On determining G-invariant von Neumann subalgebras in $L(G)$

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Overview

Joint work with Tattwamasi Amrutam

Group von Neumann algebras

Definition

$L(G) := \overline{span\{\lambda_g : g \in G\}}^{w.o.t.} \subset \mathbb{B}(\ell^2(G))$, where $\lambda_g \in \mathcal{U}(\ell^2(G))$ is defined by $\lambda_g(\delta_s) = \delta_{gs}$ for all $s \in G$.

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Example

$L(\mathbb{Z}^n) \cong L^\infty(\mathbb{T}^n, \mu)$.

Fourier expansion

Let $\tau : L(G) \to \mathbb{C}$ be defined as $\tau(a) = \langle a\delta_e, \delta_e \rangle$ for all $a \in L(G)$. Then τ is the unique trace on $L(G)$.

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Fact: $L(G) \hookrightarrow \ell^2(G)$ via the map $a \mapsto a\delta_e$. So we may write every $a \in L(G)$ as $a = \sum_{g \in G} a_g u_g$, where $u_g = \lambda_g$, $u_g \delta_e = \delta_g$. This is the so-called **Fourier expansion** of a.

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Set $\mathrm{Supp}(a) = \{g : \ a_g \neq 0\}$.

Fact: If $H \leq G$ is a subgroup, then $L(H) \leq L(G)$ naturally:

$$L(H) = \{a \in L(G) : \ Supp(a) \subseteq H\}.$$

Proposition

$L(G)$ is a factor iff G is infinite-conjugacy-class (i.c.c.), i.e. $\sharp Conj(g) = \infty$ for all $g \neq e$, where $Conj(g) := \{sgs^{-1} : s \in G\}$.

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Proof.

$\Leftarrow$: $a \in \mathcal{Z}(L(G)) \Leftrightarrow u_s a u_s^* = a, \ \forall\ s \in G, \Leftrightarrow a_{s^{-1}gs} = a_g, \ \forall\ s, g \in G.$
Note that $\infty > ||a||_2^2 = \sum_{g \in G} |a_g|^2 \geq \sum_{t \in Conj(g)} |a_t|^2 = \sharp Conj(g) \cdot |a_g|^2$.

Hence $a_g = 0$ for all $g \neq e$, i.e. $a \in \mathbb{C}id$.

$\Rightarrow$: if $g \neq e$ and $Conj(g) = \{s_1 g s_1^{-1}, \cdots, s_n g s_n^{-1}\}$, then
$a := \sum_{i=1}^{n} u_{s_i g s_i^{-1}} \in \mathcal{Z}(L(G))$. □

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RK: If $H \leq G$ and $x \in L(H)' \cap L(G)$, then

$$Supp(x) \subseteq \{g : \ \sharp\{hgh^{-1} : h \in H\} < \infty\}.$$

This observation has already been used in Dixmier's work on MASAs.

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2. $E(x) \geq 0$ for every $x \in L(G)$ with $x \geq 0$.
3. $E(m_1 x m_2) = m_1 E(x) m_2$ for every $m_1, m_2 \in M$ and $x \in L(G)$.

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The construction:

Denote by $E : \ell^2(G) \twoheadrightarrow L^2(M, \tau)$ the orthogonal projection, then show that $E(L(G)) \subseteq M$.

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Definition

We say G has the invariant von Neumann subalgebras rigidity (ISR) property if every G-invariant von Neumann subalgebra M of $L(G)$ is of the form $L(H)$ for some normal subgroup $H \lhd G$.

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Problem

Which group G has this ISR property?

Some history and known works

The study of this question stems from the work of Alekseev-Brugger ('19). Recall that

Theorem (Margulis's normal subgroup theorem)

Let G be a lattice in a higher rank simple real Lie group with trivial center, e.g. $G = SL_n(\mathbb{Z})$ for $n \geq 3$. Then every non-trivial normal subgroup in G is of finite index.

