Kannan-Lovasz-Simonovits conjecture up to polylog

Joseph Lehec
based on a joint work with Bo'az Klartag

Institute for Advanced Study in Mathematics
Harbin Institute of Technology
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Bourgain's slicing problem

Question [Bourgain early 80s]

Let K be a convex body in $\mathbb{R}^n$ of volume 1. Is it true that there exists an affine hyperplane H such that

$$\mathrm{vol}_{n-1}(K \cap H) \geq c,$$

where $c > 0$ is a universal constant?

Bourgain's estimate

Let L_n be the slicing constant:

$$\frac{1}{L_n} = \inf_K \sup_H \{\mathrm{vol}_{n-1}(K \cap H)\}$$

where the infimum is taken on all convex bodies of volume 1 in $\mathbb{R}^n$ and the supremum on all affine hyperplanes.

Theorem [Bourgain '89]

$$L_n \leq C n^{1/4} \cdot \log n$$

This essentially remained the best known estimate up until very recently (Klartag removed the log in 2006).

High dimension and universality

The philosophy of asymptotic convex geometry is that as the dimension tends to infinity some universality phenomena tend to appear.

Dvoretzky's theorem (Milman version 1971)

When n is large, most $\log n$-dimensional sections of an n-dimensional convex body are approximately Euclidean.

Central limit problem for convex sets

Is it true that most marginal of a high dimensional convex body are approximately Gaussian?

The Euclidean ball and the hypercube are interesting exemples. For the unit ball all marginals are nearly Gaussian, for X uniformly distributed on $[-1,1]^n$ the variable $\sum \theta_i X_i$ is approximately Gaussian if the mass of θ is sufficiently spread out over the coordinates.

Log-concave measures

Log-concave measures

A measure μ on $\mathbb{R}^n$ of the form

$$\mu(dx) = \mathrm{e}^{-V(x)}\, dx$$

with $V\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *convex* is called log-concave.

This class contains the class of uniform measures on convex sets and is more convenient to work with, as it is stable under various operations such as taking products and marginals. In particular the convolution of two log-concave measures is log-concave.

Thin-shell constant

A probability measure μ on $\mathbb{R}^n$ is called isotropic if

$$\int_{\mathbb{R}^n} x \, d\mu = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} x \otimes x \, d\mu = \mathrm{Id}.$$

Given an isotropic log-concave μ let

$$\sigma_\mu^2 = \frac{1}{n} \mathrm{var}_\mu |x|^2$$

be the *thin-shell* constant of μ.

Most of the mass of an isotropic log-concave vector is located at $O(\sqrt{n})$-distance from the origin. The question here is whether or not $\sigma_\mu = o(\sqrt{n})$ in which case the mass would be located in a shell whose width is of a smaller order than $\sqrt{n}$.

Slicing conjecture

The slicing constant constant put forward by Antilla-Ball-Perissinaki and Bobkov-Koldobsky around 2003 is

$$\sigma_n = \sup_{\mu} \sigma_\mu$$

where the supremum is taken on every isotropic log-concave measure μ.
They proved that the central limit theorem for convex sets would follow from the estimate

$$\sigma_n = o(\sqrt{n}).$$

They also made the following conjecture:

Thin-shell conjecture

$$\sigma_n = O(1).$$

Classical results on thin-shell

- Central limit problem

$$\sigma_n = o(\sqrt{n})$$

  was solved by Klartag in 2007
- Other proof shortly afterwards by Fleury, Guédon, Paouris
- ...
- Guédon, E. Milman '11: $\sigma_n = O(n^{1/3})$.

Kannan-Lovasz-Simonovits conjecture

Given a probability measure μ on $\mathbb{R}^n$ let $C_P(\mu)$ be the best constant in the Poincaré inequality:

$$\mathrm{var}_\mu(f) \leq C_P(\mu) \int_{\mathbb{R}^n} |\nabla f|^2 \, dx, \quad \forall f$$

Again set

$$\psi_n^2 = \sup_\mu \{C_P(\mu)\}$$

where the supremum is taken over all isotropic log-concave probability measure on $\mathbb{R}^n$.
In 1995, Kannan, Lovasz and Simonovits proved $\psi_n = O(n^{1/2})$ and made the following conjecture:

KLS conjecture

$$\psi_n = O(1)$$

Connection between the conjectures

The motivation of KLS was of algorithmic nature (speed of convergence of certain random walks on high dimensional convex sets) but the popularity of this conjecture rather comes from it implications in asymptotic convex geometry.

