# Quantum differentiation and integration for the non-commutative plane

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# Infinitesimals according to Newton

Isaac Newton made use of variables which he called "infinitesimals", and which he described as the following:

"In a certain problem, a variable is a quantity that takes an infinite number of values which are quite determined by this problem and are arranged in a definite order

...

"A variable is called infinitesimal if among its particular values one can be found such that this value itself and all following it are smaller in absolute value than an arbitrary given number"

Newton thought of infinitesimals as variable quantities: a quantity $h$ is called infinitesimal which vanishes to zero as time passes. One cannot escape the conclusion: Newton was talking about vanishing sequences!

# Infinitesimals as operators: Alain Connes' viewpoint

A. Connes has proposed an operator-theoretic rigorous notion of infinitesimals:
We shall say that an operator $T \in B(\mathcal{H})$ is infinitesimal if for any $n \geq 1$ there exists a finite dimensional subspace $E$ such that $\|T\big|_{E^\perp}\| < \frac{1}{n}$. This is equivalent to saying that $T$ is compact.
Connes' operator-theoretic infinitesimals are similar in nature to Newton's idea of an infinitesimal: it is a quantity which takes an infinite sequence of values which go to zero.
Newton: an infinitesimal is a sequence tending to zero. Connes: an infinitesimal is a compact operator (that is a bounded operator which has only discrete eigenvalues tending to zero. It will be convenient, if you recall the definition of a sequence of singular values of a compact operator and view it in "decreasing ordering". I explain on the next slide.

# Singular value sequence

Let $\mathcal{H}$ be a Hilbert space. If $T$ is a compact operator on $H$, the singular value sequence of $T$ is defined as

$$\mu(k, T) := \inf\{\|T - R\|_{\mathrm{op}} \ : \ \mathrm{rank}(R) \leq k\}, \quad k \geq 0.$$

(Equivalently, $\mu(T) = \{\mu(k, T)\}_{k=0}^{\infty}$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.)
A compact operator $T$ is said to be weak trace-class ($T \in \mathcal{L}_{1,\infty}$) if $\mu(k, T) = \mathcal{O}(k^{-1})$. Equivalently,

$$\|T\|_{1,\infty} := \sup_{k \geq 0} (1 + k)\mu(k, T) < \infty.$$

The class $\mathcal{L}_{1,\infty}$ is an an ideal. We write $T \in \mathcal{L}_{p,\infty}$ if $|T|^p \in \mathcal{L}_{1,\infty}$.

## Principal ideals $\mathcal{L}_{p,\infty}$is an important class of NC infinitesimals, so let us elaborate on it.

Let $\mathcal{L}_{p,\infty}$ be the principal ideal in $B(\ell_2)$ generated by the element $A_0 = \mathrm{diag}(\{(k+1)^{-\frac{1}{p}}\}_{k\geq 0})$. Equivalently,

$$\mathcal{L}_{p,\infty} = \{A : \sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,A) < \infty\}.$$

In Noncommutative Geometry, a compact operator $A$ is called an infinitesimal of order $\frac{1}{p}$ if for singular values of the operator $A$, we have

$$\mu(k,A) = O((k+1)^{-\frac{1}{p}}), \quad k \in \mathbb{Z}_+.$$

In other words, $\mathcal{L}_{p,\infty}$ is the set of all infinitesimals of order $\frac{1}{p}$.

# Quantized differential

Let $\mathcal{A}$ be an involutive algebra over $\mathbb{C}$. Then a **Fredholm module** over $\mathcal{A}$ is given by

1. an involutive representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,
2. an operator $F = F^*$ , $F^2 = 1$, on $\mathcal{H}$ such that $[F, \pi(a)]$ is a compact operator for any $a \in \mathcal{A}$. (Atiyah's definition of abstract elliptic)

**Quantized differential** $đa$ of $a \in \mathcal{A}$:

$$đa = i[F, a] = i(Fa - aF);$$

A general question occurring in NG: to which class of NC infinitesimals a quantum differential belongs?

