Orthogonally additive operators in vector lattices and around

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Let E be a real vector space, equipped with a partial order $\leq$, which satisfies the following conditions:

1) If $u, v \in E$, $u \leq v$, then $u + w \leq v + w$, for every $w \in E$;
2) If $u, v \in E$, $u \leq v$, then $\lambda u \leq \lambda v$, for every $\lambda \in \mathbb{R}_+$.

An element $u \in E$ is said to be positive, if $u \geq 0$.
An ordered vector space E is called *a vector lattice*, if every finite subset D of E has supremum and infimum.

F. Riesz in 1928 in his talk *"On the decomposition of linear functionals into their positive and negative parts"*, at the International Congress of Mathematicians in Bologna, Italy, marked the beginnings of the study of vector lattices (Riesz spaces) and positive operators. The theory of vector lattices in first part of 20 century was developed Freudenthal, L. V. Kantorovich, G. Birkhoff. Later important contributions came from P. P. Korovkin, M. A. Krasnoselskii, P. P. Zabreiko, W. A. J. Luxemburg and A. C. Zaanen, Schaefer, and B. Z. Vulikh, P. G. Dodds, D. H. Fremlin, Abramovich, C. D. Aliprantis, A. V. Buhvalov, O. Burkinshaw, D. I. Cartwright, J. J. Grobler, Luxemburg, M. Meyer, P. Meyer-Nieberg, R. J. Nagel, U. Schlotterbeck, H. H. Schaefer, A. R. Schep, C. T. Tucker, A. I. Veksler, A. W. Wickstead, M. Wolff, A. C. Zaanen and others.

- W. A. J. Luxemburg and A. C. Zaanen, Riesz Spaces, I, North-Holland, Amsterdam, 1971.
- L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker, Functional Analysis in Partially Ordered Spaces, Gosudarstv. Izdat. Tecn. and Teor. Lit., Moscow and Leningrad, 1950.
- D. H. Fremlin, Topological Riesz Spaces and Measure Theory, Cambridge University Press, London and New York, 1974.
- A. C. Zaanen, Riesz Spaces II , North-Holland, Amsterdam, 1983..
- P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin etc., 1991.
- C. D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Dordrecht, (2006).
- Y. A. Abramovich, C. D. Aliprantis, *An Invitation to Operator Theory*, AMS, 2002.
- A. G. Kusraev, *Dominated operators*, Kluwer Academic Publishers, (2000).

In 1936 von Neumann in the article

- J. von Neumann, Charakterisierung des Spektrums eines Integraloperators, Actualites Sci. et Ind., Paris 1935, no. 229.

raised the problem of a characterization of integral operators in $L_2(\mu)$.
A linear operator $T\colon L_2 \to L_2$ is called integral if there exists a measurable function $K(s,t)$ (the kernel) such that

$$Tf(s) = \int_A K(s,t)f(t)d\,\mu(t)$$

for any $f \in L_2$. The integral is understood in the Lebesgue sense.

In 1974 this problem was solved by the Russian mathematician A.V. Bukhvalov.

- A.V. Bukhvalov, On integral representation of linear operators, Zap. Nauchn. Sem. LOMI 47 (1974), 5-14.
- A.V. Bukhvalov, Application of methods of the theory of order-bounded operators to the theory of operators in L_p-spaces. Russian Math. Surveys 38 : 6 (1983), 43-98.

By $L_0(A,\Sigma,\mu)$ or simply L_0, we denote the set of all measurable and a.e. finite real-valued functions on (A,Σ,μ) with the usual identification of equivalent functions. Since it is important for the theory of operators that the theory should be applicable also to complex spaces, we remark at once that the condition for functions to be real-valued is nowhere essential. We are interested in the following order relation between functions in L_0 or a subspace of L_0 (say, L_p). For any two functions $f,g\in L_0$ we set $f\leq g$ if $f(t)\leq g(t)$ a.e. It is clear that then L_0 becomes an ordered set and any two functions $f,g\in L_0$ have a supremum $f\vee g\in L_0$ and an infimum $f\wedge g\in L_0$, defined by

$$f\vee g(t)=\max(f(t),g(t));\;\; f\wedge g(t)=\min(f(t),g(t))$$

we note two further simple properties of the order in L_0: $f\leq g$ implies that $f+h\leq g+h$ for any $h\in L_0$; if $0\leq g$ and $0\leq\lambda$, then $0\leq\lambda g$.

Let us introduce some more definitions. Two elements $f, g \in L_0$ are called disjoint if $|f| \wedge |g| = 0$. A set $M \subset L_0$ is called order-bounded if there is a $g \in L_0$ such that $|f| \leq |g|$ for any $f \in M$.

If $M \subset L_0$ is an order bounded subset then $g = \sup M \in L_0$ exists. Moreover, there is a countable set $(f_n)_{n \in \mathbb{N}}$ of elements of M such that $g = \sup M = \sup_n f_n$ (the latter supremum can be calculated pointwise by means of). The same is true for the infimum.

We now come to the consideration of subspaces of L_0. An ideal space on (A, Σ, μ) is a linear subspace E of L_0 such that if $f \in L_0$, $g \in E$ $|f| \leq |g| \Rightarrow f \in E$, that is, with every function E contains its modulus function and all functions that are smaller in modulus. A norm $\|\cdot\|$ on an ideal space E is called monotone if $|f| \leq |g| \Rightarrow \|f\| \leq \|g\|$ for $f, g \in E$. A Banach ideal space on (A, Σ, μ) is an ideal space E endowed with a monotone norm with respect to which E is a Banach space. All the concepts just introduced are simply stated and natural: practically all spaces in the theory of functions of a real variable are ideal Banach spaces. Such are the classical L_p-spaces and the Orlicz spaces, the Marcinkiewicz spaces, the Lorentz spaces, and the Morrey spaces.

An ideal in a vector lattice E is a linear subspace I of E such that $x \in E$, $y \in I$ $|x| \leq |y| \Rightarrow x \in I$. It is clear that any ideal space is an ideal in L_0 (hence the term). Every ideal is itself a vector lattice. Two elements $x, y \in E$ are called disjoint $x \perp y$ if $|x| \wedge |y| = 0$. We now introduce the operation of forming the disjoint complement, which associates with any $M \subset E$ the set

$$M^{\perp} := \{x : x \perp y, \forall y \in M\};\ M^{\perp\perp} := \left(M^{\perp}\right)^{\perp}.$$

We say that a subset $B \subset E$ is a *band* if $B = M^{\perp}$ for some $M \subset E$.

With any band B a Dedekind complete vector lattice E there is connected the canonical projection π_B from E to B. For an element $x \in E_+$ we set

$$\pi_B x = \sup\{0 \leq y \leq x : \ y \in B\}$$

For any $x \in E$ we set

$$\pi_B x = \pi_B x^+ - \pi_B x^-.$$

Now we give an account of the calculus of order-bounded operators in an ideal space, which was developed mainly by Kantorovich. A significant contribution was made by his students Vulikh and Pinsker by the Japanese mathematician Nakano. Let E be an ideal space on (A, Σ, μ) and F an ideal space on (B, Ξ, ν) An operator $T\colon E \to F$ is called

1. order-bounded if it transforms order bounded subsets of E into order bounded subsets of F;
2. positive if $T(E_+) \subset F_+$;
3. regular if $T = S_1 - S_2$ where $S_1, S_2\colon E \to F$ are positive operators.

