The Toms–Winter regularity conjecture

Harbin Functional Analysis Seminar

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A glimpse into classification

Problem: Given two C^*-algebras A and B, how can we decide if A is isomorphic (or not) to B?

This is a very difficult question!

In order to simplify this problem one must restrict to simple separable unital and nuclear C^*-algebras.

A is nuclear if there is a net of finite rank completely positive approximations, i.e.

- F_i is finite dimensional
- ψ_i and φ_i are cp
- $\lim_i \|a - \varphi_i \circ \psi_i(a)\| = 0$

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi_i"'] & & A \\ & F_i \arrow[ur, "\varphi_i"'] & \end{tikzcd}$$

Examples.
- Commutative C^*-algebras
- $C^*(G)$ if G is amenable

Inductive limits

Given an increasing sequence of C^*-algebras

$$A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \ldots \hookrightarrow A_n \hookrightarrow \ldots$$

we can construct a new C^*-algebra

$$A = \varinjlim A_n = \overline{\bigcup_{n=1}^{\infty} A_n}^{\|\cdot\|}$$

<u>Definition</u>. (Non commutative Cantor sets)

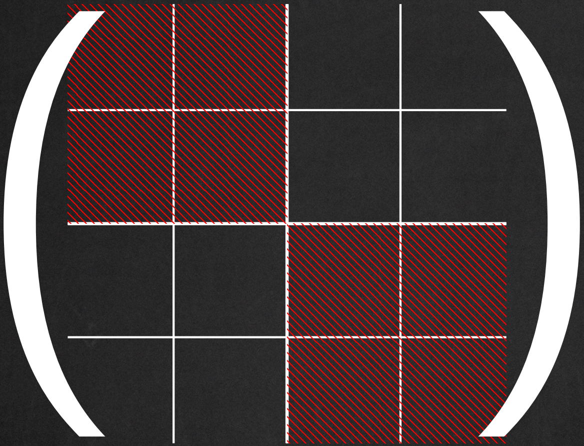
A C^*-algebra is AF it is an inductive limit of finite dimensional C^*-algebras.

Fact: $C(X)$ is AF $\iff$ X is totally disconnected

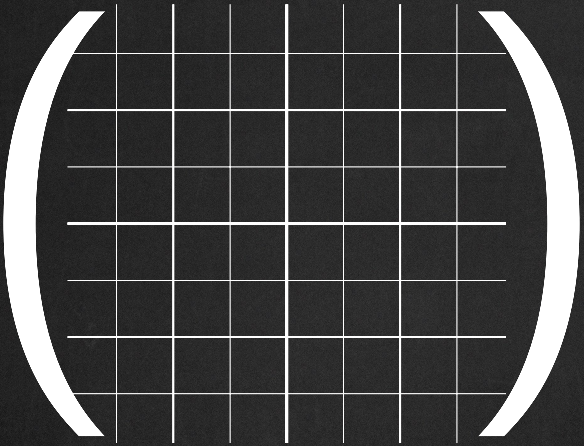
$$M_2 \left(\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right)$$

$$M_4 \left(\begin{array}{c|c|c|c} \quad & \quad & \quad & \quad \\ \hline \quad & \quad & \quad & \quad \\ \hline \quad & \quad & \quad & \quad \\ \hline \quad & \quad & \quad & \quad \end{array}\right)$$