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Trying to prove an analogue of this for subalgebras of $L(G)$, Alekseev-Brugger proved, a priori, partial analogue:

Theorem (Alekseev-Brugger, '19)

Let G be the same as the above. Then every non-trivial G-invariant von Neumann ***subfactor*** *of $L(G)$ is of finite Jones index in $L(G)$.*

Related works

Theorem (Kalantar-Panagoupolos, '21)

Let G be an irreducible lattice in a connected semisimple Lie group with trivial center, no non-trivial compact factors, and such that all its simple factors have real rank at least two. Then every G-invariant von Neumann subalgebra is of the form $L(H)$ for some normal subgroup $H \lhd G$.

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For a different class of groups, Chifan-Das proved:

Theorem (Chifan-Das, '19)

Let G be a "negatively curved" group, e.g. a non-amenable group that is either exact and acylindrically hyperbolic or has positive first L^2-Betti number, then all G-invariant subfactors are commensurable to subalgebras $L(H)$ for some $H \lhd G$.

Tools used

Results	Tools used
Alekseev-Brugger	Peterson's character rigidity
Chifan-Das	Popa's deformation/rigidity
Kalantar-Panagoupolos	Boutonnet-Houdayer's non-commutative Nevo-Zimmer thm Chifan-Das result

Remarks:

1. In Kalantar-Panagoupolos's proof, the key step is to argue every G-invariant vN subalgebra is automatically a subfactor.
2. In Chifan-Das's result, if we further assume G is i.c.c., then their proof shows every G-invariant subfactor is of the form $L(H)$ for some $H \lhd G$.

Main results

Theorem (Amrutam-J., '22)

The following groups satisfy the ISR property.

1. *torsion-free non-amenable hyperbolic groups.*
2. *torsion-free groups with positive first L^2-Betti number under a mild assumption (*) used by Peterson-Thom.*
3. *finite products of groups in either of the above two items.*

(*): every non-trivial element of $\mathbb{Z}G$ acts with zero kernel on $\ell^2(G)$.

Proposition (Amrutam-J., '22)

1. *For an infinite group G, G has ISR property implies G is i.c.c., but the converse fails.*
2. $G = \mathbb{Z} * \frac{\mathbb{Z}}{2\mathbb{Z}}$ *has ISR property.*

I.C.C. groups without ISR property

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Set

$$M = \{ \sum_{v \in \mathbb{Z}^2} \lambda_v v : \ \lambda_v = \lambda_{-v}, \ \forall \ v \in \mathbb{Z}^2 \} \subsetneq L(\mathbb{Z}^2) \subsetneq L(G).$$

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Then M is G-invariant, but $M \neq L(H)$ for any $H \lhd G$.

Similar construction works for $G = \mathbb{Z} \wr \mathbb{Z} := (\oplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$ etc.

Ideas behind the proof

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For the inclusion $L(G) \leq M \leq L^\infty(X,\mu) \rtimes G$, M is automatically G-invariant.

Strategy

Let M be a G-invariant vN subalgebra and $E : L(G) \to M$ be the c.e.

It suffices to show $E(u_g) \in \mathbb{C}u_g$ for all $g \in G$. Then it follows that $M = L(H)$ for $H = \{g \in G : \ E(u_g) \neq 0\}$ and $H \lhd G$.

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We think of $\{E(u_g): \ g \in G\}$ as unknowns and find equations involving them and solve for them. Here, we use:

1. $u_s E(u_g) u_s^* = E(u_{sgs^{-1}})$ for all $s \in G$.
2. $E(u_g E(u_h)) = E(u_g) E(u_h)$ for all $g, h \in G$.

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Next, we sketch a piece of the proof for $G = F_2 = \langle a, b \rangle$.

From now on, we simply write g for u_g.

A piece of the proof for $G = F_2$

Goal: show $E(a) \in \mathbb{C}a$.

Step 1: find the Fourier expansion of $E(a)$.

$$a^n E(a) a^{-n} = E(a) \text{ implies } E(a) \in L(\langle a \rangle)' \cap L(F_2) = L(\langle a \rangle).$$

Hence, we write

$$E(a) = \sum_{n \in \mathbb{Z}} \lambda_n a^n.$$

Note that bab^{-1} is free from a, i.e. $\langle bab^{-1}, a \rangle \cong \langle bab^{-1} \rangle * \langle a \rangle \cong F_2$, and we have

$$E(bab^{-1}) = bE(a)b^{-1} = \sum_{n \in \mathbb{Z}} \lambda_n ba^n b^{-1}.$$

Step 2: solve for λ_n, i.e. show $\lambda_n = 0$ for all $n \neq 1$.

