Noise Stability: Old and New

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Presented at Institute for Advanced Study in Mathematics,
Harbin Institute of Technology
July 13, 2022

Outline

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Definitions

- Let P_{XY} be a joint probability measure on $\mathcal{X} \times \mathcal{Y}$
- Let $P_{XY}^{\otimes n}$ be the n-product of P_{XY} (which is a joint probability measure on $\mathcal{X}^n \times \mathcal{Y}^n$)
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- The noise stability of a pair of measurable sets (A, B) with $A \subseteq \mathcal{X}^n, B \subseteq \mathcal{Y}^n$ is defined as the joint probability $P_{XY}^{\otimes n}(A \times B)$.
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  - Noise stability of a pair of sets is a measure of the resistance of this pair of sets to noise corruption
- The noise stability problem: Estimate $P_{XY}^{\otimes n}(A \times B)$ given $P_X^{\otimes n}(A)$ and $P_Y^{\otimes n}(B)$
  - Geometrically, estimate the “area” given the “length” and “width”

Doubly Symmetric Binary Distribution

Throughout this talk, we mainly consider the doubly symmetric binary distribution (unless otherwise specified)

$$P_{XY} = \begin{array}{ccc} X\backslash Y & 0 & 1 \\ 0 & \frac{1+\rho}{4} & \frac{1-\rho}{4} \\ 1 & \frac{1-\rho}{4} & \frac{1+\rho}{4} \end{array}$$

with correlation $\rho \in (0, 1)$

Maximal/minimal noise stability

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$$\overline{\Gamma}^{(n)}(a,b) := \max_{\substack{A,B\subseteq\{0,1\}^n : P_X^{\otimes n}(A)\le a,\\ P_Y^{\otimes n}(B)\le b}} P_{XY}^{\otimes n}(A\times B)$$

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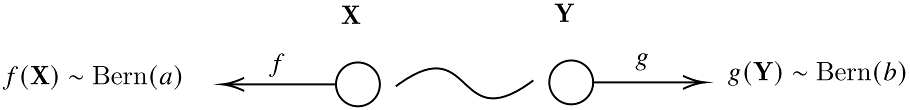
- For dyadic rationals $a = \frac{M}{2^n}, b = \frac{N}{2^n}$ (with integers M, N), the "inequalities" in the constraints can be replaced by "equalities"
  - Moreover, for this case, $\overline{\Gamma}^{(n)}(1 - a, b) = b - \underline{\Gamma}^{(n)}(a, b)$

Interpretation: Noninteractive Correlation Distillation



$\max \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}))$ or equivalently, $\max \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1)$

Background

- The notion of “noise stability” was first introduced explicitly by Benjamini, Kalai, and Schramm in 1999
- However, such a quantity was in fact first studied by Witsenhausen in his classic work in 1975, which played a key role in proving a converse result for Gács–Körner common information problem.
- Noise stability was also used by Kahn, Kalai, and Linial in 1988 to prove the famous KKL theorem
- Now, noise stability is one of central topics in analysis of Boolean functions

Existing Results

- Trivial cases: a or b is 0 or 1
- Known nontrivial cases: $\overline{\Gamma}^{(n)}\left(\frac{1}{2},\frac{1}{2}\right), \overline{\Gamma}^{(n)}\left(\frac{1}{4},\frac{1}{4}\right)$ and $\underline{\Gamma}^{(n)}\left(\frac{1}{2},\frac{1}{2}\right), \underline{\Gamma}^{(n)}\left(\frac{1}{4},\frac{3}{4}\right)$
- $\overline{\Gamma}^{(n)}(a,b)$ and $\underline{\Gamma}^{(n)}(a,b)$ for other (a,b)?—unknown and difficult!

Limiting Cases

- Central Limit (CL) regime: a, b are fixed
  - $\overline{\Gamma}^{(\infty)}(a,b), \underline{\Gamma}^{(\infty)}(a,b)$ denote the limits of $\overline{\Gamma}^{(n)}(a,b), \underline{\Gamma}^{(n)}(a,b)$ as $n \to \infty$.