An operator is order bounded if and only if it is regular.

We denote by $\mathcal{L}_r(E, F)$ the collection of all regular operators. This is obviously a linear set. We introduce an order relation in $\mathcal{L}_r(E, F)$. We write $T \geq 0$ if T is a positive operator. We write $T \geq S$ if $T - S \geq 0$. The following theorem due to Kantorovich (which was established by Riesz in for functionals on C[0, 1]) plays the most important role.

Theorem

The set $\mathcal{L}_r(E,F)$ is a Dedekind complete vector lattice. Moreover, for all $S,T \in \mathcal{L}_r(E,F)$ and $x \in E_+$ the following relations hold:

1. $(T \vee S)(f) := \sup\{Tg + Sh : 0 \leq g, h, \ f = g + h\}$;
2. $(T \wedge S)(f) := \inf\{Tg + Sh : 0 \leq g, h, \ f = g + h\}$;
3. $(T)^+(f) := \sup\{Tg : 0 \leq g \leq f\}$;
4. $(T)^-(f) := -\inf\{Tg : 0 \leq g \leq f\}$;
5. $|T|(f) := \sup\{|Tg| : |g| \leq |f|\}$;
6. $|Tf| \leq |T|(f)$.

We consider with the integral operator

$$(Tf)(s) = \int_A K(s,t)f(t)d\,\mu(t)$$

another integral operator with

$$(Wf)(s) = \int_A |K(s,t)|f(t)d\,\mu(t)$$

with the kernel $|K(s,t)|$. The function Wf is defined for all $f \in E$. Thus, W always acts from E to L_0. And what about action in F? In general W need not act from L_2 into L_2.

Lemma

The following statements are equivalent:

1. W *acts from* E *to* F;
2. T *is a regular operator from* E *to* F.

Moreover $|T| = W$.

Corollary

An integral operator T *is positive if and only if* $K(\cdot,\cdot) \geq 0$ *a.e. Also,* $T = 0$ *if and only if* $K(\cdot,\cdot) = 0$.

Theorem

Let $T\colon L_2 \to L_2$ be a bounded linear operator. Then the following statements are equivalent:

1. *T is an integral operator, that is, T admits the integral representation;*
2. *$0 \leq f_n \leq f \in L_2$ and $f_n \to 0$ in measure (or equivalently, in the L_2-norm), then $Tf_n \to 0$ almost everywhere.*

Let us discuss the statement. The implication (1) $\Rightarrow$ (2) is trivial. The principal equivalence in Theorem is (2) $\Rightarrow$ (1). First of all one may think that one theorem on integral representation for functionals or another might be helpful. One could wish to fix an $s \in B$ and introduce a functional on L_2 by setting

$$\varphi_s(f) = (Tf)(s).$$

But the very notion of the value of a measurable function (or, more accurately, of a class of equivalent functions) at a point s is not defined. There is a very non-trivial theorem of von Neumann-Maharam on the existence of a lifting, which allows us to give a meaning to the notion for functions belonging to L_∞, but in L_p with $p < \infty$ no lifting can exist in principle.

Suppose, however, that even this difficulty proves to be surmountable. A new one would appear when verifying that $\varphi_s(f)$ is continuous on L_2 for almost all s. Suppose again that we have surmounted this difficulty. Then we can apply Riesz's theorem on the general form of a functional on L_2

$$\varphi_s(f) = \int_A g_s(t)f(t)d\mu(t)$$

or, setting $K(s,t) = g_s(t)$,

$$(Tf)(s) = \varphi_s(f) = \int_A K(s,t)f(t)d\,\mu(t)$$

Now we observe that by Riesz's theorem

$$\int_A |g_s(t)|^2 d\mu(t) < \infty$$

and hence for almost all $s \in B$

$$\int_A |K(s,t)|^2 d\mu(t) < \infty.$$

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$$(Tf)(s) = \varphi_s(f) = \int_A K(s,t) f(t) d\,\mu(t)$$

Now we observe that by Riesz's theorem

$$\int_A |g_s(t)|^2 d\mu(t) < \infty$$

and hence for almost all $s \in \Xi$

$$\int_A |K(s,t)|^2 d\mu(t) < \infty.$$

Theorem

The set $K(L_2, F)$ of all regular integral operators from L_2 to F is the band in the Dedekind complete vector lattice $\mathcal{L}_r(L_2, F)$.

Proof of the implication (2) $\Rightarrow$ (1). We know that $T \in \mathcal{L}_r(L_2, L_0)$ (we cannot assert that $T \in \mathcal{L}_r(L_2, L_2)$ therefore, a proof using the technique of regular operators within the framework of L_2 is impossible). By above the set of integral operators from L_2 to L_0 is the band of $\mathcal{L}_r(L_2, L_0)$. From the basic properties of the operator calculus it follows that there is a projection operator $\pi_{K(L_2,L_0)}$ from $\mathcal{L}_r(L_2, L_0)$ onto the band $K(L_2, L_0)$. We set $W = T - \pi_{K(L_2,L_0)}T$. The operator W satisfies (2) in Theorem. W is disjoint to $K(L_2, L_0)$, and it can be shown that $W = 0$.

We begin our discussion by recalling some basic properties of binary relations. Recall that a binary relation on a (non-empty) set X is a non-empty subset $\preceq$ of $X \times X$. The membership $(x, y) \in \preceq$ is usually written as $x \preceq y$. A binary relation $\preceq$ on a set X is said to be:

1. reflexive; whenever $x \preceq x$ holds for all $x \in X$;
2. complete; whenever for each pair (x, y) of elements of X either $x \preceq y$ or $y \preceq x$ holds;
3. transitive; whenever $x \preceq y$ and $y \preceq z$ imply $x \preceq z$.

Definition

A preference relation on a set X is a reflexive, complete and transitive relation on the set X.

Definition

A neoclassical private ownership production economy E is a 4-tuple:

$$\Big(E, \{(\omega_i, \preceq_i), 1 \leq i \leq m\}, \{Y_j,\ 1 \leq j \leq k\},\\ \{\theta_{ij},\ 1 \leq i \leq m, 1 \leq j \leq k\}\Big)$$

Definition

A non-empty subset Y of a some finite-dimensional $\mathbb{R}^n$ is said to be a production set whenever

1. Y is closed;
2. Y is convex;
3. $\mathbb{R}^n \cap Y = \{0\}$;
4. Y is bounded from above.

1. A finite dimensional vector lattice $E = \mathbb{R}^l$ is the commodity space, i.e., there are r commodities in the economy.
2. There are m consumers indexed by i such that each consumer has an initial endowment $\omega_i > 0$ and a neoclassical preference $\preceq_i$. The total endowment is assumed to be strictly positive, i.e. $\omega = \sum_{i=1}^{m} \omega_i \gg 0$.
3. There are k producers indexed by j such that the j-th producer is characterized by a strictly convex production set Y_j.
4. The economy is a private ownership economy. That is, the firms are owned by the consumers. The real number θ_{ij} represents consumer i' s share of producer j' s profit.