# Traces on $\mathcal{L}_{1,\infty}$–precursors of noncommutative integrals in NG

**Definition**

A linear functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called a trace if $\varphi(AB) = \varphi(BA)$ for every $A \in \mathcal{L}_{1,\infty}$ and for every $B \in B(H)$.

The trace $\varphi$ is called normalised if $\varphi(\mathrm{diag}(\{\frac{1}{k+1}\}_{k\geq 0})) = 1$.
There exists a plethora of (normalised) traces on $\mathcal{L}_{1,\infty}$. The most famous ones are Dixmier traces.

# Dixmier traces

An extended limit is a bounded functional $\omega$ in $\ell_\infty$ which extends the "limit" functional on the subspace $c$ of convergent sequences and $\|\omega\|_{\ell_\infty^*} = 1$. Extended limits exist by the Hahn-Banach theorem.

### Definition (Dixmier)

If $\omega$ is an extended limit then the functional

$$A \to \omega\left(\frac{1}{\log(n+2)}\sum_{k=0}^{n}\mu(k,A)\right), \quad 0 \leq A \in \mathcal{L}_{1,\infty}$$

is finite and additive on the positive cone of $\mathcal{L}_{1,\infty}$. Thus, it uniquely extends to a unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$. The latter is called a Dixmier trace and is denoted by $\mathrm{tr}_\omega$.

# Quantized integration; Conformal trace formula

Alain Connes' idea (1988): "*The next result shows how to pass from quantized* 1*-forms to ordinary forms, not by a classical limit, but by a direct application of the Dixmier trace.*" may be seen as a precursor for conformal trace formulae like this one: for every $f \in C_c^\infty(\mathbb{R}^p)$, we have

$$\mathrm{tr}_\omega(|[F, 1 \otimes M_f]|^p) = c_p \int_{\mathbb{R}^p} \left( \sum_{j=1}^{p} |\partial_j f(s)|^2 \right)^{\frac{p}{2}} ds.$$

Here $\mathrm{tr}_\omega$ is an arbitrary Dixmier trace, and $F = \mathrm{sgn}(D)$ is the sign of Dirac operator.

**Question**

What are the conditions on $f$ which guarantee

$$[F, 1 \otimes M_f] \in \mathcal{L}_{p,\infty}, \quad p > 0?$$

## Example: spectral triple for $\mathbb{R}$

Let $\mathcal{A} = C_c^\infty(\mathbb{R})$, $H = L_2(\mathbb{R})$ and let $D = -i\frac{d}{dx}$ be the differentiation operator. For $F = \mathrm{sgn}(D)$, the following assertion is due to Peller.

**Theorem**

If $f$ is locally integrable on $\mathbb{R}$ then

$$[\mathrm{sgn}(D), M_f] \in B(H) \Leftrightarrow f \in BMO. \text{ [Nehari 1957]}$$

$$[\mathrm{sgn}(D), M_f] \in \mathcal{K}(H) \Leftrightarrow f \in VMO. \text{ [Coifman-Rochberg-Weiss 1976]}$$

$$[\mathrm{sgn}(D), M_f] \in \mathcal{L}_p \Leftrightarrow f \in B^{\frac{1}{p}}_{p,p}. \text{ [Peller 1980]}$$

However, as we will see later, 1-dimensional geometry is, in certain sense, an exception.

# Example: spectral triple for $\mathbb{R}^d$, $d > 1$

Let $\mathcal{A} = C_c^\infty(\mathbb{R}^d)$, $H = L_2(\mathbb{R}^d)$ and let $D_k$, $1 \leq k \leq d$, be the partial direvatives. Dirac operator is given with the help of Pauli matrices:

$$D = \sum_{k=1}^{d} \gamma_k \otimes D_k.$$

For $F = \mathrm{sgn}(D)$, the boundedness and compactness of $[F, M_f]$ remain the same as $d = 1$ case, due to Coifman-Rochberg-Weiss 1976.
What is different , due to Janson and Wolff:

**Theorem**

If $f \in L_\infty(\mathbb{R}^d)$, then

$$[\mathrm{sgn}(D), 1 \otimes M_f] \in \mathcal{L}_p, \quad p > d \Leftrightarrow f \in B^{\frac{d}{p}}_{p,p}(\mathbb{R}^d).$$

$$[\mathrm{sgn}(D), 1 \otimes M_f] \in \mathcal{L}_p, \quad p \leq d \Leftrightarrow f = \mathrm{const}.$$

## More on spectral triple for $\mathbb{R}^d$ , $d > 1$

In between the ideals $\mathcal{L}_p$ for $p > d$ and $p \leq d$ lies the ideal $\mathcal{L}_{d,\infty}$, and for this ideal we have the following result.