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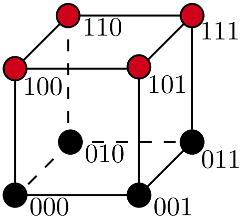
- Large Deviations (LD) regime: For $a = 2^{-n\alpha}, b = 2^{-n\beta}$ (with fixed $\alpha, \beta > 0$), denote

$$\underline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha,\beta) := -\frac{1}{n}\log\overline{\Gamma}^{(n)}\left(e^{-n\alpha}, e^{-n\beta}\right),$$
$$\overline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha,\beta) := -\frac{1}{n}\log\underline{\Gamma}^{(n)}\left(e^{-n\alpha}, e^{-n\beta}\right),$$

and their limits as $\underline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta), \overline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta)$

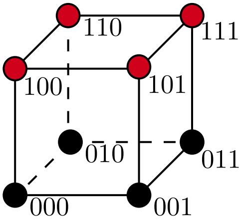
Outline

Hamming Subcubes



- An $(n-k)$-subcube C_{n-k} is a set of $\mathbf{x}$ with k components fixed (e.g., $\{\mathbf{1}_k\} \times \{0,1\}^{n-k}$)

Hamming Subcubes



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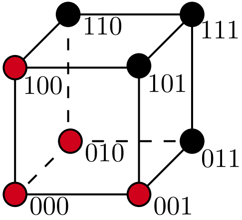
  - Special case C_{n-1}: e.g., $\{1\} \times \{0,1\}^{n-1}$ (Indicator $\mathbf{x} \mapsto x_1$ called a dictator function)
- Case of $a = b = 2^{-k}$:
  - $A = B = C_{n-k}$ (identical) $\Longrightarrow P_{XY}^{\otimes n}(A \times B) = P_{XY}(1,1)^k = \left(\frac{1+\rho}{4}\right)^k$
  - $A = \mathbf{1} - B = C_{n-k}$ (anti-symmetric) $\Longrightarrow P_{XY}^{\otimes n}(A \times B) = P_{XY}(1,0)^k = \left(\frac{1-\rho}{4}\right)^k$

Hamming Balls



- Hamming Ball: For $r \in [0, n]$, $\mathbb{B}_r(\mathbf{0}) := \{\mathbf{x} : d_{\mathrm{H}}(\mathbf{x}, \mathbf{0}) \leq r\} = \left\{\mathbf{x} : \sum_{i=1}^{n} x_i \leq r\right\}$

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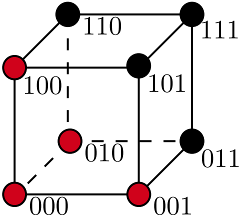

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- CL regime: Choose $A = \mathbb{B}_{r_n}(\mathbf{0}), B = \mathbb{B}_{s_n}(\mathbf{0})$ with $r_n = \frac{n}{2} + \frac{\lambda\sqrt{n}}{2}, s_n = \frac{n}{2} + \frac{\mu\sqrt{n}}{2}$ where $\lambda, \mu \in \mathbb{R}$

Hamming Balls

- By the univariate and multivariate CL theorems,

$$P_X^{\otimes n}(A) \to \Phi(\lambda), \qquad P_Y^{\otimes n}(B) \to \Phi(\mu), \qquad P_{XY}^{\otimes n}(A \times B) \to \Phi_\rho(\lambda, \mu)$$

where Φ is the CDF of the standard Gaussian, and $\Phi_\rho(\cdot, \cdot)$ is the CDF of the zero-mean bivariate Gaussian with covariance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