In mathematical economics, the Arrow-Debreu model suggests that under certain economic assumptions (convex preferences, perfect competition, and demand independence) there must be a set of prices such that aggregate supplies will equal aggregate demands for every commodity in the economy.
The model is central to the theory of general (economic) equilibrium and it is often used as a general reference for other microeconomic models. It is proposed by Kenneth Arrow, Gerard Debreu in 1954, and Lionel W. McKenzie independently in 1954, with later improvements in 1959.
Arrow (1972) and Debreu (1983) were separately awarded the Nobel Prize in Economics for their development of the model.

The following statement is called the Arrow-Debreu theorem.

Theorem

Every neoclassical private ownership production economy has an equilibrium price.

Arrow (1972) and Debreu (1983) were separately awarded the Nobel Prize in Economics for their development of the model.

We write $x = \bigsqcup_{i=1}^{n} x_i$ if $x = \sum_{i=1}^{n} x_i$ and $x_i \perp x_j$ for all $i \neq j$. In particular, if $n = 2$ we use the notation $x = x_1 \sqcup x_2$. We say that y is a *fragment* (a *component*) of $x \in E$, and use the notation $y \sqsubseteq x$, if $y \perp (x - y)$. The set of all fragments of an element $x \in E$ is denoted by $\mathcal{F}_x$. We say that $x_1, x_2 \in \mathcal{F}_x$ are *mutually complemented*, if $x = x_1 \sqcup x_2$.

Definition

Let E be a vector lattice and X a real vector space. A map $T \colon E \to X$ is called an *orthogonally additive operator* (OAO in short) provided $T(x + y) = T(x) + T(y)$ for any disjoint elements $x, y \in E$.

Definition

Let (A, Σ, μ) and (B, Ξ, ν) be finite measure spaces. By $(A \times B, \mu \otimes \nu)$ we denote the completion of their product measure space. The union $\Gamma \cup \Theta$ of two disjoint measurable sets $\Gamma, \Theta \in A$ we denote by $\Gamma \sqcup \Theta$. A map $K\colon A \times B \times \mathbb{R} \to \mathbb{R}$ is said to be a *Carathéodory function* if it satisfies the following conditions:

(C_1) $K(\cdot, \cdot, r)$ is $\mu \otimes \nu$-measurable for all $r \in \mathbb{R}$;

(C_2) $K(s, t, \cdot)$ is continuous on $\mathbb{R}$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

We say that a Carathéodory function K is *normalized* if $K(s, t, 0) = 0$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

Let E be an order ideal of $L_0(\nu)$, let $K\colon A \times B \times \mathbb{R} \to \mathbb{R}$ be a normalized Carathéodory function and for every $f \in E$ the function $K(s, \cdot, f(\cdot)) \in L_1(\nu)$ for almost all $s \in A$. Suppose that the function $s \mapsto \int_B K(s, t, f(t))\, d\nu(t)$ belongs to F. Then is defined orthogonally additive operator $T\colon E \to F$ by setting

$$Tf(s) = \int_B K(s, t, f(t))\, d\nu(t).$$

- M. A. Krasnosel'skii, P. P. Zabrejko, E. I. Pustil'nikov, P. E. Sobolevski, *Integral operators in spaces of summable functions*, Noordhoff, Leiden (1976)..

Let (A, Σ, μ) be a finite measure space. We say that $N : A \times \mathbb{R} \to \mathbb{R}$ is a *superpositionally measurable function*, or *sup-measurable* for brevity, if $N(\cdot, f(\cdot))$ is μ-measurable for every $f \in L_0(\mu)$. A sup-measurable function N is called *normalized* if $N(s, 0) = 0$ for μ-almost all $s \in A$.
With every normalized sup-measurable function N is associated an orthogonally additive operator $\mathcal{N} \colon L_0(\mu) \to L_0(\mu)$ defined by

$$\mathcal{N}(f)(s) = N(s, f(s)), \quad f \in L_0(\mu).$$

It is not hard to verify that $\mathcal{N}$ is a disjointness preserving operator. Indeed, for almost all $s \in A$ we have that

$$\mathcal{N}(f)(s) = N(s, f(s)) = N(s, f1_{\operatorname{supp} f}(s)) = N(s, f(s))1_{\operatorname{supp} f}(s).$$

Hence $\operatorname{supp} \mathcal{N}(f) \subset \operatorname{supp} f$ and $\mathcal{N}(f) \in \{f\}^{dd}$.
We note that the operator $\mathcal{N}$ is known in literature as the nonlinear superposition operator or Nemytskii operator.

- J. Appell, P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge, 1990.
- T. Runst, W. Sickel,*Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Series of Nonlinear Analysis and Applications, 1996.
- J. Appell, J. Banas, N. Merentes *Bounded variation and around*, De Gruyter, 2014.

A map $f\colon E \to \mathbb{R}$ is called valuation if

$$F(f \vee g) + F(\wedge g) = F(f) + F(g), \quad f, g \in E$$

- M. Ludwig, M. Reitzner, *A classification of $SL(n)$ invariant valuation*, Annals Math., **172** (2010), 2, 1219–1267.
- S. Alesker, *Continuous rotation invariant valuations on convex sets*, Annals Math., **149** (1999), 3, 977–1005.
- P. Tradacete, and I. Villanueva, *Valuations on Banach lattices*,Int. Math. Res. Not., **2020** (2020), 1, 287–319.

- The Riesz-Kantorovich type order calculus for regular OAOs were obtained. Note that for linear regular operators, this result is classical and fundamental. It was established by the founders of the theory of vector lattices F. Riesz and L.Kantorovich.

Definition

Let E, F be vector lattices. An OAO $T \colon E \to F$ is said to be:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$;
- *regular* if $T = S_1 - S_2$, where S_1, S_2 are positive OAOs from E to F;
- *order bounded*, or an *abstract Uryson operator*, if it maps order bounded sets in E to order bounded sets in F;
- *disjointness preserving*, if $Tx \perp Ty$ for every disjoint $x, y \in E$;
- *non-expanding*, if $E = F$ and $Tx \in \{x\}^{\perp\perp}$ for every $x \in E$;
- *C-bounded* or a *Popov operator*, if the set $T(\mathfrak{F}_x)$ is order bounded in F for every $x \in E$.

The sets of positive, regular, order bounded and C-bounded orthogonally additive operators from E to F we denote by $\mathcal{OA}_+(E,F)$, $\mathcal{OA}_r(E,F)$, $\mathcal{OA}_b(E,F)$ and $\mathcal{P}(E,F)$ respectively. There is a natural partial order on $\mathcal{OA}_r(E,F)$, namely $S \leq T \Leftrightarrow (T-S) \in \mathcal{OA}_+(E,F)$. The next assertion is the Riesz-Kantorovich type theorem for orthogonally additive operators.