Recall that

$$E(a) = \sum_{n\in\mathbb{Z}} \lambda_n a^n,$$
$$E(bab^{-1}) = \sum_{n\in\mathbb{Z}} \lambda_n b a^n b^{-1}.$$

Clearly, $\lambda_0 = 0$ by taking trace on both sides.

We compute both sides of the identity $E(a)E(bab^{-1}) = E(aE(bab^{-1}))$.

$$\text{LHS} = \sum_{i,j\in\mathbb{Z}} \lambda_i \lambda_j a^i b a^j b^{-1}.$$
$$\text{RHS} = E(a(\sum_{k\in\mathbb{Z}} \lambda_k b a^k b^{-1})) = \sum_{k\in\mathbb{Z}} \lambda_k E(a b a^k b^{-1}).$$

For any $x \in L(G)$, recall that $\text{Supp}(x) = \{g \in G : \tau(x u_g^*) \neq 0\}$, i.e. the collection of all g with non-zero coefficient in the Fourier expansion of x.

Key part

$$\mathrm{LHS} = \sum_{(i,j)\in\mathbb{Z}^2} \lambda_i\lambda_j a^i b a^j b^{-1}.$$

$$\mathrm{RHS} = \sum_{k\in\mathbb{Z}} \lambda_k E(aba^k b^{-1}).$$

(Fact 1) Since bab^{-1} is free from a, we deduce that

$$\mathrm{supp}(\mathrm{LHS}) = \{a^i b a^j b^{-1} : \ \lambda_i\lambda_j \neq 0\}.$$

(Fact 2) $\mathrm{supp}(\mathrm{RHS}) \subseteq \bigcup_{k\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} (aba^k b^{-1})^{\ell}$.

(Fact 3) If $i \neq 1$ and $j \neq 0$, then

$$a^i b a^j b^{-1} \notin \bigcup_{k\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} (aba^k b^{-1})^{\ell}.$$

Thus, $\lambda_i\lambda_j = 0$ for all $i \neq 1$ and $j \neq 0$.

Hence $\lambda_i = 0$ for all $i \neq 1, 0$ (recall that $\lambda_0 = 0$).

It is not hard to see the proof relies on two points:

1. control of $E(g)$, clearly,

$$E(g) \in L(C(g))' \cap L(G) \subseteq L(\langle g \rangle)' \cap L(G) \subseteq L(\cup_{i \geq 1} C(g^i)).$$

2. find $s \in G$ such that $sg's^{-1}$ is free from g', $\forall\, g' \in \mathrm{Supp}(E(g))$ and calculate the Fourier expansion.

On the proof for general cases

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Due to point 1, we need to assume G is torsion-free in general.

To make point 2 work, we either use property naive/Peterson-Thom theorem for the group G.

On the proof for general cases

Case 1: G is a torsion-free non-amenable hyperbolic group.

1. $\forall g(\neq e) \in G$, $C(g)$ is cyclic.
2. G satisfies property naive, i.e. for any finite subset $F \subseteq G \setminus \{e\}$, $\exists\ s \in G$ of infinite order s.t. $\forall g' \in F$, $\langle g', s \rangle \cong \langle g' \rangle * \langle s \rangle$.

Case 2: G is a torsion-free group with $\beta_2^{(1)}(G) > 0$ satisfying (*).

1. $E(g)$ lies in a subgroup.
2. freeness, i.e. Peterson-Thom theorem.