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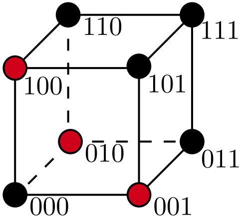
- Achievable CL probabilities:

$$\overline{\Gamma}^{(\infty)}(a,b) \ge \Lambda_\rho(a,b) \ \text{(by concentric balls } \mathbb{B}_{r_n}(\mathbf{0}), \mathbb{B}_{s_n}(\mathbf{0})\text{)}$$
$$\underline{\Gamma}^{(\infty)}(a,b) \le \Lambda_{-\rho}(a,b) \ \text{(by anti-concentric balls } \mathbb{B}_{r_n}(\mathbf{0}), \mathbb{B}_{s_n}(\mathbf{1})\text{)}$$

where bivariate normal copula (or Gaussian quadrant probability function):

$$\Lambda_\rho(a,b) := \Phi_\rho\left(\Phi^{-1}(a), \Phi^{-1}(b)\right)$$

Hamming Spheres



- Hamming Sphere: For $r \in [0:n]$, $\mathbb{S}_r(\mathbf{0}) := \{\mathbf{x} : d_{\mathrm{H}}(\mathbf{x}, \mathbf{0}) = r\} = \left\{\mathbf{x} : \sum_{i=1}^{n} x_i = r\right\}$

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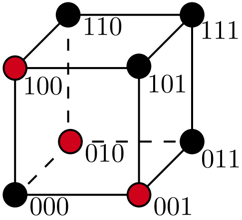

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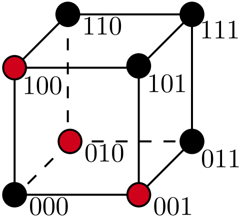

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- LD regime: Choose $A = \mathbb{S}_{r_n}(\mathbf{0})$, $B = \mathbb{S}_{s_n}(\mathbf{0})$ with $r_n = \lambda n, s_n = \mu n$ where $\lambda, \mu \in [0, 1]$

Hamming Spheres

- By LD theory (or Sanov's theorem),

$$-\frac{1}{n}\log P_X^{\otimes n}(A) \to D\left(\left(\bar{\lambda},\lambda\right)\|P_X\right) = 1 - H_2(\lambda)$$
$$-\frac{1}{n}\log P_Y^{\otimes n}(B) \to D\left((\bar{\mu},\mu)\|P_Y\right) = 1 - H_2(\mu)$$
$$-\frac{1}{n}\log P_{XY}^{\otimes n}(A\times B) \to \mathbb{D}\left(\left(\bar{\lambda},\lambda\right),(\bar{\mu},\mu)\|P_{XY}\right)$$

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  - relative entropy: $D(Q\|P) := \sum_x Q(x)\log\frac{Q(x)}{P(x)}$
  - binary entropy function: $H_2 : t \in [0,1] \mapsto -t\log_2 t - (1-t)\log_2(1-t)$
  - minimum-relative-entropy over couplings of (Q_X, Q_Y):

$$\mathbb{D}(Q_X, Q_Y\|P_{XY}) := \min_{Q_{XY}\in C(Q_X,Q_Y)} D(Q_{XY}\|P_{XY})$$

with $C(Q_X, Q_Y) := \{Q_{XY} \text{ with marginals } Q_X, Q_Y\}$ denoting the coupling set

Hamming Spheres

- Optimizing $\mathbb{D}(Q_X, Q_Y \| P_{XY})$ over feasible $Q_X := (\lambda, \bar{\lambda}), Q_Y := (\mu, \bar{\mu}) \Longrightarrow$

$$\underline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta) \le \underline{\Theta}_{\mathrm{LD}}(\alpha,\beta) := \min_{\substack{Q_X, Q_Y : D(Q_X\|P_X)\ge\alpha,\\ D(Q_Y\|P_Y)\ge\beta}} \mathbb{D}(Q_X, Q_Y \| P_{XY}),$$

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  - attained by concentric and anti-concentric spheres
  - or respectively attained by concentric and anti-concentric balls.