Theorem

Let E, F be vector lattices with F being Dedekind complete. Then $\mathcal{OA}_r(E,F) = \mathcal{P}(E,F)$, and $\mathcal{OA}_r(E,F)$ is a Dedekind complete vector lattice. Moreover, for every S, $T \in \mathcal{OA}_r(E,F)$ and every $x \in E$ the following hold:

1. $(T \vee S)x = \sup\{Ty + Sz : \, x = y \sqcup z\}$;
2. $(T \wedge S)x = \inf\{Ty + Sz : \, x = y \sqcup z\}$;
3. $T^+x = \sup\{Ty : \, y \sqsubseteq x\}$;
4. $T^-x = -\inf\{Ty : \, y \sqsubseteq x\}$;
5. $|Tx| \leq |T|x$.

- M. Pliev, K. Ramdane, *Order unbounded orthogonally additive operators in vector lattices*, Mediter. J. Math., **15** (2018), 2, article number 55.

1. Suppose Q is a compact topological space. The Banach space of all continuous functions on Q is denoted by $C(Q)$.

2. Suppose A is an open bounded subset of $\mathbb{R}^n$, Σ is σ-algebra of Lebesgue measurable sets and $\mu\colon \Sigma \to \mathbb{R}_+$ is the Lebesgue measure. Recall that $L_p(A, \Sigma, \mu)$ $(1 \leq p < \infty)$ are classical Lebesgue spaces of measurable functions, $f \leq g$, if $f(t) \leq g(t)$ μ-a.e.

Definition

A subset D of $L_p(\mu)$ is said to be *order bounded* if there exists $f \in L_p(\mu)_+$ such that $|g| \leq f$ for every $g \in D$.

Every such a space is additionally a vector lattice equipped with the natural partial order inherited from $L_0(A, \Sigma, \mu)$, the space of all (classes of) μ-measurable functions on A. This simple observation led to various modifications of the notion of compact operators.
In particular, a linear operator $T\colon E \to Y$ from a Banach lattice E to a Banach lattice Y is said to be *AM-compact* if T maps every order bounded subset of E to a relatively compact subset of Y. These operators were introduced by Dodds and Fremlin in their groundbreaking paper

- P. G. Dodds, D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math., **34** (1979), 287–320.

Later *AM*-compact operators were intensively studied by many authors.

- Y. A. Abramovich, C. D. Aliprantis, *An Invitation to Operator Theory*, AMS, 2002.
- C. D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Dordrecht, (2006).

The domination problem can be stated as follows. Let E, F be vector lattices and $S, T : E \to F$ be orthogonally additive operators with $0 \leq S \leq T$. Let $\mathcal{P}$ be some property of orthogonally additive operators $R : E \to F$, so that $\mathcal{P}(R)$ means that R possesses $\mathcal{P}$. Does $\mathcal{P}(T)$ imply $\mathcal{P}(S)$? The general methods of domination for OAOs were obtained. These methods made it possible to obtain results on the domination of AM-compact and narrow orthogonally additive operators.

Let E, F be vector lattices with F Dedekind complete and $T \in \mathcal{U}_+(E, F)$. By definition

$$\mathfrak{F}_T = \{S \in \mathcal{U}_+(E, F) : \, S \wedge (T - S) = 0\}.$$

For a subset $\mathcal{A}$ of a vector lattice W we employ the following notation:

$$\mathcal{A}^{\uparrow} = \{x \in W : \exists \text{ a net } (x_\alpha) \subset \mathcal{A} \text{ with } x_\alpha \uparrow x\}.$$

The meanings of $\mathcal{A}^{\downarrow}$ are analogous. As usual, we also write

$$\mathcal{A}^{\downarrow\uparrow} = (\mathcal{A}^{\downarrow})^{\uparrow}; \mathcal{A}^{\uparrow\downarrow\uparrow} = ((\mathcal{A}^{\uparrow})^{\downarrow})^{\uparrow}.$$

It is clear that $\mathcal{A}^{\downarrow\downarrow} = \mathcal{A}^{\downarrow}$, $\mathcal{A}^{\uparrow\uparrow} = \mathcal{A}^{\uparrow}$. Consider a positive abstract Uryson operator $T : E \to F$, where F is Dedekind complete. Under the partial ordering induced by $\mathcal{U}(E, F)$ the set $\mathcal{F}_T$ is a Boolean algebra with smallest element 0 and largest element T and $\neg S = T - S$ for every $S \in \mathcal{F}_T$.

Since $\mathfrak{F}_T$ is a Boolean algebra, it is closed under finite suprema and infima. In particular, all "ups and downs" of $\mathfrak{F}_T$ are likewise closed under finite suprema and infima, and therefore they are also directed upward and, respectively, downward.

Definition

A subset D of a vector lattice E is called a *lateral ideal if the following conditions hold*

1. *if $x \in D$ then $y \in D$ for every $y \in \mathcal{F}_x$;*
2. *if $x, y \in D$, $x \perp y$ then $x + y \in D$.*

A subset D of the vector lattice E is called laterally closed, if for every laterally convergence net $(x_\alpha)_{\alpha \in \Lambda} \subset D$ such that $x_\alpha \xrightarrow{\text{lat}} x$ we have $x \in D$. Laterally closed lateral ideal D is called laterally band in E.

Example

Let E be a vector lattice. Every order ideal in E is a lateral ideal.

Example

Let E, F be a vector lattices and $T \in \mathcal{U}_+(E, F)$. Then $\mathcal{N}_T := \{e \in E : \ T(e) = 0\}$ is a lateral ideal.

Example

Let E be a vector lattice and $x \in E$. Then $\mathcal{F}_x$ is a lateral ideal.

Example

Let E be a vector lattice. An every band D in E is a lateral band.

Let $T \in \mathcal{U}_+(E,F)$ and $D \subset E$ be a lateral ideal. Then is defined a map $\pi^D T : E \to F$ by the following formula

$$\pi^D T(y) = \sup\{Tz : z \in \mathcal{F}_y \cap D\}, \ (y \in E). \tag{1}$$

Let E, F be vector lattices with F Dedekind complete, ρ be an order projection in F, $T \in \mathcal{U}_+(E,F)$ and D be a lateral ideal. Then $\pi^D T$ is a positive abstract Uryson operator and $\rho\pi^D T \in \mathcal{F}_T$.
If $D = \mathcal{F}_x$ then the operator $\pi^D T$ is denoted by $\pi^x T$. Let F be a vector lattice. Any fragment of the form $\sum\limits_{i=1}^{n} \rho_i \pi^{x_i} T$, $n \in \mathbb{N}$, where $\rho_1, \dots, \rho_n$ is a finite family of mutually disjoint order projections in F, like in the linear case is called an *elementary* fragment T.

Definition

The set of all elementary fragments of T we denote by $\mathcal{A}_T$.

Theorem

Let E, F be vector lattices, F Dedekind complete, $T \in \mathcal{U}_+(E, F)$. Then $\mathfrak{F}_T = \mathcal{A}_T^{\uparrow\downarrow\uparrow}$.

- M. Pliev, *Domination problem for narrow orthogonally additive operators*, Positivity, **21** (2017), 1, 23–33.

Theorem

(Freudenthal Spectral Theorem) Let E be a vector lattice with the principal projection property and let $x \in E_+$. Then for every $y \in E_x$ there exists a sequence (u_n) of x-step functions satisfying $0 \leq y - u_n \leq \frac{1}{n}x$ for each n and $u_n \uparrow y$.