**Theorem**

If $f \in L_\infty(\mathbb{R}^d)$, then

$$[\operatorname{sgn}(D), 1 \otimes M_f] \in \mathcal{L}_{d,\infty}, \Leftrightarrow f \in \dot{W}_d^1(\mathbb{R}^d).$$

Here $\dot{W}_d^1(\mathbb{R}^d)$ is the homogeneous Sobolev space of functions $f$ such that $\nabla f \in L_d(\mathbb{R}^d, \mathbb{C}^d)$.
This theorem appears in a paper by Connes, Sullivan and Teleman. Similar results were also proved by Rochberg and Semmes. The recent (rather different) proof is given in [Lord,McDonald,S.,Zanin (2017)].

# Conformal trace formula II

The following follows from a simple modification of the argument in [Lord,McDonald,S.,Zanin (2017)].

**Theorem**

If $h \in L_\infty(\mathbb{R}^d)$ and $f \in \dot{W}^1_d(\mathbb{R}^d)$, then

$$\varphi((1 \otimes M_h)|[F, 1 \otimes M_f]|^d) = c_d \int_{\mathbb{R}^d} h(s) \|(\nabla f)(s)\|^d_{\ell_2} ds.$$

# Connes Integration Formula I: the classical case

For any smooth compactly supported function $f$ on $\mathbb{R}^d$, Connes established the following formula

$$\mathrm{tr}_\omega(M_f(1-\Delta)^{-\frac{d}{2}}) = \frac{\mathrm{Vol}(\mathbb{S}^{d-1})}{d(2\pi)^d}\int_{\mathbb{R}^d} f(s)ds.$$

In this talk, we provide a substantially stronger (and more general) version of the Connes Integration Formula for noncommutative Euclidean space.

# A Dictionary

| | | |
|---|---|---|
| CLASSICAL | $\leftrightarrow$ | QUANTUM |
| Complex variable | $\leftrightarrow$ | Operator on $\mathcal{H}$ |
| Real variable | $\leftrightarrow$ | Self-adjoint operator on $\mathcal{H}$ |
| Range | $\leftrightarrow$ | Spectrum of operator |
| Infinitesimal | $\leftrightarrow$ | Compact operator on $\mathcal{H}$ |
| Size of infinitesimals | $\leftrightarrow$ | Decay of eigenvalues |
| Differential | $\leftrightarrow$ | $đf = i[F,f] = i(Ff - fF)$ |
| Integral of infinitesimal | $\leftrightarrow$ | Dixmier trace |

# Cwikel-Solomyak estimates in $\mathcal{L}_{1,\infty}$

One of the most beautiful results in Noncommutative Geometry concerns estimates for the singular values of the operator $M_f g(\nabla)$ on $L_2(\mathbb{R}^d)$ in weak Schatten ideals $\mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$.
We say that $f \in \ell_1(L_2)(\mathbb{R}^d)$ if

$$\sum_{k\in\mathbb{Z}^d} \|f\chi_{k+K}\|_2 < \infty, \quad K = [0,1]^d.$$

The result below is (probably) due to Birman and Solomyak. Its clear proof is given in [LeSZ].

### Theorem

If $f \in \ell_1(L_2)(\mathbb{R}^d)$, then

$$M_f(1-\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)).$$

# Connes Integration Formula II

The following result established in [SZ-JOT-2018] (Section 4 there is fully devoted to the proof of this result).

**Theorem**

If $f \in \ell_1(L_2)(\mathbb{R}^d)$, then

$$\varphi(M_f(1-\Delta)^{-\frac{d}{2}}) = \frac{\mathrm{Vol}(\mathbb{S}^{d-1})}{d(2\pi)^d} \int_{\mathbb{R}^d} f(s)ds$$

for every continuous normalised trace on $\mathcal{L}_{1,\infty}$.

