The Brown measure of the sum of a free random variable and an elliptic deformation of Voiculescu's circular element

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Some relevant work

Free probability method: additive model

- Haagerup-Larsen 2000, R-diagonal operators
- Biane-Lehner 2001, examples of Brown measures
- Hermitian reduction: Aagaard-Haagerup 2004, Belinschi-Speicher-Śniady 2018
- *Z. 2021, Hermitian reduction, subordination functions*

PDE method: additive model or multiplicative model

- Driver-Kemp-Hall 2019
- Ho-Z. 2019, PDE method and subordination functions
- Hall-Ho, 2020 and 2021

Random matrix method: additive model

- Bordenave-Caputo-Chafai 2014, random matrix approach

Typical behavior of random matrices

- Let X_N be some random matrix model and set

$$\mu_{X_N} = \frac{1}{N}\left(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N}\right).$$

- The measure μ_{X_N} is a *random probability measure*.

Quite often, there exists some *deterministic probability measure* μ such that

$$\mu_{X_N} \to \mu$$

as $N \to \infty$.

Main questions and our goal

- Hermitian random matrices are relatively well understood.
- Non-Hermitian random matrices and non-selfadjoint operators have wild properties.

Problem

Find the eigenvalue distribution of non-Hermitian random matrices.

1. *Find their limit distributions (our main goal of this talk).*
2. *Prove convergence of empirical spectral distribution (ESD) of random matrices.*

We study explicit formula of the limit ESD of summation of two non-Hermitian random matrices, one of which has certain symmetry.

Formulation of the main questions

- Random matrix $X_N \longrightarrow$ operator $x \in \mathcal{A} \subset B(\mathrm{H})$.
- Noncommutative probability space: $(\mathcal{A}, \phi)$.
- Brown measure of a random variable in free probability can often be regarded as limit of eigenvalue distribution of suitable random matrix models.

- limit operators can help us understand random matrices;
- random matrices can help us understand operators.

Ginibre Ensemble

$$Z_N = \begin{pmatrix} x_{11} & x_{12} & \cdots x_{1N} \\ x_{21} & x_{22} & \cdots x_{2N} \\ \vdots & \vdots & \ddots \vdots \\ x_{N1} & x_{N2} & \cdots x_{NN} \end{pmatrix}$$

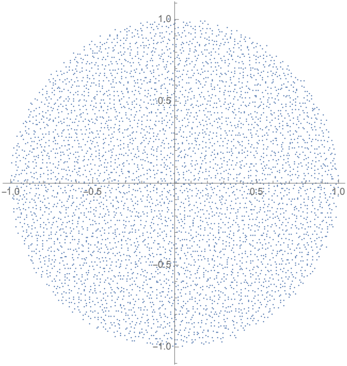
The Ginibre Ensemble Z_N has i.i.d. complex Gaussian entries with variance $1/N$.

Definition

The Empirical Spectral Distribution (ESD) of Z_N is

$$\mu_N = \frac{1}{N}\left(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N}\right).$$

The circular law



The following result is due to Ginibre, Girko, Bai, Tao, Vu and many others.

Theorem (circular law) Given any random variable X with mean zero and variance one. Let Z_N be the $n \times n$ square random matrices with i.i.d. entries that have the same distribution as $X/\sqrt{n}$. The ESD of Z_N convergences to the uniform measure on the unit disk as $n \to \infty$.

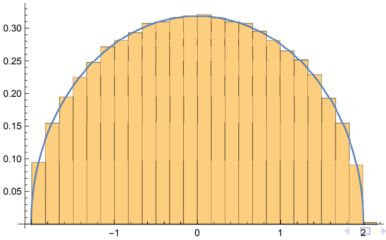
Wigner's Semicircle law

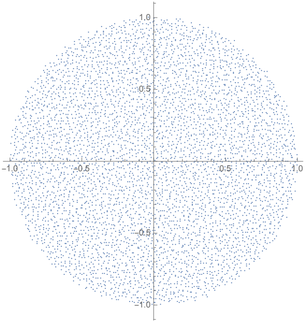
Definition

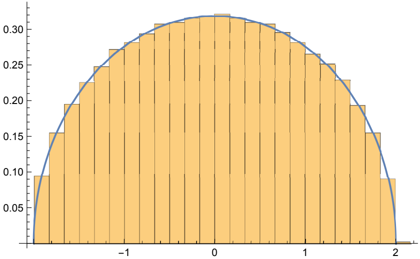
A matrix $W = \frac{1}{\sqrt{N}}(x_{kl})_{k,l=1}^{N}$ is a complex Wigner random matrix if:

- it is Hermitian: $W = W^*$, and
- $\{x_{kl} | 1 \leq k \leq l \leq N\}$ are independent, $x_{k,k} \sim \mathcal{N}(0,1)$ and $x_{k,l} \sim \mathcal{N}(0,1/2) + \mathrm{i}\,\mathcal{N}(0,1/2)$.