- Obviously $\sigma_\mu^2 \le 4 \cdot C_P(\mu)$ (just take $f(x) = |x|^2$ in Poincaré)
- In 2011 Eldan and Klartag proved

$$L_n \le C\sigma_n.$$

- Thus

$$L_n \le C\sigma_n \le C'\psi_n$$

- In terms of the conjectures

$$\text{KLS} \quad \Rightarrow \quad \text{thin-shell} \quad \Rightarrow \quad \text{slicing}$$

Eldan's Theorem

Theorem [Eldan 2013]

$$\psi_n \leq C\sigma_n \cdot \log n$$

- Thus, up to a logarithmic factor the trivial implication between KLS and thin-shell can be reversed.
- More importantly Eldan introduces a new approach for KLS: *stochastic localization*. It later turned out that this approach yielded much more than that.

Stochastic localization results

Theorem [Lee-Vempala 2017]

$$\psi_n \le C n^{1/4}$$

- This improves upon Guédon-Milman $\sigma_n \le C\, n^{1/3}$;
- And recovers Bourgain-Klartag $L_n \le C n^{1/4}$.

Theorem [Chen 2021]

$$\psi_n \le \exp\left(C\sqrt{\log n} \cdot \sqrt{\log\log n}\right)$$

- This is of a smaller order than any polynomial in n.

Main result of this talk

Theorem [Klartag, Lehec 2022]

$$\psi_n \leq C(\log n)^5$$

- We actually prove $\sigma_n \leq C(\log n)^4$,
- Hence $\psi_n \leq C'(\log n)^5$ by Eldan,
- And $L_n \leq C''(\log n)^4$ by Eldan-Klartag.

Sketch of proof

Our proof also uses stochastic localization, although in a slightly different way than in Eldan, Lee-Vempala and Chen's works. Stochastic localization is arguably less central in our proof.
The main steps are as follows:

1. The H^{-1}-inequality
2. Spectral measure and the heat flow
3. Stochastic localization tools

The Laplace operator

- Laplace (or Langevin) operator associated to $\mu(dx) = \rho(x)\,dx$:

$$Lf = \Delta f + \langle \nabla \log \rho, \nabla f \rangle$$

- $-L$ is self-adjoint positive semi-definite in $L^2(\mu)$

$$-\int (Lf) g \, d\mu = \int \langle \nabla f, \nabla g \rangle \, d\mu.$$

- 0 is a simple eigenvalue and the corresponding eigenspace is the space of constant functions.

The H^{-1}-inequality

For $f \in L^2(\mu)$ satisfying $\int_{\mathbb{R}^n} f \, d\mu = 0$ set

$$\|f\|_{H^{-1}(\mu)} = \sup \left\{ \int_{\mathbb{R}^n} f u \, d\mu, \ \int_{\mathbb{R}^n} |\nabla u|^2 \, d\mu \leq 1 \right\}.$$

H^{-1}-inequality [Klartag '09]

If μ is log-concave then for every f such that $\int_{\mathbb{R}^n} \nabla f \, d\mu = 0$

$$\mathrm{var}_\mu(f) \leq \sum_{i=1}^{n} \|\partial_i f\|_{H^{-1}(\mu)}^2.$$

Proof: Bochner + Cauchy-Schwarz.

Spectral mass and the H^{-1}-norm

Let's assume for simplicity that the spectrum of L is discrete: Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of $-L$ and (e_i) be the corresponding eigenfunctions. The spectral gap of L is

$$\lambda_1 = \frac{1}{C_P(\mu)}.$$

The H^{-1}-norm of a centred function f can then be rewritten

$$\begin{aligned}\|f\|^2_{H^{-1}(\mu)} &= \sum_{i\geq 1} \lambda_i^{-1} \langle f, e_i\rangle^2 \\ &= \int_{\lambda_1}^{\infty} \frac{1}{\lambda}\, \nu_f(d\lambda) = \int_{\lambda_1}^{\infty} \nu_f([0,\lambda])\, \frac{d\lambda}{\lambda^2},\end{aligned}$$

where $\nu_f = \sum_{i\geq 1} \langle f, e_i\rangle^2 \delta_{\lambda_i}$ is the spectral measure of f.

The heat flow

- Let X have law μ and let (B_s) be a standard Brownian motion on $\mathbb{R}^n$, independent of X.
- Let μ_s be the law of $X + B_s$.
- Then μ_s has density $P_s\rho$ where ρ is the density of μ, and P_s is the heat semi-group:

$$P_s\rho(x) = \mathbb{E}\rho(x + B_s) \ .$$

- P_s is a contraction from $L^2(\mu_s)$ to $L^2(\mu)$. It is easily seen that its adjoint is given by

$$Q_s f := (P_s)^* f = \frac{P_s(f\rho)}{P_s\rho}.$$

Spectral measure and heat flow

Proposition

If μ is log-concave then for every f such that $\|f\|_{L^2(\mu)} = 1$ and $\int_{\mathbb{R}^n} f\, d\mu = 0$ then

$$\nu_f([0,\lambda]) \leq C(\|Q_s f\|^2_{L^2(\mu_s)} + s\lambda),$$

for every $s > 0$ and $\lambda > 0$.