Definition

Let E be a vector lattice and F a Banach space. An orthogonally additive operator $T : E \to F$ is called: *AM-compact* if for every order bounded set $L \subset E$ its image $T(L)$ is a relatively compact set in F.

Let (A, Σ, μ) be a σ-finite and complete measure space and E be an order ideal in $L_0(A, \Sigma, \mu)$. If E carries a lattice norm $\|\cdot\|_E$ and E is complete with respect of $\|\cdot\|_E$ then E is called a *Banach function space*.

Example

Let E, F be a Banach function spaces, F having order continuous norm. Then every integral Uryson operator $T \in \mathcal{U}(E, F)$ is AM-compact.

The next theorem is the nonlinear version of the Dodds-Fremlin theorem.

Theorem

Let E be Dedekind complete vector lattice, F be a Banach lattice with an order continuous norm, and $T \in \mathcal{U}_+(E,F)$ be an AM-compact operator. Then every operator $S \in \mathcal{U}_+(E,F)$, such that $0 \leq S \leq T$ is AM-compact.

- V. Orlov, M. Pliev, D. Rode, *Domination problem for AM-compact abstract Uryson operators*, Arch. Math., **107** (2016), 5, 543–552.

Were obtained a number of strong results for disjointness preserving OAOs. In particular, was obtained a nonlinear version of the Kutateladze theorem for nonlinear lattice homomorphisms and an operator version of the Radon-Nikodym theorem for disjointness preserving OAOs.

- M. Pliev, F. Polat. *The Radon-Nikodym theorem for disjointness preserving orthogonally additive operators*, Operator Theory and Differential Equations. Trends in Mathematics, Springer International Publishing, (2021), 155-161.

- N. Abasov, M. Pliev, *On extensions of some nonlinear maps in vector lattices*, J. Math. Anal. Appl., **455** (2017), 1, 516–527.

Definition

Let E be a vector lattice and X a vector space. An orthogonally additive map $T : E \to X$ is called even if $T(x) = T(-x)$ for every $x \in E$. If E, F are vector lattices, the set of all even abstract Urysohn operators from E to F we denote by $\mathcal{U}^{ev}(E, F)$.

If E, F are vector lattices with F Dedekind complete, then the space $\mathcal{U}^{ev}(E, F)$ a Dedekind complete sublattice of $\mathcal{U}(E, F)$
If E, F are vector lattices with F Dedekind complete, then the space $\mathcal{U}^{ev}(E, F)$ a Dedekind complete sublattice of $\mathcal{U}(E, F)$.

Definition

Let E and F be vector lattices. Operator $T \in \mathcal{U}_+^{ev}(E, F)$ is called an *Urysohn lattice homomorphism*, if the following conditions hold

1) $T(x \vee y) = Tx \vee Ty$ for every $x, y \in E_+$;
2) $T(x \wedge y) = Tx \wedge Ty$ for every $x, y \in E_+$.

It is clear that an Urysohn lattice homomorphism is an increasing operator in E_+.

Theorem

Let E, F be vector lattices with F Dedekind complete. Let an operator $T \in \mathcal{U}_+^{ev}(E, F)$ be a strictly increasing in E_+ and a laterally full in E_+. Then the following statements are equivalent:

1. *T is an Urysohn lattice homomorphism;*
2. *For every $0 \le S \le T$, $S \in \mathcal{U}^{ev}(E, F)$, S is an increasing in E_+, there exists an increasing in $T(E_+)$ Id-operator $N : F \to F$ satisfying $S = N \circ T$.*

Every such a space is additionally a vector lattice equipped with the natural partial order inherited from $L_0(A, \Sigma, \mu)$, the space of all (classes of) μ-measurable functions on A. This simple observation led to various modifications of the notion of compact operators.
In particular, a linear operator $T\colon E \to Y$ from a Banach lattice E to a Banach lattice Y is said to be *AM-compact* if T maps every order bounded subset of E to a relatively compact subset of Y. These operators were introduced by Dodds and Fremlin in their groundbreaking paper

- P. G. Dodds, D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math., **34** (1979), 287–320.

Definition

Recall that the *support* of a μ-measurable function $f\colon A \to \mathbb{R}$ (denotation $\operatorname{supp} f$) is the μ-measurable set

$$\operatorname{supp} f := \{t \in A : \ f(t) \neq 0\}.$$

Two elements f and g of $L_p(\mu)$ is said to be *disjoint* if

$$\mu\{t \in \operatorname{supp} f \cap \operatorname{supp} g\} = 0$$

Nigel Kalton
1946-2010

Haskell Rosenthal
1940-2021

Kalton obtained the analytical representation of continuous operators on $L_1([0,1],\Sigma,\mu)$. This representation is given in the terms of weak*-measurable functions from $[0,1]$ to the set of all regular Borel measures on $[0,1]$.

- N. J. Kalton, The endomorphisms of L_p $(0 \leq p \leq 1)$, Indiana Univ. Math. J., **27** (1978), 3, 353–381.

Theorem

For every operator $T \in \mathcal{L}(L_1)$ there is a weak* measurable function $\mu_s\colon [0,1] \to \mathcal{M}[0,1]$ taking values in the space of all regular Borel measures on $[0,1]$ such that for every $f \in L_1([0,1],\Sigma,\mu)$ the following equality

$$Tf(s) = \int_{[0,1]} f(t)d\,\mu_s(t) \tag{2}$$

holds for almost all $s \in [0,1]$. Conversely, every weak*-measurable function $\mu\colon [0,1] \to \mathcal{M}[0,1]$ defines an operator $T \in \mathcal{L}(L_1)$ as above.

Definition

We say that a linear bounded operator $T\colon L_1(\mu) \to L_1(\mu)$

1. *preserves disjointness* if for all $f, g \in L_1$ the relation
$$\mu\{t \in \operatorname{supp} f \cap \operatorname{supp} g\} = 0$$
implies that
$$\mu\{t \in \operatorname{supp} Tf \cap \operatorname{supp} Tg\} = 0.$$
2. *narrow* if for every $A \in \Sigma_+$ the restriction $T|_{L_1(A)}$ is not an isomorphic embedding to L_1;
3. *pseudo embedding* if for each $\varepsilon > 0$ there exists $A \in \Sigma_+$ such that the restriction $T|_{L_1(A)}$ is an into isomorphism with
    1. $\|T|_{L_1(A)}\| \geq \|T\| - \varepsilon$;
    2. there exists disjointness preserving operator $U\colon L_1(A) \to L_1$ such that $\|T|_{L_1(A)} - U\| < \varepsilon$.

- H. P. Rosenthal, *Embeddings of L^1 in L^1*, Contemp. Math., **26** (1984), 335–349.

Theorem

Every operator $T \in \mathcal{L}(L_1(\mu))$ has a unique representation

$$T = T_{\mathcal{N}} + T_{\mathcal{H}},$$

where $T_{\mathcal{N}}$ is a narrow operator, $T_{\mathcal{H}}$ is a pseudo-embedding and $T_{\mathcal{N}}, T_{\mathcal{H}} \in \mathcal{L}(L_1)$.

