Env}_{\mathcal{O}} A = \mathsf{Env}^{\mathsf{DEpi}}_{\mathsf{Ban}} A$$

Example: $\mathsf{Env}_{\mathcal{O}} \mathcal{P}(M) = \mathcal{O}(M)$.

Duality for finite groups



Theorem

If G is a Moore group, then

$$\begin{array}{ccc} \mathcal{C}^\star(G) & \overset{\mathrm{Env}_{\mathcal{C}}}{\longmapsto} & \mathrm{Env}_{\mathcal{C}}\mathcal{C}^\star(\mathrm{G}) \\ {\scriptstyle\star}\uparrow & & \downarrow{\scriptstyle\star} \\ \mathcal{C}(G) & \overset{\mathrm{Env}_{\mathcal{C}}}{\longleftarrow} & (\mathrm{Env}_{\mathcal{C}}\mathcal{C}^\star(\mathrm{G}))^\star \end{array}$$

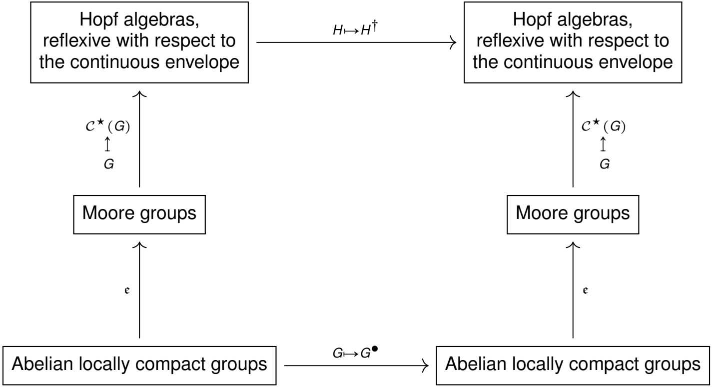
If we denote

$$H^\dagger = (\mathrm{Env}_{\mathcal{C}}\mathrm{H})^\star,$$

then $\mathcal{C}^\star(G)$ becomes a Hopf algebra “reflexive with respect to the continuous envelope”:

$$H^{\dagger\dagger} \cong H,$$

and we receive the following “diagram of functors”, which means that † generalizes the usual Pontryagin duality •:


Hopf algebras, reflexive with respect to the continuous envelope
$H \mapsto H^{\dagger}$
Hopf algebras, reflexive with respect to the continuous envelope
$\mathcal{C}^{\star}(G)$
G
$\mathcal{C}^{\star}(G)$
G
Moore groups
Moore groups
$\mathfrak{e}$
$\mathfrak{e}$
Abelian locally compact groups
$G \mapsto G^{\bullet}$
Abelian locally compact groups

Theorem

If $G = C \times K$, where C is a compactly generated Abelian Lie group, and K a compact Lie group, then

$$\begin{array}{ccc} \mathcal{E}^\star(G) & \xmapsto{\mathrm{Env}_\mathcal{E}} & \mathrm{Env}_\mathcal{E}\mathcal{E}^\star(\mathrm{G}) \\ {\scriptstyle\star}\big\uparrow & & \big\downarrow{\scriptstyle\star} \\ \mathcal{E}(G) & \xleftarrow{\mathrm{Env}_\mathcal{E}} & \left(\mathrm{Env}_\mathcal{E}\mathcal{E}^\star(\mathrm{G})\right)^\star \end{array}$$