Note that this proposition gives an inequality between two seemingly unrelated objects. The left hand-side is concerned with the spectral distribution of f for the Laplace operator L. In the right-hand side we are only looking at f and μ along the heat flow.

Stochastic localization enters the picture

- Backward martingale formulation of Q_s:

$$Q_s f(X + B_s) = \mathbb{E}[f(X) \mid (X + B_r)_{r \geq s}]$$

- If we change s to $1/s$ the right-hand-side becomes of forward martingale, which is related to Eldan's stochastic localization:

$$Q_s f(X + B_s) = \int_{\mathbb{R}^n} f(x) p_t(x)\, dx, \quad t = \frac{1}{s},$$

  where (p_t) is the localization process initiated at μ.
  [Klartag-Putterman '21]

- In particular

$$\|Q_s f\|^2_{L^2(\mu_s)} = \mathbb{E}\left(\int_{\mathbb{R}^n} f(x) p_t(x)\, dx \right)^2 .$$

Eldan's stochastic localization

- (p_t) is a martingale taking values in the space of log-concave probability densities on $\mathbb{R}^n$
- The process (p_t) solves the Eldan equation:

$$\begin{cases} p_0(x) = \rho(x) \\ dp_t(x) = p_t(x)\langle x - a_t, dB_t\rangle \end{cases}$$

  where $a_t = \int_{\mathbb{R}^n} x p_t(x)\, dx$ is the barycenter of p_t.
- Also p_t has the form

$$p_t(x) = Z_t^{-1} \mathrm{e}^{\langle \theta_t, x\rangle - t|x|^2/2} \rho(x).$$

  In particular when ρ is log-concave, p_t is t-uniformly log-concave.

Estimate for the norm of the covariance for small time

Set

$$A_t = \int_{\mathbb{R}^n} (x - a_t)^{\otimes 2} p_t(x)\, dx = \operatorname{cov}(p_t),$$

and $\|A_t\|_p = \operatorname{Tr}(A_t^p)^{1/p}$.

Lemma (essentially due to Eldan)

Suppose we start from an isotropic log-concave μ on $\mathbb{R}^n$. Then

$$\mathbb{E}\|A_t\|_p^p \le C_p n$$

as long as $t \le (C\sigma_n^2 \cdot \log^2 n)^{-1}$.

- From this estimate, it is pretty easy to get Eldan's theorem $\psi_n \le C\sigma_n \log n$.
- Actually we will only use this for $p = 2$.

Controlling the variation of A_t

We also need to control the evolution of $\mathbb{E}\|A_t\|_2^2$ beyond time $t_0 := (C\sigma_n^2 \cdot \log^2 n)^{-1}$.

Lemma

For every $t_1 \le t_2$

$$\mathbb{E}\|A_{t_2}\|_2^2 \le \left(\frac{t_2}{t_1}\right)^3 \mathbb{E}\|A_{t_1}\|_2^2.$$

- Chen proves the analogue for $\|A_t\|_p$ for $p \ge 3$.
- This is the key to his $n^{o(1)}$ bound for ψ_n.
- His method seems to break down for $p < 3$, our proof of this lemma is quite different from his.

Estimate for the norm of the barycenter

From Eldan's equation we have $da_t = A_t(dB_t)$, hence:

$$\mathbb{E}|a_t|^2 = \int_0^t \mathbb{E}\|A_r\|_2^2 \, dr.$$

Putting the two lemmas together we thus get the following:

Corollary

For every $t > 0$

$$\mathbb{E}|a_t|^2 \leq Cn \cdot t \cdot \max(1, (t/t_0)^3)$$

where $t_0 = (C\sigma_n^2 \cdot \log^2 n)^{-1}$.

Proof of the main result

Let μ be log-concave and isotropic.
Applying the H^{-1} inequality to $f(x) = |x|^2$ we get

$$\sigma_\mu^2 = \frac{1}{n} \cdot \mathrm{var}_\mu |x|^2 \leq \frac{4}{n} \sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2$$
$$= 4 \int_{\lambda_1}^\infty F(\lambda) \, \frac{d\lambda}{\lambda^2},$$

where

$$F(\lambda) = \frac{1}{n} \sum_{i=1}^n \nu_{x_i}([0;\lambda])$$

is the average spectral distribution of the coordinate functions.