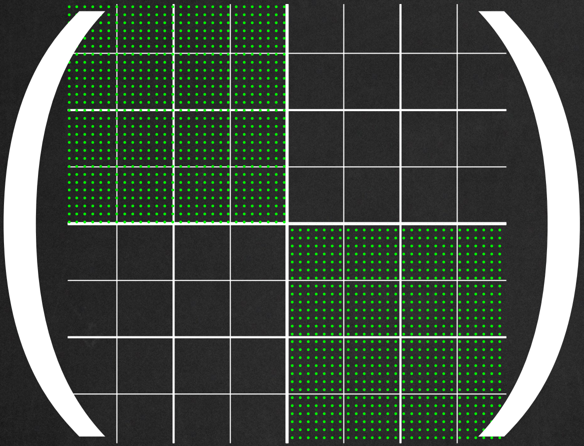
M_4



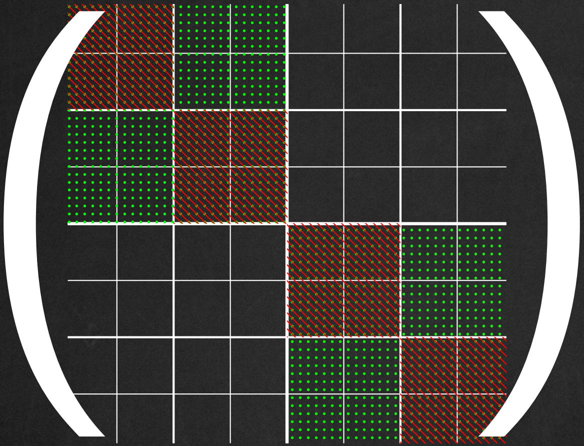
M_8



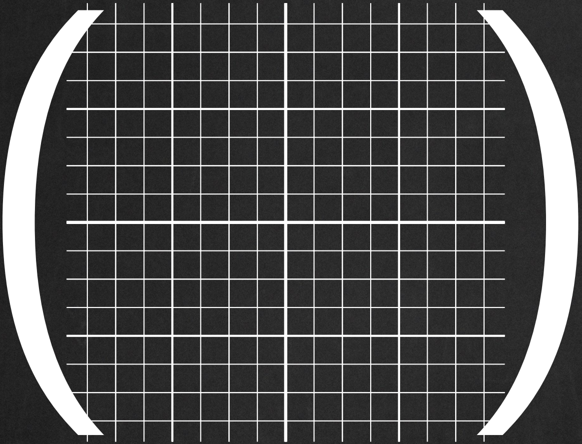
M_8



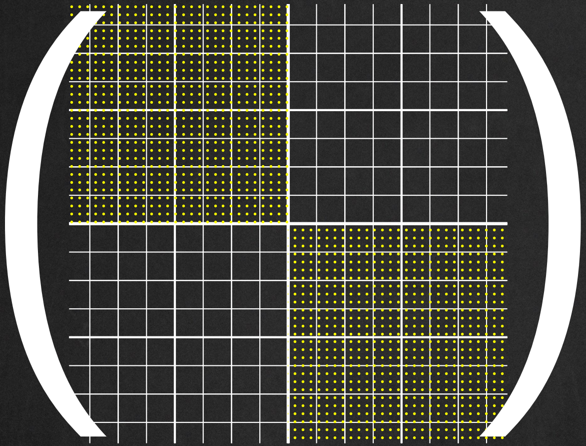
M_8



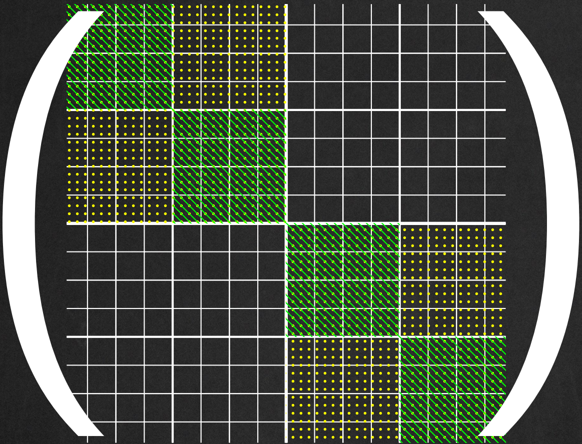
M_{16}



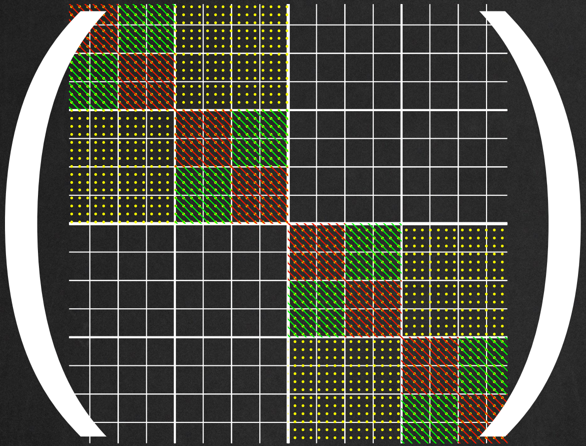
M_{16}

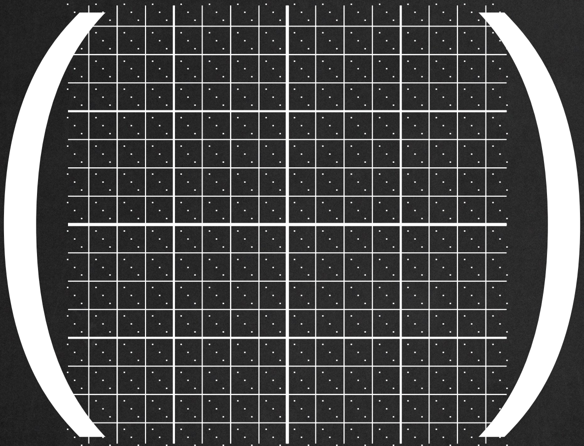


M_{16}



M_{16}



M_{2^∞} 

Formally M_{2^∞} is the inductive limit of the sequence

$$M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \dots$$

In general, given $r \in \mathbb{N}$ we can construct M_{r^∞} by considering the sequence

$$M_r(\mathbb{C}) \to M_{r^2}(\mathbb{C}) \to M_{r^3}(\mathbb{C}) \to \dots$$

Question: $M_{2^\infty} \cong M_{3^\infty}$?

To answer this question, we can use some covariant functors

$$\{C^*\text{-algebras}\} \;\to\; \{\text{abelian groups}\} \quad A \mapsto K_*(A)$$

extending the classical functors of topological K-theory

$$K_0(M_{2^\infty}) = \mathbb{Z}[1/2] \;\text{ and }\; K_0(M_{3^\infty}) = \mathbb{Z}[1/3] \implies M_{2^\infty} \ncong M_{3^\infty}$$

Elliott conjecture 80's:

Simple separable unital and nuclear C^*-álgebras are classified by an invariant constructed with K-theory and tracial information, i.e.