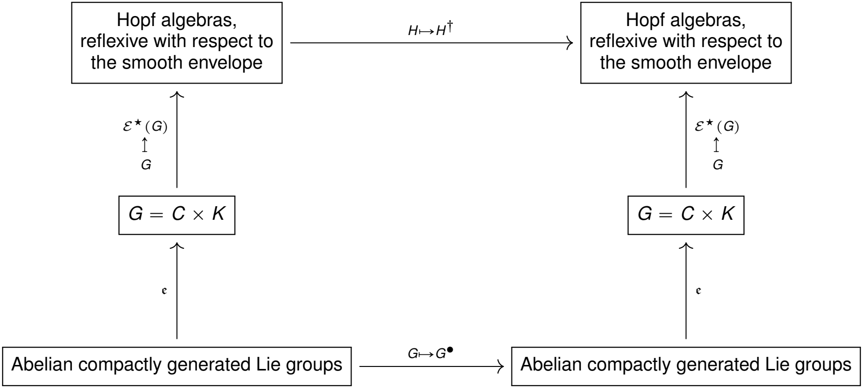
If we denote

$$H^\dagger = \left(\mathrm{Env}_\mathcal{E}\mathrm{H}\right)^\star,$$

then $\mathcal{E}^\star(G)$ becomes a Hopf algebra “reflexive with respect to the smooth envelope”:

$$H^{\dagger\dagger} \cong H,$$

and we receive the following “diagram of functors”, which means that † generalizes the usual Pontryagin duality •:


Hopf algebras, reflexive with respect to the smooth envelope
$H \mapsto H^\dagger$
Hopf algebras, reflexive with respect to the smooth envelope
$\mathcal{E}^\star(G)$
G
$\mathcal{E}^\star(G)$
G
$G = C \times K$
$G = C \times K$
$\mathfrak{e}$
$\mathfrak{e}$
Abelian compactly generated Lie groups
$G \mapsto G^\bullet$
Abelian compactly generated Lie groups

Theorem

If G is a finite extension of a connected complex linear group, then

$$\begin{array}{ccc} \mathcal{O}^\star(G) & \stackrel{\mathrm{Env}_{\mathcal{O}}}{\longmapsto} & \mathrm{Env}_{\mathcal{O}}\mathcal{O}^\star(\mathrm{G}) \\ {\scriptstyle\star}\uparrow & & \downarrow{\scriptstyle\star} \\ \mathcal{O}(G) & \stackrel{\mathrm{Env}_{\mathcal{O}}}{\longleftarrow} & \left(\mathrm{Env}_{\mathcal{O}}\mathcal{O}^\star(\mathrm{G})\right)^\star \end{array}$$

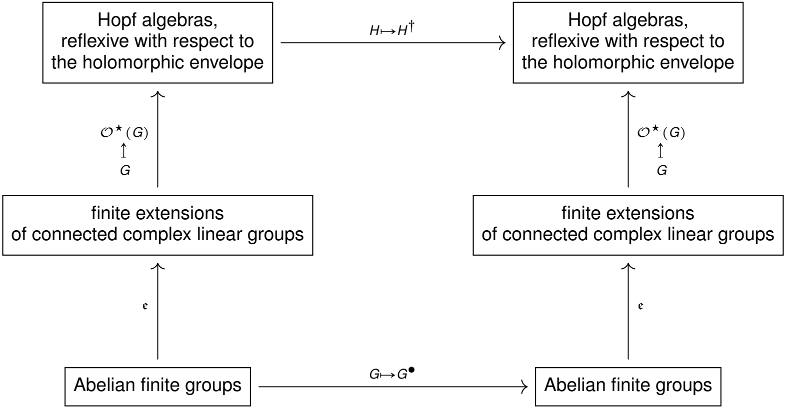
If we denote

$$H^\dagger = \left(\mathrm{Env}_{\mathcal{O}}\mathrm{H}\right)^\star,$$

then $\mathcal{O}^\star(G)$ becomes a Hopf algebra “reflexive with respect to the smooth envelope”:

$$H^{\dagger\dagger} \cong H,$$

and we receive the following “diagram of functors”, which means that † generalizes the usual Pontryagin duality •:


Hopf algebras, reflexive with respect to the holomorphic envelope
$H \mapsto H^\dagger$
Hopf algebras, reflexive with respect to the holomorphic envelope
$\mathcal{O}^\star(G)$
G
$\mathcal{O}^\star(G)$
G
finite extensions of connected complex linear groups
finite extensions of connected complex linear groups
$\mathfrak{e}$
$\mathfrak{e}$
Abelian finite groups
$G \mapsto G^\bullet$
Abelian finite groups

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