Large N limit of eigenvalue distribution μ_W is the semicircle law.







Distribution of real part in circular law "=" semicircular law

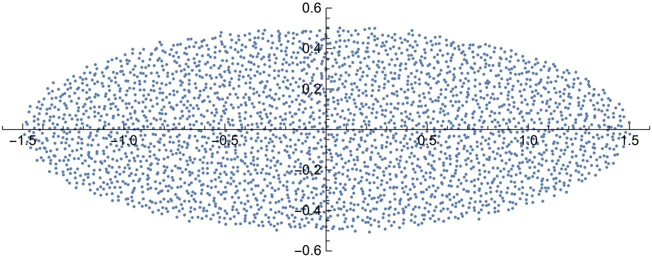
Elliptic deformation

- An elliptic random matrix X_N is a square matrix whose (i,j)-entry $X_N(i,j)$ is independent of every other entry except possibly $X_N(j,i)$.
- Elliptic random matrices generalize Wigner matrices and non-Hermitian random matrices with i.i.d. entries.

$$\begin{pmatrix} x_{11} & \color{red}{x_{12}} & \cdots & \color{blue}{x_{1N}} \\ \color{red}{x_{21}} & x_{22} & \cdots & \color{green}{x_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ \color{blue}{x_{N1}} & \color{green}{x_{N2}} & \cdots & x_{NN} \end{pmatrix}$$

Circular and elliptic random matrices

- Given two independent i.i.d. Wigner random matrices W_n, W_n'.
- Circular random matrix $= W_n + iW_n'$
- Elliptic random matrix $= e^{i\theta}(\alpha W_n + i\beta W_n')$, where $\alpha, \beta \geq 0$.



The Brown measure of a square matrix

The characteristic polynomial of a matrix $T \in M_n(\mathbb{C})$ is

$$P(\lambda) = det(\lambda I - T) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n).$$

The eigenvalue distribution is

$$\mu_T = \frac{1}{n}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_n}).$$

Consider $\log|P(\lambda)| = \log|\det(\lambda I - T)| = \sum_{i=1}^n \log|\lambda - \lambda_i|$. Note

$$\Delta \log|\lambda| = 2\pi\delta_0.$$

Then

$$\mu_T = \frac{1}{2\pi n}\Delta \log|\det(\lambda I - T)|.$$

Noncommutative probability space and Brown measure

- Noncommutative probability space: $(\mathcal{A}, \phi)$:

$\mathcal{A} \subset B(\mathrm{H})$ operator algebra (finite von Neumann algebra),

and $\phi : \mathcal{A} \to \mathbb{C}$ is a replacement of trace.

- The Fuglede-Kadison determinnat of $x \in (\mathcal{A}, \phi)$ is defined as

$$\mathcal{D}(x) = \exp[\phi(\log(|x|))] \in [0, \infty).$$

Definition (Brown, 1983)

The Brown measure of x is the distributional Laplacian,

$$\mu_x = \frac{1}{2\pi} \Delta \log \mathcal{D}(x - \lambda).$$

- When $A \in M_n(\mathbb{C})$, then μ_A is the eigenvalue distribution of A.

Operator models

- semicircular element g_t: selfadjoint, μ_{g_t} is semicircular law
- Voiculescu's circular element:

$$c_t = \frac{1}{\sqrt{2}}(g_t + ig_t').$$

- elliptic deformations $y = g_{t,\gamma}$ ($|\gamma| \le t$):

$$g_{t,\gamma} = e^{i\theta}(\alpha g_t + \beta g_t'),$$

such that all non-zero free cumulants of y are given by

$$\kappa(y, y^*) = \kappa(y^*, y) = t, \kappa(y, y) = \gamma, \kappa(y^*, y^*) = \overline{\gamma}.$$

Convergence of empirical spectral distributions

Let X_N be a sequence of $N \times N$ Hermitian matrices (either random but independent with Z_N, or deterministic) that *converges* to some limit.

Question (deformed random matrix model)

- What is the limit ESD of $X_N + W_N$ (sum of two Hermitian matrices)?
- What is the limit ESD of $X_N + Z_N$ (Hermitian + non-normal matrix)?

Theorem (Corollary of (Voiculescu, 90s))

The limit distribution of $X_N + W_N$ is the distribution of two selfadjoint random variables that are independent in the sense of Voiculescu's free independence. That is,

$$\mu_{X_N+W_N} \to \mu_{x+g}.$$

Asymptotic freeness and convergence in $*$-moments

Theorem (Voiculescu 1991)

For a suitable family of independent random matrices $(X_i^{(N)})_{i\in\mathcal{I}}$, all mixed moments

$$tr\left(X_{i_1}^{(N)}\cdots X_{i_k}^{(N)}\right) \to \phi\left(x_{i_1}\cdots x_{i_k}\right)$$

almost surely as $N\to\infty$, where $i_1,\cdots,i_k\in I$ and $\{x_i\}_{i\in\mathcal{I}}$ is a family of freely independent random variables in certain non-commutative probability space $(\mathcal{A},\phi)$.