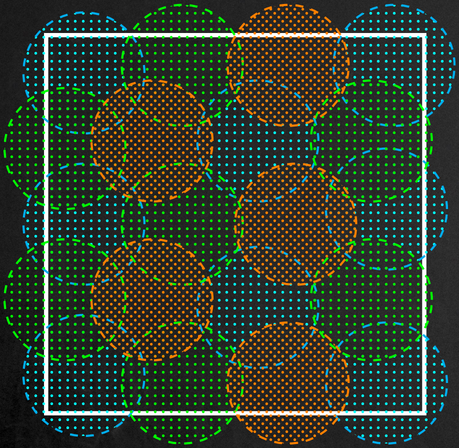
$$A \cong B \qquad \Longleftrightarrow \qquad \mathrm{Ell}(A) \cong \mathrm{Ell}(B)$$

The conjecture is false!

- Rørdam (Acta Math. '03)
- Toms (Ann. of Math. '08)

These counterexamples are inductive limits using $M_k(C(\prod_{\mathbb{N}} \mathbb{S}^2))$

The covering dimension of a topological space is at most n if any open cover has a finite refinement that can be coloured with $n+1$ colours in such a way that open sets with the same colour do not intersect each other. The covering dimension is the minimum n.



- $\dim \mathbb{R}^n = n$

- $\dim X = 0$ iff X is totally disconnected

- $\dim \prod_{\mathbb{N}} \mathbb{S}^2 = \infty$

Non commutative topological dimension

Definition. (Winter-Zacharias '10) A C^*-algebra A has nuclear dimension at most n, $\dim_{\mathrm{nuc}} A \leq n$, if for any finite set $\mathfrak{F} \subset A$ and $\epsilon > 0$ there exist finite dimensional C^*-algebras $F_1, \ldots, F_{n+1}$ and cp maps $\psi : A \to \bigoplus_{k=1}^{n} F_k$, $\phi : \bigoplus_{k=1}^{n} F_k \to A$ such that

- $\|a - \phi \circ \psi(a)\| < \epsilon$ for all $a \in \mathfrak{F}$,
- $\phi_k := \phi|_{F_k}$ preserves orthogonality.