Theorem

Let $T \in \mathcal{L}(L_1)$. Then the following statements are equivalent:

1. *T is a narrow operator;*
2. *if for every $f \in L_1(\mu)$ and $\varepsilon > 0$ there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $\|T(f_1 - f_2)\|_Y < \varepsilon$*

Definition

Let Y be a normed space. A bounded linear operator $T\colon L_p(\mu) \to Y$ is said to be

1. *narrow* if for every $f \in L_p(\mu)$ and $\varepsilon > 0$ there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $\|T(f_1 - f_2)\|_Y < \varepsilon$;
2. *strictly narrow* if for every $f \in L_p(\mu)$ there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $Tf_1 = Tf_2$.

Narrow operators were explicitly articulated by Plichko and Popov in

- Plichko A., M.Popov, *Symmetric function spaces on atomless probability spaces,* Dissertationes Math. (Rozprawy Mat.), **306** (1990), 1–85.

For a detailed historical account, we refer to

- M. Popov, B. Randrianantoanina, *Narrow operators on function spaces and vector lattices*, 45, De Gruyter Studies in Mathematics, 2013 Walter de Gruyter & Co., Berlin/Boston.

On the other hand, in

- O. Maslyuchenko, V. Mykhaylyuk, M. Popov, *A lattice approach to narrow operators*, Positivity, **13** (2009), 459–495.

Maslyuchenko, Mykhaylyuk and Popov demonstrated that the notion of narrow operators admits a natural extension in the setting of vector lattices and that this general approach has a serious advantage.

Namely, they proved the following remarkable theorem which shows that narrow operators can be regarded as a generalization of *AM*-compact operators.

Theorem

Let Y be a Banach space. Then every AM-compact order-to-norm continuous linear operator $T\colon L_p(\mu) \to Y$ is narrow.

Let (A, Σ, μ) be a finite measure space and X be a Banach space. We say that a function $f: A \to X$ is *strongly μ-measurable* if there is a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} \|f(\cdot) - f_n(\cdot)\|_X = 0$ μ-almost everywhere. By $L_0(\mu, X)$ we denote the space of all (equivalence classes of) strongly μ-measurable X-valued functions defined on A. A Lebesgue-Bochner space $L_p(\mu, X)$ on (A, Σ, μ) as defined as

$$L_p(\mu, X) := \{f \in L_0(\mu, X) : \ \|f(\cdot)\|_X \in L_p\}.$$

We observe that the space $L_p(\mu, X)$ is equipped with a "mixed"norm

$$\|f\|_{L_p(\mu,X)} := \|\|f(\cdot)\|_X\|_E, \; f \in L_p(\mu, X).$$

it is turned out to be a Banach space.
For the detailed exposition of the theory of more general Köthe-Bochner spaces we to monograph

- P. K. Lin, *Köthe-Bochner function spaces*, Birkhäuser, Boston, (2004).

Definition

Recall that the *support* of a strongly μ-measurable X-valued function $f : A \to X$ (denotation $\operatorname{supp} f$) is the μ-measurable set

$$\operatorname{supp} f := \{t \in A : \ f(t) \neq 0\}.$$

Two elements f and g of $L_p(\mu, X)$ is said to be *disjoint* if

$$\mu\{t \in \operatorname{supp} f \cap \operatorname{supp} g\} = 0$$

Definition

Let X be a Banach space and Y be a vector space. An operator $T\colon L_p(\mu, X) \to Y$ is said to be *orthogonally additive* if

$$T(x+y) = Tx + Ty \text{ for all disjoint } x, y \in L_p(\mu, X).$$

Clearly $T(0) = 0$.

Definition

Let X be a Banach space and Y be a vector space. An operator $T\colon L_p(\mu, X) \to Y$ is said to be *orthogonally additive* if

$$T(x+y) = Tx + Ty \text{ for all disjoint } x, y \in L_p(\mu, X).$$

Clearly $T(0) = 0$.

Definition

Let X be a Banach space and Y be a normed space. An orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ is said to be

1. *narrow* if for every $f \in L_p(\mu, X)$ and $\varepsilon > 0$ there exist there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $\|Tf_1 - Tf_2)\|_Y < \varepsilon$;
2. *strictly narrow* if for every $f \in L_p(\mu, X)$ there exist there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $Tf_1 = Tf_2$.

Definition

Let $f \in L_p(\mu, X)$. By $\mathfrak{F}_f$ we denote the following set:

$$\mathfrak{F}_f := \{g \in L_p(\mu, X) : (f - g) \perp g\}$$

Elements of $\mathfrak{F}_f$ are said to be fragments of f (notations $g \sqsubseteq f$).

Definition

We say that a net $(f_\alpha)_{\alpha\in A}$ in $L_p(\mu, X)$ *order* converges to $f \in L_p(\mu, X)$ (notation $f_\alpha \xrightarrow{o} f$) if the net $(\|f - f_\alpha\|_X)_{\alpha\in A}$ order converges to 0.

Definition

We say that a net $(f_\alpha)_{\alpha\in A}$ in $L_p(\mu, X)$ *laterally* converges to $f \in L_p(\mu, X)$ (notation $f_\alpha \xrightarrow{l} f$) if the net $(f_\alpha)_{\alpha\in A}$ order converges to f and $f_\alpha \sqsubseteq f_\beta \sqsubseteq f$ for all $\alpha, \beta \in A$ with $\alpha \leq \beta$.

Definition

Let Y be a normed space. A mapping $T\colon L_p(\mu, X) \to Y$ is said to be:

1. *laterally-to-norm continuous* if for a net $(f_\alpha)_{\alpha\in\Lambda}$ in $L_p(\mu, X)$ laterally convergent to $f \in L_p(\mu, X)$ a net $(Tf_\alpha)_{\alpha\in\Lambda}$ norm converges to Tf;
2. *laterally-to-norm σ-continuous* if for every sequence $(f_n)_{n\in\mathbb{N}}$ in $L_p(\mu, X)$ laterally convergent to $f \in L_p(\mu, X)$ the sequence $(Tf_n)_{n\in\mathbb{N}}$ norm converges to Tf;
3. *order-to-norm continuous* if for every net $(f_\alpha)_{\alpha\in\Lambda}$ in $L_p(\mu, X)$ order convergent to $f \in L_p(\mu, X)$ the net $(Tf_\alpha)_{\alpha\in\Lambda}$ norm converges to Tf;
4. *AM-compact* if T maps order bounded subsets in $L_p(\mu, X)$ to a relatively compact subsets of Y;
5. *C-compact* if $T(\mathfrak{F}_f)$ is a relatively compact subset of Y for all $x \in L_p(\mu, X)$;
6. *disjointness preserving* if Y is additionally a lattice-normed space and $Tx \perp Ty$ for all disjoint $x, y \in L_p(\mu, X)$.

Definition

We observe that laterally-to-norm continuity of an orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ implies its σ-laterally-to-norm continuity. Below we consider some examples of laterally-to-norm continuous and C-compact OAOs.

Example

Since $\mathfrak{F}_f$ is an order bounded subset of $L_p(\mu, X)$ for all $f \in L_p(\mu, X)$ we have that every AM-compact OAO is C-compact.

Definition

Example

Let (A, Σ, μ), (B, Ξ, ν) be finite measure spaces and $p, q \in [1, \infty)$. Then every order bounded Uryson integral operator $T\colon L_p(\mu) \to L_q(\nu)$ is AM-compact.