• The convergence of random matrices in the sense of Brown measure does not follow from convergence in $*$-moments.

Theorem (Śniady 2001 and Tao-Vu 2010)

The empirical spectral distribution of $X_N + Z_N(t)$ converges to the Brown measure of $x_0 + c_t$, where

- $X_N \to x_0$ *in $*$-moments,*
- c_t *is Voiculescu's circular element,*
- *and $\{x_0, c_t\}$ are freely independent.*

• Biane-Lehner (2001) calculated $Brown(x_0 + c_t)$ for some special x_0.
• Bordenave-Caputo-Chafai (2014) obtained Brown measure formula for normal operator ($x_0^* x_0 = x_0 x_0^*$) with Gaussian distribution (related to Laplacian matrix of the oriented Erdös-Rényi random graph).

Main results

Let x_0 be an arbitrary operator that is $*$-free from $\{x_0, g_{t,\gamma}\}$.

Theorem (Z. 2021)

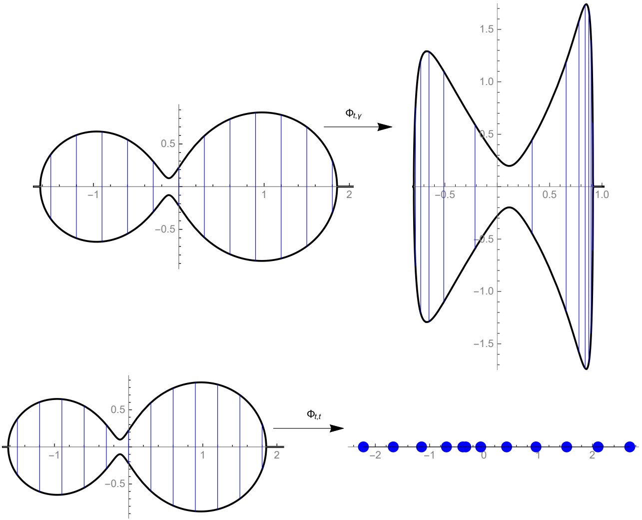
The Brown measure of $x_0 + c_t$ is absolutely continuous in some open set Ξ_t and is supported in its closure $\overline{\Xi_t}$. The density of the Brown measure can be expressed explicitly by certain subordination functions.

Theorem (Z. 2021)

The Brown measure of $x_0 + g_{t,\gamma}$ is the push-forward measure of the Brown measure of $x_0 + c_t$ by certain explicitly constructed map $\lambda \mapsto \Phi_{t,\gamma}(\lambda)$. That is,

$$\mu_{x_0+c_t}((\Phi_{t,\gamma}^{-1}(\cdot)) = \mu_{x_0+g_{t,\gamma}}(\cdot). \tag{1}$$

The pushforward map from $x_0 + c_t$ to $x_0 + ig_t$ and $x_0 + g_t$



From circular to elliptic: selfadjoint case

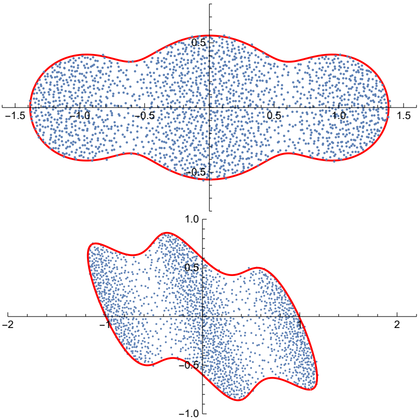


Figure: The Brown measures of $x_0 + c_t$ and $x_0 + g_{t,\gamma}$ for $t = 0.5$, $\gamma = -0.25 - 0.25i$, and x_0 distributed as $0.25\delta_{-1} + 0.5\delta_0 + 0.25\delta_1$.

Brown measure support of addition with a circular element

Theorem (Z., 2021)

The Brown measure of $x_0 + c_t$ is supported in the closure of the open set

$$\Xi_t = \left\{ \lambda \in \mathbb{C} : \phi\left[\left((x_0 - \lambda)^* (x_0 - \lambda) \right)^{-1} \right] > \frac{1}{t} \right\}.$$

The density formula can be expressed in terms of subordination functions.

- We believe that $\mu_{x_0+c_t}(\Xi_t) = 1$ (all our examples support this).