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi"'] \arrow[rr, phantom, shift right=8ex, "\approx"] & & A \\
& \bigoplus\limits_{k=1}^{n+1} F_k \arrow[ur, "\phi = \sum\limits_{k=1}^{n+1} \phi_k"'] &
\end{tikzcd}$$

Basic facts and examples

- $\dim_{\mathrm{nuc}} C_0(X) = \dim X$ (Winter '03)
- If $\dim_{\mathrm{nuc}} A < \infty$ then A is nuclear.
- Converse is false! $\dim_{\mathrm{nuc}} C([0,1]^{\mathbb{N}}) = \infty$
- $\dim_{\mathrm{nuc}} A = 0 \iff A$ es AF (Winter '03)
- Γ abelian group $\implies \dim_{\mathrm{nuc}} C_r^*(\Gamma) = \dim \widehat{\Gamma}$
- Γ non amenable $\implies \dim_{\mathrm{nuc}} C_r^*(\Gamma) = \infty$
- Γ virtually nilpotent and finitely generated
  $\implies \dim_{\mathrm{nuc}} C_r^*(\Gamma) < \infty$ (Eckhardt-Gillaspy-McKenney '17)
- $\dim_{\mathrm{nuc}} C_0(X) \rtimes_\alpha \mathbb{Z} \leq 2(\dim X)^2 + 6\dim X + 4$ (Hirshberg-Wu '17)
- $\dim_{\mathrm{nuc}} A \rtimes_\alpha \Gamma < \infty$ with extra hypotheses.

The Classification Theorem(Many hands)

The class of simple separable unital (UCT) C^*-algebras with finite nuclear dimension is classified with the Elliott invariant; i.e.

$$A \cong B \qquad \Longleftrightarrow \qquad \mathrm{Ell}(A) \cong \mathrm{Ell}(B)$$

Technical hurdle: Verifying that a C^*-algebra has finite nuclear dimension can be quite challenging!

The Toms–Winter regularity conjecture predicts that other regularity conditions are equivalent to finite nuclear dimension.

The catch is that these conditions (at priori) are easier to verify!

Dimension drop algebra

$$Z_{p,q} = \{f : [0,1] \to M_p \otimes M_q \mid f(0) \in M_p \otimes \{1\}, f(1) \in \{1\} \otimes M_q\}$$

The Jiang-Su algebra $\mathcal{Z}$ is constructed as some inductive limit of the form

$$Z_{p_1,q_1} \xrightarrow{\phi_1} Z_{p_2,q_2} \xrightarrow{\phi_2} Z_{p_3,q_3} \xrightarrow{\phi_2} \ldots \to \mathcal{Z}.$$

It is simple, separable, unital, nuclear, with unique trace and with no non-trivial projections.

We view it as an infinite dimensional version of $\mathbb{C}$.

In fact, $\mathrm{Ell}(\mathbb{C}) \cong \mathrm{Ell}(\mathcal{Z})$ and under mild conditions $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{Z})$.

It satisfies $\mathcal{Z} \cong \mathcal{Z}^{\otimes 2} \cong \mathcal{Z}^{\otimes 3} \cong \ldots \cong \mathcal{Z}^{\otimes \infty}$ and $\dim_{\mathrm{nuc}} \mathcal{Z} = 1$.

It is the right C^*-analogue of the hyperfinite II_1 factor $\mathcal{R}$.

A C^*-algebra A is called $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \cong A$.

Important property: If A is $\mathcal{Z}$-stable, then there is a *-isomorphism $\phi : A \to A \otimes \mathcal{Z}$ such that $\phi \approx_{\text{u.e.}} \mathrm{id}_A \otimes 1_{\mathcal{Z}}$

Reminder. Let ω be a free ultrafilter on $\mathbb{N}$.

$$A_\omega = \ell^\infty(A)/\{(a_n) \mid \lim_{n\to\omega} \|a_n\| = 0\}$$

$$A_\omega \cap A' = \{(x_n) \in A_\omega \mid \lim_{n\to\omega} \|x_n a - a x_n\| = 0 \text{ for all } a \in A\}$$

Theorem (Rørdam-Winter, Toms-Winter)
Let A be unital and separable. TFAE

(i) A is $\mathcal{Z}$-stable,

(ii) $\mathcal{Z} \hookrightarrow A_\omega \cap A'$ unitally,

(iii) $Z_{n,n+1} \hookrightarrow A_\omega \cap A'$ for all n.