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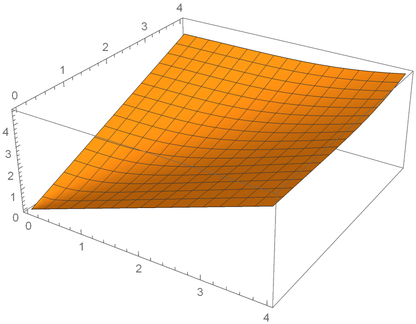
Conjecture (Ordentlich–Polyanskiy–Shayevitz (OPS, 2019))

For $\alpha, \beta \in (0,1)$,

$$\underline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta) = \underline{\Theta}_{\mathrm{LD}}(\alpha,\beta), \qquad \overline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta) = \overline{\Theta}_{\mathrm{LD}}(\alpha,\beta).$$

Exponents induced by balls/spheres for $\rho = 0.9$



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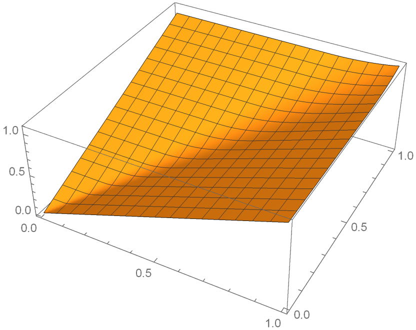

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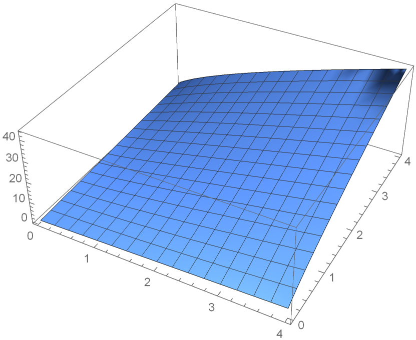

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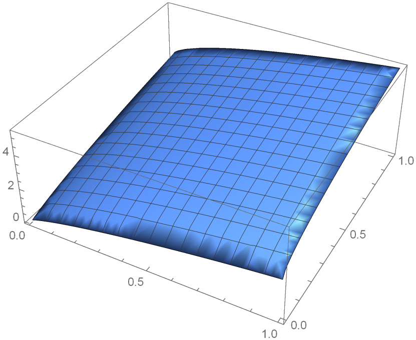

$\overline{\Theta}_{\mathrm{LD}}(\alpha,\beta)$

Comparison: Subcubes vs. Balls/Spheres

Regime	Central Limit		Large Deviations
a, b	fixed and large a, b	fixed but small a, b	exp. vanishing a, b
Subcubes	Better		
Balls		Better	Better

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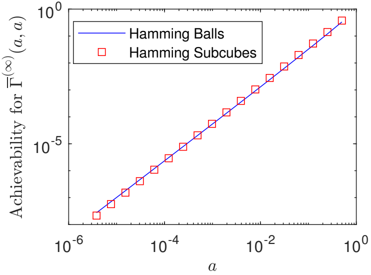

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- We answer these questions in the following.

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  - $\rho(U;V) := \frac{\mathrm{cov}(U,V)}{\sqrt{\mathrm{var}(U)\mathrm{var}(V)}}$ is the Pearson correlation coefficient
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- Data Processing Inequality (DPI): For a Markov chain $U - X - Y - V$,

$$\rho_{\mathrm{m}}(U;V) \le \rho_{\mathrm{m}}(X;Y).$$

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- For binary X, Y, $\rho_{\mathrm{m}}(X;Y) = |\rho(X;Y)|$.

Optimality for $a = b = \frac{1}{2}$: subcubes (dictators) optimal?

Theorem ([Witsenhausen, 1975])

For any A, B *with* $P_X^{\otimes n}(A) = a, P_Y^{\otimes n}(B) = b$,

$$ab - \rho\sqrt{a\bar{a}b\bar{b}} \leq P_{XY}^{\otimes n}(A \times B) \leq ab + \rho\sqrt{a\bar{a}b\bar{b}}, \qquad \text{where } \bar{x} = 1 - x.$$

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- Proof: Setting $U = 1_A(\mathbf{X}), V = 1_B(\mathbf{Y}) \implies U - \mathbf{X} - \mathbf{Y} - V \implies$

$$\frac{\left|P_{XY}^{\otimes n}(A \times B) - ab\right|}{\sqrt{a\bar{a}}\sqrt{b\bar{b}}} = |\rho(U;V)| = \rho_{\mathrm{m}}(U;V) \overset{\mathrm{DPI}}{\leq} \rho_{\mathrm{m}}(\mathbf{X};\mathbf{Y}) \overset{\mathrm{Tensorization}}{=} \rho_{\mathrm{m}}(X;Y)$$