Fundamental domain (circular): selfadjoint case

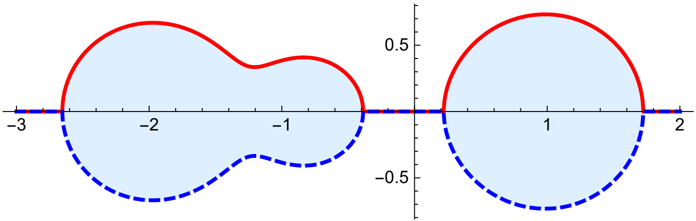


Figure: The domain Ξ_t for $t = 1$ and x_0 distributed as $0.4\delta_{-2} + 0.1\delta_{-0.8} + 0.5\delta_1$. The graph of v_t is the solid read curve above the x-axis.

$$\Xi_t = \left\{ \lambda = a + bi : \int_{\mathbb{R}} \frac{1}{(a-u)^2 + b^2} d\mu_{x_0}(u) > \frac{1}{t} \right\}$$

Main results: formulas

Theorem

The density of the Brown measure at $\lambda \in \Xi_t$ is given by

$$\frac{1}{\pi}\left(\frac{|\phi((\lambda - x_0)(h^{-1})^2)|^2}{\phi((h^{-1})^2)} + w_t(\lambda)^2\phi(h^{-1}k^{-1})\right)$$

where $w_t(\lambda)$ is determined by

$$\phi((x_0 - \lambda)^*(x_0 - \lambda) + w_t(\lambda)^2)^{-1}) = \frac{1}{t},$$

and $h = h(\lambda, w_t(\lambda))$ and $k = k(\lambda, w_t(\lambda))$ for

$$h(\lambda, w_t) = (\lambda - x_0)^*(\lambda - x_0) + w_t(\lambda)^2$$

and

$$k(\lambda, w_t) = (\lambda - x_0)(\lambda - x_0)^* + w_t(\lambda)^2.$$

Brown measure of the addition with an elliptic deformation

We denote

$$\Phi_{t,\gamma}(\lambda) = \lambda + \gamma \cdot p_\lambda^{(0)}(w_t), \qquad \lambda \in \Xi_t,$$

where

$$p_\lambda^{(0)}(w_t) = -\phi\left[(x_0 - \lambda)^*\left((x_0 - \lambda)(x_0 - \lambda)^* + w_t(\lambda)^2\right)^{-1}\right].$$

Let $g_{t,\gamma}$ be an elliptic operator $e^{i\theta}(s_1 + is_2)$, where s_1, s_2 are semicircular family.

Theorem (Z. 2021)

The Brown measure of $x_0 + g_{t,\gamma}$ is the push-forward measure of the Brown measure of $x_0 + c_t$ by the map $\lambda \mapsto \Phi_{t,\gamma}(\lambda)$. That is,

$$\mu_{x_0+c_t}((\Phi_{t,\gamma}^{-1}(\cdot)) = \mu_{x_0+g_{t,\gamma}}(\cdot). \tag{2}$$

Another formula for the pushforwrd map

- The pushforward map between $\mu_{x_0+c_t}$ and $\mu_{x_0+g_{t,\gamma}}(|\gamma| \leq t)$ is

$$\Phi_{t,\gamma}(\lambda) = \lambda + 2\gamma \cdot \frac{\partial}{\partial \lambda} \log \Delta(x_0 + c_t - \lambda), \qquad \lambda \in \mathbb{C}.$$

Problem

Is $\Phi_{t,\gamma}$ some optimal transport map?

- For some special cases, we can show $\Phi_{t,\gamma}(\lambda)$ is the gradient of some convex function.

Examples

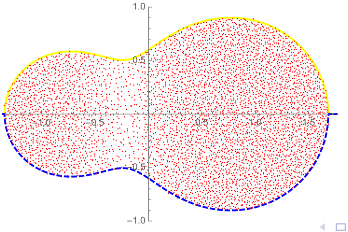
We can calculate explicit formulas when

- x_0 is selfadjoint;
- x_0 is Haar unitary/R-diagonal operator;
- x_0 is quasi-nilpotent DT operator.

The Brown measure of free circular Brownian motion with selfadjoint initial condition x_0

Theorem (Ho-Z., 2019 (based on PDE method of Driver-Kemp-Hall))

- *Brown($x_0 + c_t$) is symmetric with respect to the x-axis.*
- *The boundary of the support is the graph of a function, related to the subordination function of $x_0 + s_t$ with respect to x_0.*
- *The density is constant along vertical lines, and can be expressed explicitly by the boundary set.*



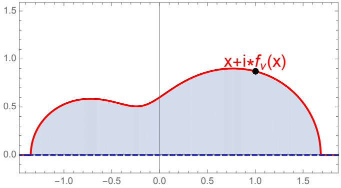
The density formula: selfadjoint+circular

Theorem (Ho-Z., 2019)

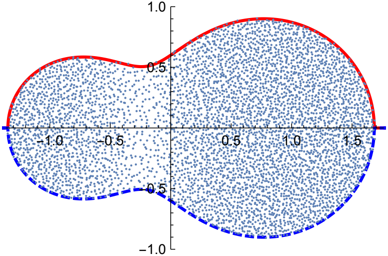
The Brown measure $\mu_{x_0+c_t}$ is absolutely continuous and its density formula (within the support) is

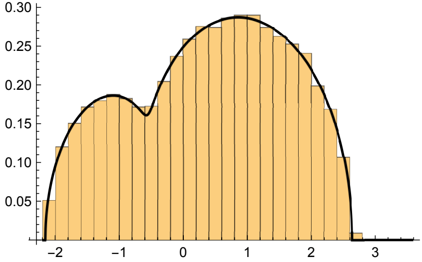
$$d\rho_t(a+ib) = \frac{1}{\pi t}\left(1 - \frac{t}{2}\frac{d}{da}\int_{\mathbb{R}} \frac{x}{(a-x)^2 + f_\nu(a)^2}\, d\nu(x)\right) db\, da,$$

where $\nu = \nu_{x_0}$ the spectral distribution of x_0.