Observe that we have significantly weakened the assumption imposed on the function $f$ comparatively with the original version due to Connes and significantly extended the stock of singular traces for which the equality above holds.

# What is Noncommutative Euclidean space I

The algebra of polynomial functions on $\mathbb{R}^d$ is the $*-$algebra generated by elements $\{x_k\}_{k=1}^d$ satisfying the conditions

$$x_k = x_k^*, \quad 1 \leq k \leq d,$$

$$[x_{k_1}, x_{k_2}] = 0, \quad 1 \leq k_1, k_2 \leq d.$$

Now, let us deform the latter relations as follows:

$$[x_{k_1}, x_{k_2}] = i\theta_{k_1,k_2}, \quad 1 \leq k_1, k_2 \leq d.$$

Here, $\theta \in M_d(\mathbb{R})$ is some skew-symmetric matrix.
It is natural to treat the algebra generated by $\{x_k\}_{k=1}^d$ as the (algebra of polynomial functions on) Noncommutative Euclidean space.

# What is Noncommutative Euclidean space II

For analytic reasons it is easier to deal with the algebra of bounded functions than the algebra of polynomial functions.
For $t \in \mathbb{R}^d$, formally set

$$U(t) = \exp(i \sum_{k=1}^{d} t_k x_k).$$

A formal manipulation with $U(t)'$s (using the Baker-Campbell-Hausdorff formula) yields

$$U(t+s) = e^{-\frac{i}{2}\langle t, \theta s\rangle} U(t)U(s), \quad t, s \in \mathbb{R}^d.$$

# Formal definition of Noncommutative Euclidean space

The algebra $L_\infty(\mathbb{R}^d_\theta)$ is defined as a von Neumann algebra generated by unitaries $\{U(t)\}_{t\in\mathbb{R}^d}$ obeying the relation on the previous slide.
For convenience we choose a concrete family of unitary operators to generate our algebra. Namely, we set

$$(U(t)\xi)(u) = e^{-\frac{i}{2}\langle t,\theta u\rangle}\xi(u-t), \quad \xi \in L_2(\mathbb{R}^d), \quad u,t\in\mathbb{R}^d. \tag{1}$$

### Definition

The von Neumann subalgebra in $B(L_2(\mathbb{R}^d))$ generated by $\{U(t)\}_{t\in\mathbb{R}^d}$ in (1), is called the NC Euclidean space and denoted by $L_\infty(\mathbb{R}^d_\theta)$.

Note that when $\theta = 0$, $U(t)$ is a translation operator and $L_\infty(\mathbb{R}^d_0)$ is the algebra of all bounded Fourier multipliers of $L_2(\mathbb{R}^d)$.

# NC algebra of continuous functions

The subalgebra $C_0(\mathbb{R}^d_\theta)$ of $L_\infty(\mathbb{R}^d_\theta)$ is defined as the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^d_\theta)$ in the uniform norm. Here,

$$\mathcal{S}(\mathbb{R}^d_\theta) = \{\int_{\mathbb{R}^d} U(t)f(t)\,dt \ :\ f \in \mathcal{S}(\mathbb{R}^d)\} \subset L_\infty(\mathbb{R}^d_\theta).$$

### Theorem

If $\theta$ is non-degenerate, then

$$C_0(\mathbb{R}^d_\theta) \cong \mathcal{K}.$$

That is, as a $C^*$-algebra $C_0(\mathbb{R}^d_\theta)$ is isomorphic to the algebra of compact operators on a Hilbert space.

# Faithful normal semifinite trace on $L_\infty(\mathbb{R}^d_\theta)$

The following assertion is well-known. In [LeSZ], a *spatial* isomorphism is constructed.

**Theorem**

For every non-degenerate antisymmetric real matrix $\theta$, the algebra $L_\infty(\mathbb{R}^d_\theta)$ is $*$-isomorphic to $B(L_2(\mathbb{R}^{\frac{d}{2}}))$.