Notation. $T(A)$ is the set of tracial states.

Fact. If A is unital, $T(A)$ is compact.

Definition. Let $a, b \in A_+$. It is said a is Cuntz subequivalent to b, $a \precsim b$, if there is a sequence (x_n) such that $a = \lim x_n^* b x_n$.

Example. $f \precsim g$ in $C(X)$ if and only if supp$(f) \subset$ supp(g).

Definition. A has strict comparison if for all $a, b \in A_+$ that satisfy

$$\lim_{n\to\infty} \tau(a^{1/n}) < \lim_{n\to\infty} \tau(b^{1/n}), \qquad \tau \in T(A)$$

then $a \precsim b$.

Idea: Strict comparison is a technical condition that allows us to recover the Cuntz-order from tracial information.

Toms–Winter regularity conjecture

Let A be separable simple nuclear and non elementary. TFAE

(i) $\dim_{\mathrm{nuc}} A < \infty$,

(ii) $A \otimes \mathcal{Z} \cong A$,

(iii) A has strict comparison.

Progress

- (i)$\Longrightarrow$(ii) Winter '11, Tikuisis '14
- (ii)$\Longrightarrow$(iii) Rørdam '04
- (iii)$\Longrightarrow$(ii) Known for some cases (Kirchberg, Matui, Sato, Rørdam, Thiel, Toms, White, Winter, Zhang)
  - $T(A)$ Bauer or tight with finite covering dimension
  - stable rank one with locally finite nuclear dimension
- (ii)$\Longrightarrow$(i) Known for $T(A)$ Bauer (Bosa, Brown, Matui, Sato, Tikuisis, White, Winter)

<u>Theorem</u> (Matui–Sato, Sato–White–Winter)
Let A be separable simple, unital and nuclear with unique trace. If A is $\mathcal{Z}$-stable then $\dim_{\mathrm{nuc}} A \leq 1$.

Sketch. Using that $\pi_\tau(A)'' \cong \mathcal{R} \cong \pi_\tau(M_{2^\infty})''$, we can produce diagrams

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A \otimes 1_{\mathcal{Z}}"] \arrow[dr, "\psi"'] & & A \otimes \mathcal{Z} \\ & F \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \tau(a) \approx \tau(\sigma\psi(a))$$

Let $h \in \mathcal{Z}$ be a positive element with spectrum $[0,1]$. Then

$$\mathrm{id}_A \otimes h \approx_{ue} \sigma\psi \otimes h, \qquad \mathrm{id}_A \otimes (1_{\mathcal{Z}} - h) \approx_{ue} \sigma\psi \otimes (1_{\mathcal{Z}} - h)$$

$$\begin{aligned} \Longrightarrow a \otimes 1_{\mathcal{Z}} &= a \otimes h + a \otimes (1_{\mathcal{Z}} - h) \\ &\approx u_1(\sigma\psi(a) \otimes h)u_1^* + u_2(\sigma\psi(a) \otimes (1-h))u_2^* \end{aligned}$$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\psi \oplus \psi"'] & & A \otimes \mathcal{Z} \\ & F \oplus F \arrow[ur, "\phi_1 + \phi_2"'] & \end{tikzcd} \qquad \phi_i(x) = u_i(\sigma(x) \otimes h_i)u_i^*$$

$$\Longrightarrow \dim_{\mathrm{nuc}} A \leq 1$$

How do we handle more than one trace? For each trace $\tau \in T(A)$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\psi_\tau"'] & & A \otimes \mathcal{Z} \\ & F_\tau \arrow[ur, "\sigma_\tau"'] & \end{tikzcd}$$

$\tau(a) \approx \tau(\sigma_\tau \psi_\tau(a))$

By compactness of $T(A)$, there is a finite cover $\{U_i\}$ and $\tau_i \in U_i$ such that $\tau(a) \approx \tau_i(a)$ if $\tau \in U_i$. Let (h_i) be a PoU subordinated to $\{U_i\}$.