Brown(x_0 + circular) and distribution of x_0 + semicircular





The distribution of "x_0 + semicircular" is the pushforward measure of "Brown(x_0 + circular)" under some natural map (related to subordination functions).

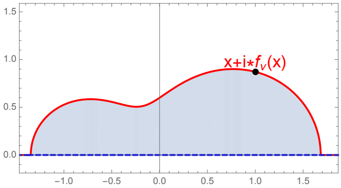
Connection with range of subordination function

Subordination function

- Cauchy transform: $G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-u} d\mu(u)$.
- Subordination function $G_{\mu_{x_0+g_t}}(z) = G_{\mu_{x_0}}(\omega(z))$.
- Then $\omega : \mathbb{C}^+ \to \mathbb{C}^+$, and

$$\omega(\mathbb{C}^+) = \mathbb{C}^+ \backslash \mathrm{supp}(Brown(x_0 + c_t)).$$

- Inverse ω^{-1} coincides with the pushforward map on the boundary.



The pushforward property

- Let x_0 be a selfadjoint operator, free from $\{c_t, g_t\}$.
- Hall-Ho (2020) calculated Brown$(x_0 + ig_t)$ for x_0 selfadjoint.

Combining Ho-Z. 2019 and Hall-Ho 2020

$$\begin{tikzcd} Brown(x_0 + c_t) \arrow[rr, "\Phi_3"] \arrow[dr, "\Phi_1"'] & & Brown(x_0 + ig_t) \arrow[dl, "\Phi_2"] \\ & Brown(x_0 + g_t) = \mu_{x_0+g_t} & \end{tikzcd}$$

Remark

The pushforward map Φ_3 is nonsingular; Φ_1, Φ_2 are singular.

Sum of a Haar unitary and an elliptic operator

Let u be a Haar unitary. Let c_t be a circular operator with variance t and g_t be a semicircular operator with variance t.

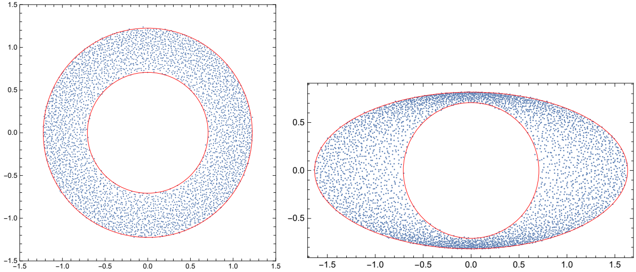


Figure: The random matrix simulation for $u + c_t$ and $u + g_t$ with $t = 0.5$.

- Similar phenomena holds for R-diagonal operator + $c_t/g_{t,\gamma}$.

Deformed random matrix model

- Our results potentially unify various *deformed random matrix models:*
  1. (finite rank/full rank) deformed Wigner random matrix (well-studied)
  $$A_N + W_N;$$
  2. (finite rank/full rank) deformed i.i.d. random matrix (Bai, Tao-Vu, Tao, Bordenave-Caputo-Chafai, Capitaine-Bordenave, etc)
  3. finite rank deformed elliptic random matrix (only finite rank perturbation was studied so far)
- work in preparation (with Yin): convergence of *full rank* deformed elliptic random matrix
- work in progress: outliers in *full rank* deformed elliptic random matrix

free probability
= non-commutative probability + freeness

Definition (Voiculescu 1985)

Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. Unital subalgebras $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ are free or freely independent, if

$$\left.\begin{array}{l} a_i \in \mathcal{A}_{j(i)}, \quad j(i) \in \mathcal{I} \\ j(1) \neq j(2), j(2) \neq j(3), \cdots, j(n-1) \neq j(n) \\ \phi(a_i) = 0, \qquad \forall i \end{array}\right\} \Rightarrow \phi(a_1 \cdots a_n) = 0.$$

Random variable $\{x_i\}_{i\in\mathcal{I}} \subset \mathcal{A}$ are free if subalgebras $\mathcal{A}_i := alg\{x_i, 1_{\mathcal{A}}\}$ are free.

Review on free additive convolution

- Given a probability measure μ on $\mathbb{R}$, define its Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-u} d\mu(u), \qquad z \in \mathbb{C}^+;$$

and Voiculescu's R-transform $R_\mu(z) = G_\mu^{\langle -1 \rangle}(z) - 1/z$.