Note that $\det(\theta) \neq 0$ implies that $d$ is even (this why we speak of NC *plane*).

Having established the $*$-isomorphism between $r : L_\infty(\mathbb{R}^d_\theta) \to B(L_2(\mathbb{R}^{\frac{d}{2}}))$ we now equip $L_\infty(\mathbb{R}^d_\theta)$ with a faithful normal semifinite trace $\tau_\theta = \mathrm{Tr} \circ r$. We can now define $L_p-$spaces on $L_\infty(\mathbb{R}^d_\theta)$.

$$L_p(\mathbb{R}^d_\theta) = \Big\{ x \in L_\infty(\mathbb{R}^d_\theta) : \ \tau_\theta(|x|^p) < \infty \Big\}.$$

# Partial derivatives on $L_\infty(\mathbb{R}^d_\theta)$ I

Let $D_k$, $1 \leq k \leq d$ be multiplication operators on $L_2(\mathbb{R}^d)$

$$(D_k\xi)(t) = t_k\xi(t), \quad \xi \in L_2(\mathbb{R}^d).$$

For brevity, we denote $\nabla = (D_1, \cdots, D_d)$ and $-\Delta = \sum_{k=1}^d D_k^2$.

# Partial derivatives on $L_\infty(\mathbb{R}^d_\theta)$ II

If $[D_k, x] \in B(L_2(\mathbb{R}^d))$ for some $x \in L_\infty(\mathbb{R}^d_\theta)$, then $[D_k, x] \in L_\infty(\mathbb{R}^d_\theta)$. This crucial observation allows us to introduce mixed partial derivative $\partial^\alpha x$ of $x \in L_\infty(\mathbb{R}^d_\theta)$.

### Definition

Let $\alpha$ be a multiindex and let $x \in L_\infty(\mathbb{R}^d_\theta)$. If every repeated commutator $[D_{\alpha_j}, [D_{\alpha_{j+1}}, \cdots, [D_{\alpha_n}, x]]]$, $1 \le j \le n$, is a bounded operator on $L_2(\mathbb{R}^d)$, then the mixed partial derivative $\partial^\alpha x$ of $x$ is defined as

$$\partial^\alpha x = [D_{\alpha_1}, [D_{\alpha_2}, \cdots, [D_{\alpha_n}, x]]].$$

In this case, we have that $\partial^\alpha x \in L_\infty(\mathbb{R}^d_\theta)$. As usual, $\partial^0 x = x$.

# Sobolev spaces in $L_\infty(\mathbb{R}^d_\theta)$

NC Sobolev space $W^m_p(\mathbb{R}^d_\theta)$ associated with the NC plane are defined naturally.

**Definition**

For $m \in \mathbb{Z}_+$ and $p \geq 1$, the space $W^m_p(\mathbb{R}^d_\theta)$ is the space of $x \in L_p(\mathbb{R}^d_\theta)$ such that every partial derivative of $x$ up to order $m$ is also in $L_p(\mathbb{R}^d_\theta)$. This space is equipped with the norm,

$$\|x\|_{W^m_p} = \sum_{|\alpha|\leq m} \|\partial^\alpha x\|_p, \quad x \in W^m_p(\mathbb{R}^d_\theta).$$

Similarly, the homogeneous Sobolev space $\dot{W}^m_p(\mathbb{R}^d)$ is equipped with the seminorm

$$\|x\|_{\dot{W}^m_p} = \sum_{0<|\alpha|\leq m} \|\partial^\alpha x\|_p, \quad x \in \dot{W}^m_p(\mathbb{R}^d_\theta).$$

# Cwikel estimates for Noncommutative Euclidean space

Given $x \in L_1(\mathbb{R}_\theta^d)$, when do we have $x(1-\Delta)^{-\frac{d}{2}} \in \mathcal{L}_1$? In the NC case, it is not clear how to define $\ell_1(L_2)(\mathbb{R}_\theta^d)$. The following type estimate for the ideal $\mathcal{L}_{1,\infty}$ and for the NC Euclidean space is established in [LeSZ].