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For any A, B *with* $P_X^{\otimes n}(A) = a, P_Y^{\otimes n}(B) = b$,

$$ab - \rho\sqrt{a\bar{a}b\bar{b}} \le P_{XY}^{\otimes n}(A \times B) \le ab + \rho\sqrt{a\bar{a}b\bar{b}}, \qquad \text{where } \bar{x} = 1 - x.$$

- Proof: Setting $U = 1_A(\mathbf{X}), V = 1_B(\mathbf{Y}) \Longrightarrow U - \mathbf{X} - \mathbf{Y} - V \Longrightarrow$

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  - For $a = b = 1/2$, $\frac{1-\rho}{4} \le P_{XY}^{\otimes n}(A \times B) \le \frac{1+\rho}{4}$.

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- Define the k-degree Fourier weight as

$$\mathbf{W}_k(f) := \sum_{|\mathbf{y}|=k} \hat{f}(\mathbf{y})^2$$

where $|\mathbf{y}|$ denotes the Hamming weight of $\mathbf{y}$.

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- However, $\underline{\Gamma}^{(n)}\left(\frac{1}{4}, \frac{1}{4}\right)$ is still open!

Another bound better for small a, b

Theorem (Small-Set Expansion [Ahlswede and Gács, 1976, Kahn et al., 1988, Mossel et al., 2006, O'Donnell, 2014])

For any $n \geq 1$ and $\alpha, \beta > 0$,

$$\underline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha,\beta) \geq \underline{\theta}(\alpha,\beta),$$
$$\overline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha,\beta) \leq \overline{\theta}(\alpha,\beta),$$

where

$$\underline{\theta}(\alpha,\beta) = \begin{cases} \frac{\alpha+\beta-2\rho\sqrt{\alpha\beta}}{1-\rho^2}, & \rho^2\alpha \leq \beta \leq \frac{\alpha}{\rho^2}, \\ \alpha, & \beta < \rho^2\alpha, \\ \beta, & \alpha < \rho^2\beta \end{cases},$$

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  - Substituting $f \leftarrow 1_A, g \leftarrow 1_B$ and optimizing p, q, SSE theorem follows.

An even better one

Theorem (Strong SSE [Yu et al., 2021, Yu, 2021c, Yu, 2021a])

For any $n \geq 1$ *and* $\alpha, \beta \in (0, 1]$,

$$\underline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha, \beta) \geq \underline{\Theta}_{\mathrm{LD}}(\alpha, \beta),$$
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- Notice: OPS's conjecture was proven previously for
  - limiting cases as $\rho \to 0$ or 1 in [Ordentlich et al., 2020]
  - the case $\alpha = \beta$ in [Kirshner and Samorodnitsky, 2021]

Summary of bounds

<table>
<tr><th rowspan="2">Bounds</th><th colspan="2">Central Limit</th><th>Large Deviations</th></tr>
<tr><th>fixed and large a, b</th><th>fixed but small a, b</th><th>exp. vanishing a, b</th></tr>
<tr><td>Maximal Correlation</td><td>Sharp for $a = b = 1/2$ (Subcubes)</td><td></td><td></td></tr>
<tr><td>Fourier Analysis</td><td>Sharp for $a = b = 1/2$ and $a = b = 1/4$ (Subcubes)</td><td></td><td></td></tr>
<tr><td>SSE</td><td></td><td rowspan="2">Almost sharp (Balls)</td><td></td></tr>
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</table>

Outline

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- Tightness: The bounds above are asymptotically sharp as $n \to \infty$.
- For the doubly symmetric binary distribution, $\overline{\Theta}_{\mathrm{LD}} = \overline{\varphi}_{\mathrm{LD}}$.