- Let x, y be operators in $\mathcal{A}$ that are free to each other, then

$$R_{\mu_{x+y}}(z) = R_{\mu_x}(z) + R_{\mu_y}(z).$$

Hence, the R-transform linearizes free additive convolution.

Subordination functions

- Let x, y be selfadjoint operators in $\mathcal{A}$ that are free to each other.

Theorem (Voiculescu 1991, Biane 1997)

There exists analytic functions $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$G_{\mu_{x+y}}(z) = G_{\mu_x}(\omega_1(z)) = G_{\mu_y}(\omega_2(z)), \quad z \in \mathbb{C}^+.$$

Theorem (Belinschi-Bercovici 2007)

The functions ω_1, ω_2 can be obtained from the following fixed point equations

$$\omega_1(z) = z + H_{\mu_y}(z + H_{\mu_x}(\omega_1(z))), \qquad \omega_2(z) = z + H_{\mu_x}(z + H_{\mu_y}(\omega_2(z))),$$

where $H_\mu(z) = 1/G_\mu(z) - z$

Operator-valued free probability

- An *operator-valued W^*-probability space* $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ consists of a von Neumann algebra $\mathcal{A}$, a unital $*$-subalgebra $\mathcal{B} \subset \mathcal{A}$, and a *conditional expectation* $\mathbb{E} : \mathcal{A} \to \mathcal{B}$, which satisfies
  1. $\mathbb{E}(b) = b$ for all $b \in \mathcal{B}$, and
  2. $\mathbb{E}(b_1 x b_2) = b_1 \mathbb{E}(x) b_2$ for all $x \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$.
- A family of subalgebras $(\mathcal{A}_i)_{i \in I}$ ($\mathcal{B} \subset \mathcal{A}_i \subset \mathcal{A}$) is *free with amalgamation* over $\mathcal{B}$ with respect to the conditional expectation $\mathbb{E}$ if

$$\mathbb{E}(x_1 x_2 \cdots x_n) = 0$$

for every $n \geq 1$, there are indices $i_1, i_2, \cdots, i_n \in I$ such that $i_1 \neq i_2, i_2 \neq i_3, \cdots, i_{n-1} \neq i_n$, and for $j = 1, 2, \cdots, n$, we have $x_j \in \mathcal{A}_{i_j}$ such that $\mathbb{E}(x_1) = \mathbb{E}(x_2) = \cdots = \mathbb{E}(x_n) = 0$.

Operator-valued subordination functions

- Let X be a selfadjoint operator in W^*-probability space $(\mathcal{A}, \mathbb{E}, \mathcal{B})$.
- The Cauchy transform is defined in $\mathbb{H}^+(\mathcal{B}) = \{b \in \mathcal{B} : \Im b > 0\}$

$$G_X(b) = \mathbb{E}(b - X)^{-1}, \qquad \Im b > 0.$$

- Let X, Y be free with amalgamation in $(\mathcal{A}, \mathbb{E}, \mathcal{B})$.

Theorem (Voiculescu, Biane)

There exists two analytic self-maps Ω_1, Ω_2 of $\mathbb{H}^+(\mathcal{B})$, such that

$$G_{X+Y}(b) = G_X(\Omega_1(b)) = G_Y(\Omega_2(b)).$$

Some ingredients of the proof: Hermitian reduction

- Operator-valued W^*-probability space $(M_2(\mathcal{A}), \mathbb{E}, M_2(\mathbb{C}))$, where the conditional expectation $\mathbb{E} : M_2(\mathcal{A}) \to M_2(\mathbb{C})$ is

$$\mathbb{E}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \phi(a_{11}) & \phi(a_{12}) \\ \phi(a_{21}) & \phi(a_{22}) \end{bmatrix}.$$

- Hermitian reducation: for $x \in \mathcal{A}$,

$$x \longrightarrow X = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \in M_2(\mathcal{A}).$$

- Voiculescu's R-transform linearizes the addition

$$R_{X+Y}(b) = R_X(b) + R_Y(b),$$

where $R_X(b) := G_X^{\langle -1 \rangle}(b) - b.$

The operator-valued Cauchy transform

Cauchy transform

$$G_X(b) = \mathbb{E}(b - X)^{-1}.$$

We have

$$G_X\left(\begin{bmatrix} i\varepsilon & \lambda \\ \overline{\lambda} & i\varepsilon \end{bmatrix}\right) = \mathbb{E}\begin{bmatrix} i\varepsilon & \lambda - x \\ \overline{\lambda} - x^* & i\varepsilon \end{bmatrix}^{-1} = \begin{bmatrix} g_{X,11}(\lambda,\varepsilon) & g_{X,12}(\lambda,\varepsilon) \\ g_{X,21}(\lambda,\varepsilon) & g_{X,22}(\lambda,\varepsilon) \end{bmatrix}$$

where

$$\begin{aligned}
g_{X,11}(\lambda,\varepsilon) &= -i\varepsilon\phi\left(\left((\lambda-x)(\lambda-x)^* + \varepsilon^2\right)^{-1}\right) \\
g_{X,12}(\lambda,\varepsilon) &= \phi\left((\lambda-x)\left((\lambda-x)^*(\lambda-x) + \varepsilon^2\right)^{-1}\right) \\
g_{X,21}(\lambda,\varepsilon) &= \phi\left((\lambda-x)^*\left((\lambda-x)(\lambda-x)^* + \varepsilon^2\right)^{-1}\right) \\
g_{X,22}(\lambda,\varepsilon) &= -i\varepsilon\phi\left(\left((\lambda-x)^*(\lambda-x) + \varepsilon^2\right)^{-1}\right).
\end{aligned}$$