Let (A, Σ, μ) a finite measure space, and X, Y be Banach spaces. Then every bounded linear operator $T \colon L_p(\mu, X) \to Y$ is laterally-to-norm σ-continuous.

The next example shows that the class of C-compact OAOs strictly includes the class of AM-compact orthogonally additive operators even in the setting of the one-dimensional vector lattice $\mathbb{R}$.

Example

There is a C-compact OAO $T\colon \mathbb{R} \to \mathbb{R}$ which is not AM-compact. Indeed, Consider a mapping $T\colon \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Theorem

Let X, Y be Banach spaces. Then every laterally-to-norm continuous C-compact orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ is narrow.

First, we need to introduce nonlinear superposition operators on $E(X)$. Suppose that (A, Σ, μ) is a finite measure space and X is a Banach space. Recall that the *support* of a strongly μ-measurable X-valued function $f\colon A \to X$ (denotation $\operatorname{supp} f$) is the μ-measurable set

$$\operatorname{supp} f := \{t \in A:\ f(t) \neq 0\}.$$

Definition

Let (A, Σ, μ) be a finite measure space and X be a Banach space. A function $N : A \times X \to X$ is said to be:

1. *superpositionally measurable* (or *super-measurable* for brevity) if $N(t, f(\cdot)) \in L_0(\mu, X)$ for every $f \in L_0(\mu, X)$
2. $L_p(\mu, X)$*-super-measurable* if $N(t, f(\cdot)) \in L_p(\mu, X)$ for every $f \in L_p(\mu, X)$;
3. normalised if $N(\cdot, 0) = 0$ for μ-almost all $t \in A$.

Let (A, Σ, μ) be a finite measure space X be a Banach space and $N : A \times X \to X$ be a normalised $L_p(\mu, X)$-super-measurable function. Then there is a map $T_N\colon L_p(\mu, X) \to L_p(\mu, X)$ defined by the setting

$$T_N(f)(\cdot) = N(\cdot, f(\cdot)), \;\; f \in L_p(\mu, X).$$

We note that T_N is one of the most important operators of a nonlinear analysis. It is known in literature as a nonlinear superposition operator or Nemyskij operator.

- J. Appell, P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge, 1990.
- T. Runst, W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Series of Nonlinear Analysis and Applications, 1996.
- J. Appell, J. Banas, N. Merentes *Bounded variation and around*, De Gruyter, 2014.

Theorem

Let (A, Σ, μ) be a finite atomless measure space, E be a Köthe-Banach space on (A, Σ, μ) with an order continuous norm, X be a Banach space and $N : A \times X \to X$ be a normalised $L_p(\mu, X)$-super-measurable function. Then for the a nonlinear superposition operator $T_N : L_p(\mu, X) \to L_p(\mu, X)$ the following assertions are equivalent:

1. *T_N is a C-compact operator;*
2. *$N(\cdot, f(\cdot)) = 0$ for all $f \in L_p(\mu, X)$.*

Definition

Let X, Y be Banach spaces. An orthogonally additive operator $T\colon L_p(\mu, X) \to L_q(\nu, Y)$ is said to be *dominated* if there exists $S \in \mathcal{OA}_+(L_p(\mu), L_q(\nu))$ such that

$$\|Tf(\cdot)\| \leq S\|f(\cdot)\| \text{for all } f \in L_p(\mu, X).$$

The operator S is called a *dominant* of T. The set of all dominants of a dominated orthogonally additive operator T is denoted by $\mathfrak{D}(T)$. We note that $\mathfrak{D}(T)$ is a partially ordered set with respect to the partial order induced by $\mathcal{OA}_r(L_p(\mu), L_q(\nu))$. If $\mathfrak{D}(T)$ has a minimal element, then this element is called the *exact dominant* of T and is denoted by $|||T|||$. The vector space of all dominated OAOs between $L_p(\mu, X)$ and $L_q(\nu, Y)$ we denote by $\mathcal{OA}_D(L_p(\mu, X), L_q(\nu, Y))$.

Suppose that X and Y are coincide with the field $\mathbb{R}$. Then the space $\mathcal{OA}_D(L_p(\mu), L_q(\nu))$ coincides with the space $\mathcal{OA}_r(L_p(\mu), L_q(\nu))$ of all regular orthogonally additive operators from $L_p(\mu)$ to $L_q(\nu)$. Moreover the exact dominant of a dominated OAO T exists and $|||T||| = |T|$, where $|T|$ is the modulus of T.

Definition

Let (A, Σ, μ) be a finite measure space and X, Y be Banach spaces. A function $K\colon A \times X \to Y$ is said to be a *weakly μ-super measurable* if $\langle z, K(t, f(t))\rangle \in L_0(\mu)$ for all $z \in Y^*$ and $f \in L_0(\mu, X)$. We recall that K is a normalised function if $K(t, 0) = 0$ for μ-almost all $t \in A$. We say that $K\colon A \times X \to Y$ belongs to the class $\mathfrak{A}$ if it satisfies the following conditions:

1. K is a weakly μ-super measurable function;
2. K is a normalised function;
3. the following inequality

$$\langle z, K(\cdot, f(\cdot))\rangle \leq w(\cdot)\varphi(\cdot), \ \ \varphi \in L_0(\mu)$$

holds for every $z \in B_{Y^*}$, $f \in L_0(\mu, X)$ such that $\|f(\cdot)\|_X = \varphi$ and some $w \in L_0(\mu)$.

With every $K \in \mathfrak{A}$ is associated a μ-measurable function $|K|\colon A \times \mathbb{R} \to \mathbb{R}$ defined by the setting

$$|\mathcal{K}|(t,r) := \sup\{\langle z, K(t,x)\rangle : \, x \in S_X^r, \; z \in B_{Y^*}\}$$

where the supremum is taken in $L_0(\mu)$ (the existence of the supremum is guaranteed by the assumption (3) above). We say that $T\colon L_p(\mu, X) \to Y$ is a *weakly integral* operator if there exists $K \in \mathfrak{A}$ such that the equality

$$\langle z, Tf\rangle = \int_A \langle z, K(t, f(t))\rangle d\mu(t), \quad f \in L_p(\mu, X).$$

holds for every $z \in Y^*$. The function K is called the *kernel* of an integral operator T.

Lemma

Let (A, Σ, μ) *be a finite measure space,* X, Y *be Banach spaces and* $T\colon L_p(\mu, X) \to Y$ *be a weakly integral operator with the kernel* K*. Then* T *is an orthogonally additive operator.*

Definition

A weakly integral operator $T\colon L_p(\mu, X) \to Y$ with the kernel K is called *regular* if there exists an integral functional $S\colon L_p(\mu) \to \mathbb{R}$ with the kernel $|K|$. We note that S is a positive orthogonally additive integral functional on E.

The following theorem is first main result of this section. It states necessary and sufficient conditions for a weakly integral operator from $L_p(\mu, X)$ to Y to be dominated.