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Hypercontractivity (HC) Inequalities

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- Strong SSE + "level-partition" technique $\Longrightarrow$ Strong HC [Kirshner and Samorodnitsky, 2021, Yu et al., 2021]
    - Strong HC: An improved version of HC inequalities with the support sizes of f, g known

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$$f \approx \sum_{i=1}^{M} c_i 1_{A_i}, \qquad g \approx \sum_{j=1}^{N} d_j 1_{B_j}$$

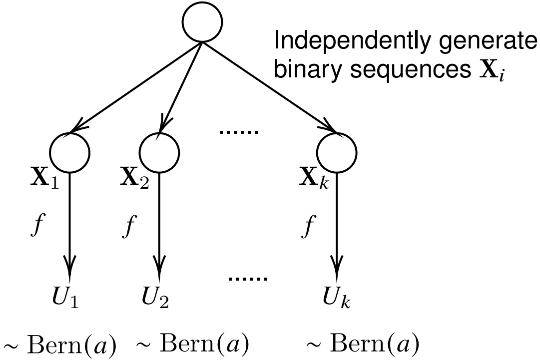
- Roughly speaking, applying SSE to each pair of $(A_i, B_j) \Longrightarrow$ HC
- Strong SSE + "level-partition" technique $\Longrightarrow$ Strong HC [Kirshner and Samorodnitsky, 2021, Yu et al., 2021]
  - Strong HC: An improved version of HC inequalities with the support sizes of f, g known

(Strong) SSE $\Longleftrightarrow$ (Strong) HC (in some sense)!

Outline

NICD with k Users

$$\mathbf{Y} \sim \mathrm{Bern}^{\otimes n}(1/2)$$



Asymmetric Version: $\max \mathbb{P}(U_1 = U_2 = \ldots = U_k = 1)$

Symmetric Version: $\max \mathbb{P}(U_1 = U_2 = \ldots = U_k)$

q-Stability

- (Asymmetric) max q-stability [Li and Médard, 2021]: For $q \in [1, \infty)$,

$$\overline{\Gamma}_q(a) := \max_{A: P_X^{\otimes n}(A)=a} \mathbb{E}_{\mathbf{Y}} \left[P_{X|Y}^{\otimes n}(A|\mathbf{Y})^q \right]$$

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- We only consider $a = 1/2$.

Dictator functions optimal?

Lemma ([Barnes and Özgür, 2020])

For $a = 1/2$, there are two thresholds $1 \leq q_{\min} < 2 < q_{\max}$ *such that dictators (i.e., $(n-1)$-subcubes) are optimal if and only if $q \in [q_{\min}, q_{\max}]$.*

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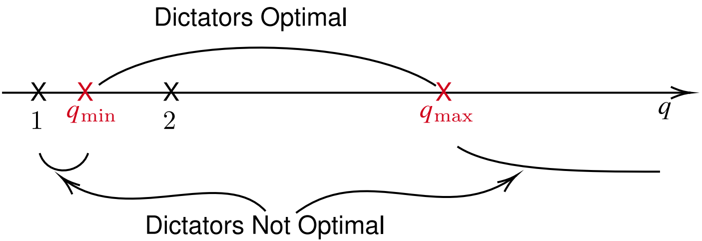

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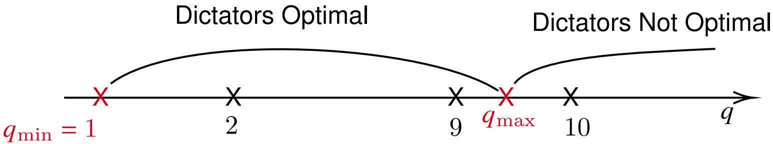


