Measure equivalence superrigidity for some generalized Higman groups

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joint work with Camille Horbez

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Definition

$f : X_1 \to X_2$ is a quasi-isometry iff there are constants $L, A > 0$ s.t.
(a)$L^{-1}d(x, y) - A \leqslant d(f(x), f(y)) \leqslant Ld(x, y) + A$ for all $x, y \in X_1$.
(b)$f(X_1)$ is A-dense in X_2.

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A *measure equivalent coupling* between two countable groups G and H is a measurable and measure-preserving action of $G \times H$ on some measure space such that the action of each factor is free and admits finite measure fundamental domains.

Two countable groups G and H are measure equivalent (ME) if there is a ME coupling between them.

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e.g. F_2 and $\pi_1(S_g)$ ($g \geqslant 2$) are ME, as they are lattices in $\mathrm{Isom}(\mathbb{H}^2)$.

ME from the viewpoint of ergodic theory

In this slide we consider probability measure preserving (p.m.p.), ergodic action on probability measure space (X, μ).

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Two p.m.p. actions $H \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ are orbit equivalence (OE) if there is a measure space isomorphism $T : (X, \mu) \to (Y, \nu)$ sending H-orbits to G-orbits.

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Two p.m.p. actions $H \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ are stably orbit equivalence (SOE) if there is positive measure subsets $X' \subset X$ and $Y' \subset Y$ and a measure scaling isomorphism $T' : X' \to Y'$ such that T' sends H-orbits in X' to G-orbit in Y'.

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(Furman): Two countable groups H and G are ME iff they admit free, ergodic, p.m.p. actions $H \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ that are SOE.

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More examples of ME:

Theorem (Ornstein-Weiss 1980)

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Rigidity and invariants

Theorem (Furman 1999)

Let G' be a higher rank simple Lie group, and let $G \leqslant G'$ be an irreducible lattice. Then any countable group H measure equivalent to G is virtually a lattice in G'.

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ME invariants: amenability, property (T), Haagerup property

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Summary: We need to find a robust collection of subgroups which are QI or ME invariants.

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Actually... one can only hope this is true up to a countable partition of the base space X.

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Goal of this talk:

1. A general criterion guarantee vertex stabilizers are ME-invariants (in an appropriate sense) when X is "negatively curved" and the action of G on X is acylindrical.
2. A ME-superrigid result for most generalized Higman groups.

Higman groups

Recall that the Baumslag–Solitar group $BS(n, m) = \langle a, b \mid ab^n a^{-1} = b^m \rangle$
When $n = 1, m = 2$; $BS(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle$.
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For each integer $k \geqslant 4$, Higman defined the following group:

$$\mathrm{Hig}_k = \langle a_1, \ldots, a_k | \{a_i a_{i+1} a_i^{-1} = a_{i+1}^2\}_{i \in \mathbb{Z}/k\mathbb{Z}} \rangle$$

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1. Higman groups are the first examples of infinite finitely presented groups without any nontrivial finite quotient.
2. Higman groups play a key role in the construction of Grothendieck pairs (G, H) by Platonov and Tavgen' ($G = F_n \times F_n$, $H < G$).
3. They are considered as potential examples for non-sofic groups (still open...)

Main results

Generalized Higman groups: Let $k \geqslant 4$, and let $\sigma = ((m_1, n_1), \ldots, (m_k, n_k))$ be a k-tuple of pairs of non-zero integers.

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Speculations: the theorem should still be true when $k = 4$.

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The theorem fails if $m_i = n_i = 1$.

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Observation: as long as each G_i is infinite and amenable, then H is OE to G. Hence H is ME to G.

Corollary of the main theorem

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(under the same assumption as before) Let $\mathrm{Hig}_\sigma \curvearrowright X$ *be a free, ergodic, p.m.p. action on* X*. Let* $\Gamma \curvearrowright Y$ *be a free, ergodic, p.m.p. action on* Y*. If these two actions are SOE, then they are virtually conjugate.*

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Given $\Gamma \curvearrowright Y$, there is a *cross-product von Neumann algebra*, namely the weak closure in bounded operators on $L^2(\Gamma \times Y)$ of the algebra generated by the operators $\{f(g,x) \to f(\gamma g, \gamma x) : \gamma \in \Gamma\}$ and $\{f(g,x) \to \phi(x) f(g,x) : \phi \in L^\infty(X,\mu)\}$.

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Given $\Gamma \curvearrowright Y$, there is a *cross-product von Neumann algebra*, namely the weak closure in bounded operators on $L^2(\Gamma \times Y)$ of the algebra generated by the operators $\{f(g,x) \to f(\gamma g, \gamma x) : \gamma \in \Gamma\}$ and $\{f(g,x) \to \phi(x) f(g,x) : \phi \in L^\infty(X,\mu)\}$. Combing our result with work of Adrian Ioana, we have the following:

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Proof sketch

$$G = \langle a_1, \ldots, a_k | \{a_i a_{i+1}^{m_i} a_i^{-1} = a_{i+1}^{n_i}\}_{i \in \mathbb{Z}/k\mathbb{Z}} \rangle$$

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Rmk: The assumption $k \geqslant 5$ is used in Step 1.