Proof of the main result (continued)

By the proposition and the corollary:

$$
\begin{aligned}
F(\lambda) &\le C\left(\frac{1}{n}\|Q_s x\|^2_{L^2(\mu_s)} + \lambda s\right) \\
&= C\left(\frac{1}{n}\mathbb{E}|a_t|^2 + \frac{\lambda}{t}\right), \quad t = 1/s \\
&\le C'\left(\frac{t^4}{t_0^3} + \frac{\lambda}{t}\right) \\
&= C''\lambda^{4/5}\cdot t_0^{-3/5}, \quad \text{choosing } t = \lambda^{1/5}\cdot t_0^{3/5}.
\end{aligned}
$$

Proof of the main result (end)

Combining

$$F(\lambda) \leq C\lambda^{4/5} \cdot t_0^{-3/5}.$$

with the H^{-1}-inequality we thus get

$$\begin{aligned}
\sigma_\mu^2 &\leq C \int_{\lambda_1}^{\infty} F(\lambda)\,\frac{d\lambda}{\lambda^2} \\
&\leq C' \cdot \lambda_1^{-1/5} \cdot t_0^{-3/5} \\
&= C'' \cdot C_P(\mu)^{1/5}(\sigma_n^2 \cdot \log^2 n)^{3/5} \\
&\leq C_1(\sigma_n^2 \cdot \log^2 n)^{4/5} \quad \text{(by Eldan's theorem).}
\end{aligned}$$

Choosing μ extremal in thin-shell then gives

$$\sigma_n^2 \leq C(\sigma_n^2 \cdot \log^2 n)^{4/5},$$

hence $\sigma_n \leq C'(\log n)^4$.

Proof of the Proposition

Claim (Klartag-Putterman)

When μ is log-concave, for every g such that $\int_{\mathbb{R}^n} g\, d\mu = 0$:

$$\|Q_s g\|^2_{L^2(\mu_s)} \geq \|g\|^2_{L^2(\mu)} \exp\left(-s\frac{\|g\|^2_{H^1(\mu)}}{\|g\|^2_{L^2(\mu)}}\right).$$

We apply this to g which is a spectral truncation of f:

$$g = \sum_{i:\ \lambda_i \leq \lambda} \langle e_i, f\rangle e_i$$

We get

$$\|Q_s g\|^2_{L^2(\mu_s)} \geq \nu_f([0,\lambda]) \cdot \mathrm{e}^{-\lambda s} \geq \nu_f([0,\lambda]) \cdot (1-\lambda s).$$

After some manipulations we can upgrade this to

$$\|Q_s f\|^2_{L^2(\mu_s)} \geq c \cdot \nu_f([0,\lambda]) - \lambda s.$$

Proof of the Chen type lemma

Eldan's equation and Itô's formula show that

$$\frac{d}{dt}\mathbb{E}\|A_t\|_2^2 \leq \mathbb{E} \text{ some order } 3 \text{ tensor of } p_t$$

which, conditioned on (B_t), looks like $\mathbb{E}\langle X, Y\rangle^3$ with X, Y two independent random vectors with density $p_t(\cdot + a_t)$.

Lemma 2

If X, Y are i.i.d. t-uniformly log-concave centred random vectors then

$$\mathbb{E}\langle X, Y\rangle^3 \leq \frac{3}{t} \cdot \mathbb{E}\langle X, Y\rangle^2 = \frac{3}{t} \cdot \|\mathrm{cov}(X)\|_2^2$$

Taking this for granted we get

$$\frac{d}{dt}\mathbb{E}\|A_t\|_2^2 \leq \frac{3}{t} \cdot \mathbb{E}\|A_t\|_2^2$$

and the result follows by integrating this differential inequality.

Proof of Lemma 2

By homogeneity we can assume $t = 1$. The proof is based on the following.

Claim

A 1-uniformly log concave centred random vector can be represented

$$X = \int_0^1 Q_s dB_s$$

with Q_s an adpapted process of matrices such that

$$0 \leq Q_s \leq \mathrm{Id}, \quad \forall s \in [0, 1]$$

almost surely (in the p.s.d. sense).

Proof of Lemma 2 (continued)

Use the claim and set $X_t = \int_0^t Q_s dB_s$. By Itô's formula

$$\mathbb{E}\langle X, Y\rangle^3 \leq 3 \int_0^1 \mathbb{E}[\langle X_t, Y\rangle \cdot |Q_t Y|^2] \, dt$$

Then write

$$\mathbb{E}[\langle x, Y\rangle \cdot |QY|^2] \leq \mathbb{E}\langle x, Y\rangle^2]^{1/2} \cdot \mathrm{var}(|QY|^2)^{1/2}$$

and apply Poincaré for Y: since Y is 1-uniformly log-concave it satisfies Poincaré with constant 1. Then it is straightforward to get the desired inequality with some constant.
Getting constant 3 requires a couple of additional tricks
(the value of that constant matters if we care about the power of the log in the main result)