Theorem

Let (A, Σ, μ) be a finite measure space, X, Y be Banach spaces and $T\colon L_p(\mu, X) \to Y$ be a weakly integral operator with the kernel K. Then the following statements are equivalent:

1. *T is a regular operator;*
2. *T is a dominated operator.*

Moreover the exact dominant $|T|$ of T is the integral functional $S\colon L_p(\mu) \to \mathbb{R}$ with the kernel $|\mathcal{K}|$.

Let Y be a vector lattice of real-valued functions $f\colon \mathbb{N} \to \mathbb{R}$ equipped with the pointwise order and a lattice norm $\|\cdot\|_Y$. We say that Y is a *Banach sequence space with an order compatible basis* if the the following conditions hold:

1. Y is a Banach lattice with respect to the norm $\|\cdot\|_Y$;
2. unit vectors $(e_i)_{i=1}^{\infty}$, where $e_i = (\delta_{ij})_{j=1}^{\infty}$, form a Schauder basis in Y;
3. projections $\pi_n\colon Y \to Y_n$, where Y_n is the linear span of $e_1, \dots, e_n$, are positive linear operators with $0 \leq \pi_n \leq Id$ for all $n \in \mathbb{N}$.

Theorem

Let X be a Banach space and Y be a Banach sequence space with an order compatible basis. Then every laterally-to-norm continuous dominated orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ is narrow.

By $E(M,\tau)_{sa}$ we denote a real vector space of self-adjoint elements of $E(M,\tau)$. We note that $E(M,\tau)_{sa}$ is an ordered Banach space with respect to the partial order defined by: $x \leq y \Leftrightarrow (y-x) \in E(M,\tau)_{+}$. For every $h \in E(\mathcal{M},\tau)$ there exists the modulus $|h| := (h^* h)^{\frac{1}{2}}$.

Suppose that $a \in (0, \infty]$, $I = (0, a)$ and Σ is the σ-algebra of Lebesgue measurable subsets of I. By (I, m) we denote the measure space (I, Σ, m) equipped with Lebesgue measure m. Let $L_0(I, m)$ (or $L_0(I)$ for brevity) be the space of all equivalence classes of almost everywhere finite Lebesgue measurable real-valued functions on I. For $x \in L_0(I)$, we denote by $\mu(x)$ the decreasing rearrangement of the function $|x|$. That is,

$$\mu(t; x) = \inf \{s \geq 0 : \; m(\{|x| > s\}) \leq t\}, \quad t > 0.$$

Definition

We say that $(E(I), \|\cdot\|_E)$ (or $(E, \|\cdot\|_E)$) is a symmetric Banach function on I if the following hold:

1. $E(I)$ is a subspace of $L_0(I)$;
2. $(E, \|\cdot\|_E)$ is a Banach space;
3. If $x \in E$ and if $y \in L_0(I)$ are such that $|y| \leq |x|$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$;
4. If $x \in E$ and if $y \in L_0(I)$ are such that $\mu(y) = \mu(x)$, then $y \in E$ and $\|y\|_E = \|x\|_E$.

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ. Let $\mathbf{1}$ be the identity. A closed and densely defined operator A affiliated with $\mathcal{M}$ is called τ-measurable if $\tau(E_{|x|}(s,\infty)) < \infty$ for sufficiently large s, where $E_{|x|}$ stands for the spectral measure of $|x|$. We denote the set of all τ-measurable operators by $L_0(\mathcal{M},\tau)$. Let $\mathcal{P}(\mathcal{M})$ denote the lattice of all projections in $\mathcal{M}$. We denote by $L_0(M,\tau)_+$ the collection of all non-negative operators in $L_0(M,\tau)$.

For every $x \in L_0(\mathcal{M}, \tau)$, we define its singular value function $\mu(x)$ by setting

$$\mu(t; x) = \inf\{\|x(\mathbf{1} - p)\|_{\mathcal{M}} : p \in P(\mathcal{M}), \quad \tau(p) \leq t\}, \quad t > 0.$$

For more details on generalised singular value functions, we refer the reader to

- S. Lord, F. Sukochev, D. Zanin, *Singular traces: Theory and applications*, De Gruyter, Berlin, 2013.

If $\mathcal{M}$ is the algebra $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ and τ is the standard trace, then $L_0(M, \tau) = \mathcal{M}$, the measure topology coincides with the operator norm topology.

Definition

Let $\mathfrak{A}$ be a linear subspace of $L_0(\mathcal{M}, \tau)$ equipped with a complete norm $\|\cdot\|_{\mathfrak{A}}$. We say that $\mathfrak{A}$ is a symmetric operator space if for $x \in \mathfrak{A}$ and for every $y \in L_0(\mathcal{M}, \tau)$ with $\mu(x) \leq \mu(y)$, we have $x \in \mathfrak{A}$ and $\|x\|_{\mathfrak{A}} \leq \|y\|_{\mathfrak{A}}$.

Recall the following construction of a symmetric Banach operator space (or non-commutative symmetric Banach space) $E(\mathcal{M},\tau)$. Set

$$E(\mathcal{M},\tau)=\Big\{x\in L_0(\mathcal{M},\tau):\ \mu(x)\in E\Big\}$$

(respectively, $E(\mathcal{M},\tau)=\Big\{x\in L_0(\mathcal{M},\tau):\ \{\mu(n;x)\}_{n\geq 0}\in E\Big\}$).

There exists the natural norm on $E(\mathcal{M},\tau)$ defined by

$$\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E\,, \quad x \in E(\mathcal{M},\tau).$$

We note that $E(\mathcal{M},\tau)$ is a Banach space with respect to $\|\cdot\|_{E(\mathcal{M},\tau)}$ and it is called the (non-commutative) symmetric operator space associated with $(\mathcal{M},\tau)$ corresponding to $(E,\|\cdot\|_E)$. If $M = B(H)$, then we denote $E(M,\tau)$ by C_E. If E is the Lebesgue space L_p, $1 \leq p < \infty$, the space C_E is familiar Schatten-von Neumann ideal denoted by C_p for brevity.

Definition

Let $E(M,\tau) \subset L_0(M,\tau)$ be is a symmetric operator space. The norm $\|\cdot\|_{E(M,\tau)}$ is called *order continuous* if $\|a_\beta\|_{E(M,\tau)} \to 0$ whenever $\{a_\beta\}$ is a downwards directed net in $E(M,\tau)_+ := E(M,\tau) \cap L_0(M,\tau)_+$ satisfying $a_\beta \downarrow 0$.

Definition

Let X be a Banach space and $E(M,\tau)$ be a symmetric operator space. An orthogonally additive operator $T\colon L_p(\mu,X)\to E(M,\tau)_{sa}$ is called *dominated* if there exists a positive orthogonally additive operator $S\colon L_p(\mu)\to E(M,\tau)_+$ such that

$$|Tf|\leq S\|f(\cdot)\|_X;\ f\in L_p(\mu,X).$$

An operator S is called a *dominant* of T.

Lemma

Suppose that M is a von Neumann algebra and X is a Banach space. Then every bounded linear operator $T\colon L_p(\mu, X) \to E(M,\tau)_{sa}$ is laterally-to-norm continuous.

Theorem

Let X be a Banach space. Then every laterally-to-norm continuous dominated orthogonally additive operator $T\colon L_p(\mu, X) \to (C_E)_{sa}$ is narrow, whenever C_E is separable.