Progress on Courtade–Kumar/Li–Médard Conjecture

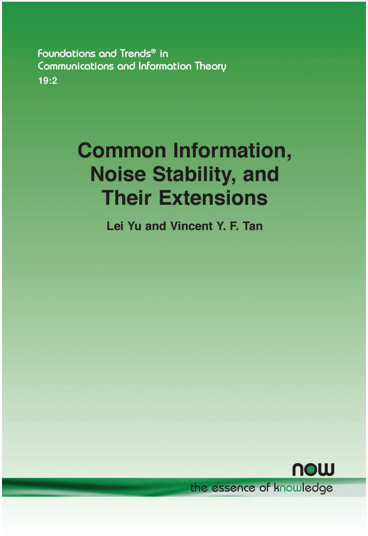
Related Works	Upper Bounds on $\max_{\text{Boolean } f:\mathbb{E}f=1/2} I(f(\mathbf{X});\mathbf{Y})$	Tools
CK/LM Conjecture	$1-H_2\left(\frac{1-\rho}{2}\right)$	
[Witsenhausen and Wyner, 1975]	ρ^2	Mrs. Gerber's lemma (or HC)
[Ordentlich et al., 2016]	$\frac{\log_2 e}{2}\rho^2+9\left(1-\frac{\log_2 e}{2}\right)\rho^4$ for $0\le\rho\le\frac{1}{\sqrt{3}}$ (asymp. tight as $\rho\to 0$)	Fourier analysis + HC
[Samorodnitsky, 2016]	tight bound for $\rho\in[0,\rho_0]$ with some $0<\rho_0<1$	Fourier analysis + Random restrictions + ...
[Yu, 2021b]	tight bound for $\rho\in[0,\rho_1]$ with $\rho_1\approx 0.46$ (explicitly given)	Fourier analysis + KKT conditions
[Pichler et al., 2018]	A weaker version: $\max_{\text{Boolean } f,g} I(f(\mathbf{X});g(\mathbf{Y}))=1-H_2\left(\frac{1-\rho}{2}\right)$	Fourier analysis + Partition technique

Progress on Mossel–O'Donnell Conjecture

Related Works	Dictators are optimal for ...	Tools
Mossel–O'Donnell Conjecture	$\mathbf{2 < q \leq 9}$ (both symmetric and asymmetric max q-stability)	
[Mossel and O'Donnell, 2005]	$2 < q \leq 3$ (symmetric)	Reducing $q = 3$ to $q = 2$
[Witsenhausen, 1975]	$q = 2$ (asymmetric)	Maximal correlation
[Yu, 2021b]	$2 < q \leq 5$ (symmetric); $2 < q \leq 3$ (asymmetric)	Fourier analysis + KKT conditions
[Mossel and O'Donnell, 2005, Li and Médard, 2021]	$\rho \to 0$ or 1 (symmetric and asymmetric)	Fourier analysis

Summary of tools

Methods		Central Limit		Moderate Deviations	Large Deviations
		fixed and large a (and b)	fixed but small a (and b)	subexp. vanishing a (and b)	exp. vanishing a (and b)
Information-Theoretic Methods	Maximal Correlation	Sharp for noise stability with $a = b = 1/2$			
	HC/SSE (stronger than MC)		Almost sharp	Sharp	
	Strong HC/SSE (stronger than HC/SSE)				Sharp
Fourier Analysis	Combined with Optimization Theory (LP or KKT)	Sharp for noise stability with $a = b = 1/2, 1/4$; Sharp for q-stability with $a = 1/2$ and certain (q, ρ)			



Common Information, Noise Stability, and Their Extensions

Lei Yu and Vincent Y. F. Tan

Common information measures the amount of matching variables in two or more information sources. It is ubiquitous in information theory and related areas such as theoretical computer science and discrete probability. However, because there are multiple notions of common information, a unified understanding of the deep interconnections between them is lacking. In this monograph the authors fill this gap by leveraging a small set of mathematical techniques that are applicable across seemingly disparate problems.

The reader is introduced in Part I to the operational tasks and properties associated with the two main measures of common information, namely Wyner's and Gács–Körner–Witsenhausen's (GKW). In the subsequent two Parts, the authors take a deeper look at each of these. In Part II they discuss extensions to Wyner's common information from the perspective of distributed source simulation, including the Rényi common information. In Part III, GKW common information comes under the spotlight. Having laid the groundwork, the authors seamlessly transition to discussing their connections to various conjectures in information theory and discrete probability.

This monograph provides students and researchers in Information Theory with a comprehensive resource for understanding common information and points the way forward to creating a unified set of techniques applicable over a wide range of problems.

This book is originally published as
Foundations and Trends® in Communications and Information Theory
Volume 19 Issue 2, ISSN: 1567-2190.

ISBN 978-1-63828-014-9

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Thank you for your attention!