**Theorem**

If $x \in W_1^d(\mathbb{R}_\theta^d)$, then

$$x(1-\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)).$$

Note that this is consistent with the condition $f \in \ell_1(L_2)(\mathbb{R}^d)$ from the commutative case, because we have the Sobolev embedding

$$W_1^d([0,1]^d) \hookrightarrow L_2([0,1]^d).$$

# Connes Integration Formula III : NC Euclidean space

The following trace theorem is proved in [SZ-cmp].

**Theorem**

If $x \in W_1^d(\mathbb{R}_\theta^d)$, then

$$\varphi(x(1-\Delta)^{-\frac{d}{2}}) = \tau_\theta(x)$$

for every normalised trace $\varphi$ on $\mathcal{L}_{1,\infty}$.

Note that the functional $\tau_\theta$ correponds to $\frac{\mathrm{Vol}(S^{d-1})}{d(2\pi)^d}\int_{\mathbb{R}^d}$ in the commutative case.

# Conformal trace formula III: for $[F, a]$ on $\mathbb{R}^d_\theta$

Consider a Dirac operator for $\mathbb{R}^d_\theta$

$$D = \sum_{j=1}^{d} \gamma_j \otimes D_j$$

on $\mathbb{C}^{2^{\lfloor d/2 \rfloor}} \otimes L_2(\mathbb{R}^d_\theta)$. Let $F = \operatorname{sgn}(D) = \frac{D}{|D|}$. Then $(H, F) = (\mathbb{C}^{2^{\lfloor d/2 \rfloor}} \otimes L_2(\mathbb{R}^d_\theta), F)$ is a Fredholm module for $A = \mathbb{C} \otimes C_0(\mathbb{R}^d_\theta)$. To link $[F, x]$ with the gradient $\nabla x$, we have the following formula

**Theorem (S.-McDonald-Xiong (2020))**

Let $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$. Then $[F, 1 \otimes x] \in \mathcal{L}_{d,\infty}$ and for every continuous normalised trace $\varphi$ on $\mathcal{L}_{1,\infty}$ we have

$$\varphi(|[F, 1 \otimes x]|^d) = c_d \int_{\mathbb{S}^{d-1}} \tau_\theta\left( \Big( \sum_{j=1}^{d} |D_j x - s_j \sum_{k=1}^{d} s_k D_k x|^2 \Big)^{\frac{d}{2}} \right) ds.$$

Note that this is a noncommutative version of the commutative conformal trace formula

$$\varphi(|[F, 1 \otimes M_f]|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f(s)\|_{\ell_2}^d \, ds$$

However, if the partial derivatives $\{D_j x\}_{j=1}^d$ commute with each other and their adjoints, then we get the same expression.

# Localisation: Conformal trace formula IV

**Theorem (S.-McDonald-Xiong (2022))**

Let $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$ and $y \in \mathbb{C} + C_0(\mathbb{R}^d_\theta)$. Then for every continuous normalised trace $\varphi$ on $\mathcal{L}_{1,\infty}$ we have

$$\varphi(y|[F, 1 \otimes x]|^d) = c_d \int_{\mathbb{S}^{d-1}} \tau_\theta\Bigg( y \Big( \sum_{j=1}^{d} |D_j x - s_j \sum_{k=1}^{d} s_k D_k x|^2 \Big)^{\frac{d}{2}} \Bigg) ds.$$

1. [LeSZ] Levitina G., Sukochev F., Zanin D. *Cwikel estimates revisited.* Proc. Lond. Math. Soc. (3) **120**, no. 2, 265–304, (2020).
2. [SZ-JOT] Sukochev, F., Zanin, D. *A $C^*$-algebraic approach to the principal symbol. I.* J. Operator Theory **80**, no. 2, 481–522 (2018).
3. [SZ-cmp] Sukochev F., Zanin D. *Connes Integration Formula for the noncommutative plane.* Comm. Math. Phys. **359**, 449–466 (2018)
4. [MSX-cmp] McDonald E., Sukochev F., Xiong X. *Quantum differentiability on noncommutative Euclidean spaces.* Comm. Math. Phys. **379**, no. 2, 491–542 (2020).

# Thank you for your attention