Theorem (Peterson-Thom)

Let G be a torsion-free countable discrete group. Then there exists a family of subgroups $\{G_i : \ i \in I\}$, such that

1. *We can write G as the disjoint union: $G = \{e\} \cup \bigcup_{i \in I} \dot{G}_i$, where $\dot{G}_i = G_i \setminus \{e\}$.*
2. *The groups G_i are mal-normal in G, for $i \in G$.*
3. *If G satisfies (*), then G_i is free from G_j, for $i \neq j$.*
4. $\beta_1^{(2)}(G_i) = 0$*, for all $i \in I$.*

A general framework

Theorem

Let G be a torsion free group and $M \subseteq L(G)$ be a G-invariant vN subalgebra. Denote by $E : L(G) \twoheadrightarrow M$ the conditional expectation. Assume that $\forall g \in G \setminus \{e\}$, $\exists$ subgroup $H_g \leq G$ s.t.

1. $g \in H_g$,
2. $E(u_g) \in L(H_g)$,
3. $\exists s \in G$ s.t. $s\dot{H}_g s^{-1}$ is free from $\dot{H}_g$, i.e. $\langle s\dot{H}_g s^{-1}, \dot{H}_g \rangle = sH_g s^{-1} * H_g$, where $\dot{H}_g = H_g \setminus \{e\}$.
4. $\forall t_1, t_2 \in H_g$, if $t_1 \neq g, e$ and $t_2 \neq e$, then $t_1 s t_2 s^{-1} \notin \cup_{t \in H_g} H_{gsts^{-1}}$.

Then $E(u_g) \in \mathbb{C} \cdot u_g$ for all $g \in G$, and hence $M = L(H)$ for some normal subgroup $H \leq G$.

If $H_g := \cup_{i \geq 1} C(g^i)$, then only need to check (3).

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2. How to characterize groups with ISR property?

The guess is that G has ISR property if G has trivial amenable radical, i.e. G contains no $\neq \{e\}$ amenable normal subgroups.

Question

Are there any infinite amenable groups which satisfy ISR property?

3. About 10 days ago, a preprint appeared on arXiv (ID: 2309.10494). Denote by $Rad(G)$ the amenable radical of G.

Theorem (Amrutam-Hartman-Oppelmayer, '23)

Let G be any countable discrete group. Then $L(Rad(G))$ is the maximal amenable G-invariant von Neumann subalgebra in $L(G)$.

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- This implies that for a group G with $Rad(G) = \{e\}$, if M is G-invariant, then M is a subfactor.
- At the group level, their proof boils down to a dynamical proof of the fact that $Rad(G)$ is the unique maximal amenable normal subgroup of G.
- The above theorem can be generalized to the setting $P \rtimes G$ for any trace preserving action $G \curvearrowright (P, \tau)$ on an amenable tracial vN alg (P, τ) showing that $P \rtimes Rad(G)$ is the maximal amenable G-inv. vN. subalg. in $P \rtimes G$.

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Proof.

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Proof.

Let $G \curvearrowright X$ be the action on its Furstenberg boundary $X = \partial_F G$. Now, assume that $Rad(G) \subseteq H \lhd G$ and H is amenable.

First, since H is amenable, then $Prob_H(X) \neq \emptyset$. Fix any $\mu \in Prob_H(X)$. Since H is normal, we deduce G acts on $Prob_H(X)$ naturally.

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Since X is a boundary, we get that $\{\delta_x : x \in X\} \subseteq \overline{G\mu}^{w*} \subseteq Prob_H(X)$ for any $\mu \in Prob(X)$. Therefore, we deduce that $\{\delta_x : x \in X\} \subseteq Prob_H(X)$.

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Hence, $H \subseteq Ker(G \curvearrowright X) = Rad(G)$, where the last equality is a theorem of Furman. □

Theorem (J., 2023)

Let $G = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$. *Then*

$$\begin{aligned}&\{P : \ P \subseteq L(G) \text{ is } G\text{-invariant}\}\\ &= \{L(H) : \ H \lhd G\} \cup \{A_n : \ n \geq 1\},\end{aligned}$$

where $A_n \subseteq L(n\mathbb{Z}^2)$ *is the von Neumann subalgebra defined by*

$$A_n = \{x \in L(n\mathbb{Z}^2) : \ \tau(xu_g) = \tau(xu_{g^{-1}}), \ \forall \ g \in n\mathbb{Z}^2\}.$$

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Step 3: notice that $P \cap L(\mathbb{Z}^2) = L(n\mathbb{Z}^2)$ or A_n, then split the proof.

□

Thank you for your attention!