An auxiliary complex for generalized Higman groups

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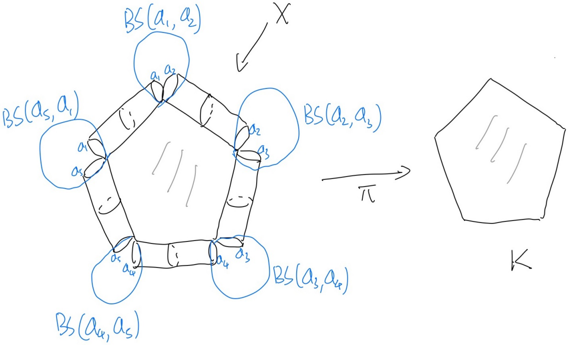

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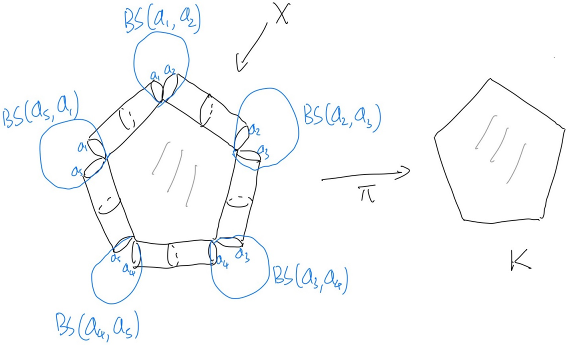

Note that $\pi_1 X = G$. Let $\widetilde{X}$ be the universal cover of X. We collapse each vertex space of $\widetilde{X}$ into a point, collapse each $[0,1] \times \mathbb{R}$ (which is a lift a cylinder in X) into the $[0,1]$ factor. The resulting space is denoted by $\hat{K}$.

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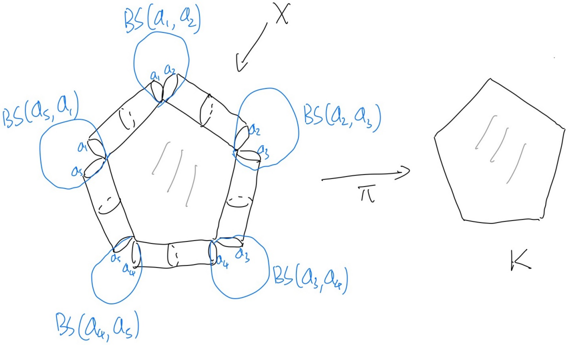

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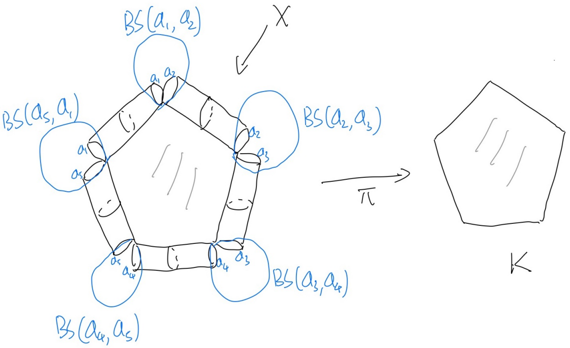

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Negative curvature of $\hat{K}$

Observation: If we metric each pentagon of $\hat{K}$ as a right-angled regular pentagon in the hyperbolic plane, then $\hat{K}$ is CAT(-1).

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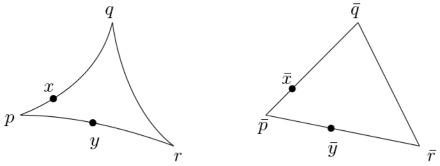
A geodesic metric space X is $CAT(-1)$ if triangles in X are thinner than those in the hyperbolic plane.



$d(x,y) \leq d(\bar{x},\bar{y})$

A general criterion for invariance of vertex groups

Theorem

[Horbez-H.] Let X be a connected $\mathrm{CAT}(-1)$ piecewise hyperbolic polyhedral complex with countably many cells in finitely many isometry types. Let G be a torsion-free countable group acting by cellular isometries on X. Assume that

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Then the collection of vertex group of G are SOE invariants in the sense explained before.

Non-isolation

Consider $G = \mathbb{Z}^2 * \mathbb{Z}^2$. Then G acts on the Bass-Serre tree with amenable stabilizers. Any tree is CAT(-1).

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Another example to have in mind: uniform lattice and non-uniform lattice acting on $\mathbb{H}^n$ are ME.

Characterization of vertex stabilizer sub-relation

Key statement: Under the assumption of the previous theorem, given a free, p.m.p., ergodic action on a probability measure space $\rho : G \curvearrowright W$ with orbit equivalence relation $\mathcal{R}$, then subrelations arising from action of vertex stabilizers can be characterized as maximal amenable subrelations which are not isolated. (up to countable partition of W)

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Let H be a group acting on a metric space Z. The H-action on Z is said to be *weakly acylindrical* if there exist $L > 0, N > 0$ such that for any two points $x, y \in Z$ with $d(x, y) \geqslant L$, the common stabilizer of x and y has cardinality at most N.

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2. If every A-invariant μ has support at most 2 pts. Then the weak acylindricity implies that A is isolated, contradiction.

Thank you!