(Naive PoU) There are positive contractions $e_1, \ldots, e_k \in A_\omega \cap A'$ such that $\tau(e_i a) = h_i(\tau)\tau(a)$.

Set $\sigma : \bigoplus F_{\tau_i} \to A \otimes \mathcal{Z}$ by $\sigma(x_1, \ldots, x_k) = \sum e_i \sigma_{\tau_i}(x_i)$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\psi = \oplus \psi_{\tau_i}"'] & & A \otimes \mathcal{Z} \\ & \bigoplus F_{\tau_i} \arrow[ur, "\sigma = \sum e_i \sigma_{\tau_i}"'] & \end{tikzcd}$$

$\tau \approx \tau \circ \sigma \circ \psi \quad \forall \tau$

As in the monotracial case, $\dim_{\mathrm{nuc}} A \leq 1$.

Naive partitions of unity do not exist in general :(but a weaker form of partitions of unity suffices at the cost of making the proof more difficult.

Definition

A has complemented partitions of unity (CPoU) if for any family of positive contractions $a_1, \ldots, a_k \in A$ and $\delta > 0$ such that

$$\delta > \sup_{\tau \in T(A)} \min\{\tau(a_1), \ldots, \tau(a_k)\},$$

there exist pairwise orthogonal contractions $e_1, \ldots, e_k \in A_\omega \cap A'$ with

$$\tau(\sum e_i) = 1, \quad \tau(e_i a_i) \leq \delta \tau(e_i), \quad i = 1, \ldots, n, \ \tau \in T(A).$$

When do we have CPoU?

Definition

Suppose A is unital and simple. A has uniform property Γ if for any n there exist pairwise orthogonal positive contractions $e_1, \ldots, e_n \in A_\omega \cap A'$ such that

$$\tau\left(\sum e_i\right) = 1, \quad \tau(ae_i) = \frac{1}{n}\tau(a), \qquad a \in A, \ \tau \in T(A).$$

This notion is a C^*-star version of the property Γ for II_1 factors.

Examples

- The universal UHF algebra $\mathbb{Q} = \bigotimes_{n\in\mathbb{N}} M_n$
- The Jiang-Su algebra $\mathcal{Z}$
- $\mathcal{Z}$-stable C^*-algebras
- the non-$\mathcal{Z}$-stable Villadsen algebras contructed by Toms and Winter

Theorem (C-Evington-Tikuisis-White)

Let A be separable nuclear and unital. TFAE

1. A has CPoU,
2. A has uniform property Γ,
3. A is uniformly McDuff, i.e.
   for all $n \in \mathbb{N}$ there exists an order zero map $\varphi : M_n(\mathbb{C}) \to A_\omega \cap A'$ such that $\tau \circ \phi(1) = 1$

Theorem (Carrión-C-Evington-Gabe-Schafhauser-Tikuisis-White)
Let A be simple separable with uniform property Γ. Then A has CPoU.

Theorem (C-Evington-Tikuisis-White-Winter, C-Evington)
Let A be simple and nuclear. If A is $\mathcal{Z}$-stable then $\dim_{\mathrm{nuc}} A \leq 1$.

Corollary Let A be separable simple nuclear. Then
$A \cong A \otimes \mathcal{Z} \iff \dim_{\mathrm{nuc}} A < \infty$.

Corollary Separable simple unital nuclear $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by their Elliott invariant.

Corollary Let A be a simple C^*-algebra. Then

$$\dim_{\mathrm{nuc}} A = \begin{cases} 0 & A \text{ is AF} \\ 1 & A \text{ is nuclear } \mathcal{Z}\text{-stable but not AF} \\ \infty & \text{otherwise} \end{cases}$$

Theorem (C-Evington-Tikuisis-White, C-Evinton, Lin)
Let A be simple separable and nuclear. If A has uniform property Γ and strict comparison then A is $\mathcal{Z}$-stable.

Theorem (Toms–Winter)
Let A be separable simple nuclear and non-elementary. TFAE

1. $\dim_{\mathrm{nuc}} A < \infty$,
2. $A \otimes \mathcal{Z} \cong A$,
3. A has strict comparison and uniform property Γ.
4. $\dim_{\mathrm{nuc}} A \leq 1$.

Thank you!