The regularized Brown measures

- The regularized Fuglede-Kadison determinnat of $x \in (\mathcal{A}, \phi)$ is defined as

$$\mathcal{D}_\varepsilon(x) = \exp\left[\frac{1}{2}\phi(\log(|x|^2 + \varepsilon^2))\right] \in (0, \infty).$$

Definition

The regularized Brown measure of x is the distributional Laplacian,

$$\mu_{x,\varepsilon} = \frac{1}{2\pi}\Delta \log \mathcal{D}_\varepsilon(x - \lambda).$$

Proposition (Haagerup-Larsen-Schultz)

The measure $\mu_{x,\varepsilon}$ is a probability measure and $\mu_{x,\varepsilon} \to \mu_x$ weakly as $\varepsilon \to 0$.

Cauchy transform and Brown measures

Cauchy transform carries important information

- Let $L_{x,\varepsilon}(\lambda) = 2\log \mathcal{D}_\varepsilon(x-\lambda) = \phi[\log((x-\lambda)^*(x-\lambda)+\varepsilon^2)]$

$$\frac{1}{2}\frac{\partial}{\partial \varepsilon} L_{x,\varepsilon}(\lambda) = i\varepsilon\phi\left(\left((\lambda-x)(\lambda-x)^*+\varepsilon^2\right)^{-1}\right) = ig_{X,11}(\lambda,\varepsilon)$$

$$\frac{\partial}{\partial \lambda} L_{x,\varepsilon}(\lambda) = \phi\left((\lambda-x)^*\left((\lambda-x)(\lambda-x)^*+\varepsilon^2\right)^{-1}\right) = g_{X,21}(\lambda,\varepsilon)$$

Free probability approach to Brown measures

How to calculate the Brown measure of $x + y$?

Dreaming some algorithm of calculation

- Find a nice formula of the matrix-valued Cauchy transform of $X + Y$,
  or find a nice formula of the FK-determinant $\mathcal{D}_\varepsilon(x + y - \lambda)$.
- Study the limit $\lim_{\varepsilon \to 0} g_{x+y,21}(\lambda, \varepsilon)$, or $\lim_{\varepsilon \to 0} \mathcal{D}_\varepsilon(x + y - \lambda)$.
- Calculate the derivative $\frac{\partial}{\partial \overline{\lambda}}$ of the limit, or the Laplacian.

Belinschi-Speicher-Śniady: for any polynomial of x, y, it is possible to calculate its Brown measure by some numerical algorithm.

Hermitian reduction of $x_0 + g_{t,\gamma}$

$$x_0 \longrightarrow X = \begin{bmatrix} 0 & x_0 \\ x_0^* & 0 \end{bmatrix} \in M_2(\mathcal{A}).$$

$$g_{t,\gamma} \longrightarrow Y = \begin{bmatrix} 0 & g_{t,\gamma} \\ g_{t,\gamma}^* & 0 \end{bmatrix} \in M_2(\mathcal{A}).$$

Proposition

The operator Y is an operator-valued semicircular element. The R-transform of Y is given by

$$R_Y(b) = \mathbb{E}(YbY) = \begin{bmatrix} a_{22}\phi(yy^*) & a_{21}\phi(yy) \\ a_{12}\phi(y^*y^*) & a_{11}\phi(y^*y) \end{bmatrix},$$

where $y = g_{t,\gamma}$ and $b = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Why Hermitian reduction method works?

$$G_{X+Y}(b) = G_X(\Omega_1(b)) = G_Y(\Omega_2(b))$$

$$R_X(b) = G_X^{\langle -1\rangle}(b) - b$$

$$R_X(b) + R_Y(b) = R_{X+Y}(b)$$

$$G_X^{\langle -1\rangle}(b) + R_Y(b) = G_{X+Y}^{\langle -1\rangle}(b)$$

$$G_X^{\langle -1\rangle}(G_{X+Y}(b)) + R_Y(G_{X+Y}(b)) = G_{X+Y}^{\langle -1\rangle}(G_{X+Y}(b))$$

- We can express the subordination function Ω_1 as

$$\Omega_1(b) = b - R_Y(G_{X+Y}(b)),$$

which is defined for all b satisfying $\Im b > \varepsilon I$ for $\varepsilon > 0$.

The Fuglede-Kadison determinant formula: circular case

Theorem

1. *If* $\phi\left[\left((x_0-\lambda)^*(x_0-\lambda)\right)^{-1}\right] > \frac{1}{t}$, *then*

$$\begin{aligned}\Delta\left(x_0+c_t-\lambda\right)^2 = {} & \Delta\left((x_0-\lambda)^*(x_0-\lambda)+w(0;\lambda,t)^2\right) \\ & \times \exp\left(-\frac{(w_t(\lambda))^2}{t}\right),\end{aligned} \tag{3}$$

where $w_t(\lambda)$ *is determined by*

$$\phi\left[\left((x_0-\lambda)^*(x_0-\lambda)+w_t(\lambda)^2\right)^{-1}\right] = \frac{1}{t}. \tag{4}$$

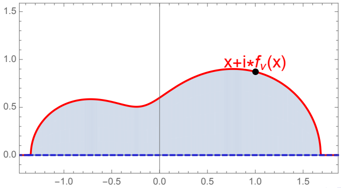
2. *If* $\phi\left[\left((x_0-\lambda)^*(x_0-\lambda)\right)^{-1}\right] \leq \frac{1}{t}$, *then*

$$\Delta(x_0+c_t-\lambda) = \Delta(x_0-\lambda).$$

The Fuglede-Kadison determinant and subordination functions

Example (Ho-Z.)

1. Given $\lambda = a + bi \in \Omega_t$, then $w_t(\lambda)^2 = f_v(a)^2 - b^2$, then
$$\Delta(x_0 + c_t - \lambda) = \left(\Delta\left((x_0-\lambda)^*(x_0-\lambda) + w_t(\lambda)^2\right)\right)^{\frac{1}{2}} \exp\left(-\frac{w_t(\lambda)^2}{2t}\right). \tag{5}$$
2. If $\lambda \in \mathbb{C}\backslash\overline{\Omega_t}$, then
$$\Delta(x_0 + c_\varepsilon - \lambda) = \Delta(x_0 - \lambda). \tag{6}$$



Strong convergence of regularized Brown measures

$$\begin{aligned}\Phi^{(\varepsilon)}_{t,\gamma}(\lambda) &= \lambda + \gamma \cdot \phi\left((\lambda - x_0)^* \left((\lambda - x_0)(\lambda - x_0)^* + w(\varepsilon; \lambda, t)^2 \right)^{-1} \right) \\ &= \lambda + \gamma \cdot \phi\left((\lambda - x_0 - c_t)^* \left((\lambda - x_0 - c_t)(\lambda - x_0 - c_t)^* + \varepsilon^2 \right)^{-1} \right)\end{aligned}$$

Lemma

The function $\Phi^{(\varepsilon)}_{t,\gamma}(\lambda)$ converges to $\Phi_{t,\gamma}(\lambda)$ uniformly in $\mathbb{C}$ as $\varepsilon \to 0$.

$$\begin{tikzcd} \mu_{x_0+c_t,\varepsilon} \arrow[r, Rightarrow, "\Phi^{(\varepsilon)}_{t,\gamma}"'] \arrow[d, "\varepsilon \to 0"'] & \mu_{x_0+g,\varepsilon} \arrow[d, "\varepsilon \to 0"] \\ \mu_{x_0+c_t} \arrow[r, Rightarrow, "\Phi_{t,\gamma}"'] & \mu_{x_0+g} \end{tikzcd}$$

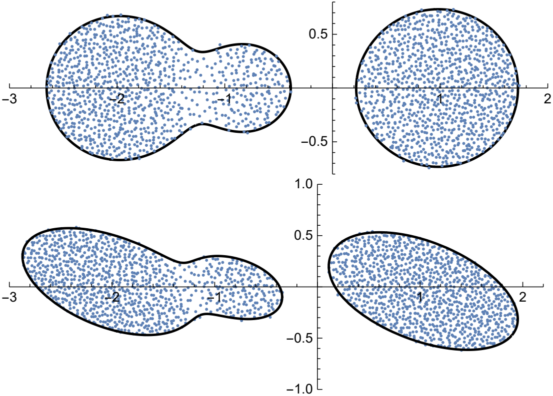
Triangular elliptic deformation

- A triangular elliptic random matrix X_N is a square matrix whose (i,j)-entry $X_N(i,j)$ is independent of every other entry except possibly $X_N(j,i)$. It generalizes elliptic model ($\alpha = \beta$).

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NN} \end{pmatrix}$$

$$\mathbb{E}(x_{ij}\overline{x_{ij}}) = \alpha, (\text{if } i < j); \qquad \mathbb{E}(x_{ij}\overline{x_{ij}}) = \beta, (\text{if } i > j).$$

- (Belinschi-Yin-Z. 2022): Brown$\left(x_0 + \text{triangular elliptic operator}\right)$ for unbounded x_0.


−3
−2
−1
1
2
0.5
−0.5
1.0
−1.0


Thank you!