# Fusion Algebraic Actions on Compact Quantum Spaces

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# Fusion algebra



## Definition (Hiai-Izumi, 1998)

A **fusion algebra** $R(\mathcal{J})$ is a unital algebra over $\mathbb{Z}$ with a basis $\mathcal{J}$ such that

(1) The unit $e$ is in $\mathcal{J}$.

(2) The abelian monoid $\mathbb{N}[\mathcal{J}]$ is closed under multiplication, that is, for all $\alpha, \beta$ in $\mathcal{J}$, there exists uniquely a family of nonnegative integers $(N_{\alpha,\beta}^{\gamma})_{\gamma\in\mathcal{J}}$ such that

$$\alpha\beta = \sum_{\gamma\in\mathcal{J}} N_{\alpha,\beta}^{\gamma}\gamma.$$

(3) There exists a map (called **conjugation**) $x \to \bar{x}$ on $\mathcal{J}$ such that $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$ for all $\alpha, \beta$ in $\mathcal{J}$.

(4) **Fronenius reciprocity** holds:

$$N_{\alpha,\beta}^{\gamma} = N_{\gamma,\bar{\beta}}^{\alpha} = N_{\bar{\alpha},\gamma}^{\beta}$$

for all $\alpha, \beta, \gamma \in \mathcal{J}$.

(5) There exists a map $d : \mathcal{J} \to [1, \infty)$ such that $d(x) = d(\bar{x})$ and

$$d(\alpha)d(\beta) = \sum_{\gamma \in \mathcal{J}} N_{\alpha,\beta}^{\gamma} d(\gamma),$$

for all $x, \alpha, \beta \in \mathcal{J}$. We call $d$ the **dimension function**.

# An example

Let $\Gamma$ be a discrete group. The integer group ring $\mathbb{Z}\Gamma$ is a fusion algebra.

Here $N_{s,t}^{u} = \delta_{st,u}$, $\bar{s} = s^{-1}$ and $d_s = 1$ for all $s, t, u$ in $\Gamma$.

# Fusion algebraic actions on compact quantum spaces

## Definition (Chen-Goswami-H.)

Let $A$ be a unital $C^*$-algebra, $R(\mathcal{J})$ be a fusion algebra with a basis $\mathcal{J}$, and $\mathrm{SG}_{\mathcal{J}}$ be the semigroup generated by $\mathcal{J}$ with respect to ring multiplication, i.e., $\mathrm{SG}_{\mathcal{J}} = \{\gamma_1\gamma_2\cdots\gamma_k \mid \gamma_1, ..., \gamma_k \in \mathcal{J}\}$. **An action of $R(\mathcal{J})$ on $A$** is a map $(\alpha, a) \mapsto \sigma_\alpha(a)$ from $\mathrm{SG}_{\mathcal{J}} \times A$ to $A$ such that

(1) For any fixed $\alpha$ in $\mathrm{SG}_{\mathcal{J}}$, the map $\sigma_\alpha : A \to A,\ a \mapsto \sigma_\alpha(a)$ is $\mathbb{C}$-linear, unital, norm-contractive and preserves $*-$operation.

(2) $\sigma_\alpha\sigma_\beta = \sigma_{\alpha\beta}$ for any $\alpha, \beta$ in $\mathrm{SG}_{\mathcal{J}}$.

(3) $\sigma_e$ is the identity map on $A$.

# Amenable fusion algebraic action

**Definition**

An action $\sigma : R(\mathcal{J}) \curvearrowright A$ of a fusion algebra $R(\mathcal{J})$ on a unital $C^*$-algebra $A$ is said to be **amenable**, if there exist a sequence $\{\xi_n\}_{n\geq 1}$ in $C_c(\mathcal{J}) \otimes A$ such that

(1) $\langle \xi_n, \xi_n \rangle_A = 1_A$.

(2) $\xi_n a = a\xi_n$ for any $a \in A$.

(3) $\|\delta_\gamma *_\sigma \xi_n - \xi_n\|_{2,A} \to 0$ as $n \to +\infty$, for any $\gamma \in \mathcal{J}$.

For $f = \sum_{\alpha \in \mathcal{J}} \delta_\alpha \otimes a_\alpha$, $g = \sum_{\beta \in \mathcal{J}} \delta_\beta \otimes b_\beta$ in $C_c(\mathcal{J}) \otimes A$, define the $A$-valued inner product on $C_c(\mathcal{J}) \otimes A$ by

$$\langle f, g \rangle_A := \sum_{\alpha \in \mathcal{J}} a_\alpha b_\alpha^* \in A,$$

and

$$fc := \sum_{\alpha \in \mathcal{J}} \delta_\alpha \otimes a_\alpha c \ \left( cf := \sum_{\alpha \in \mathcal{J}} \delta_\alpha \otimes c a_\alpha \right), \ \forall c \in A,$$

and a norm

$$\|f_{2,A}\| := \|\langle f, f \rangle_A\|_A^{\frac{1}{2}}.$$

The **twisted convolution** related to a fusion algebraic action $\sigma : R(\mathcal{J}) \curvearrowright A$ by

$$f *_\sigma g := \sum_{\alpha,\beta\in\mathcal{J}} l_\alpha(\delta_\beta) \otimes a_\alpha \sigma_\alpha(b_\beta),$$

Here $l_\alpha : \ell^2(\mathcal{J}) \to \ell^2(\mathcal{J})$ is given by $l_\alpha(\delta_\beta) = \frac{1}{d_\alpha} \sum_{\xi\in\mathcal{J}} N^{\xi}_{\alpha,\beta} \delta_\xi$.

# Definition of CQG



A **compact quantum group(CQG)** is a unital C*-algebra $A$ with a unital $*$-homomorphism $\Delta : A \to A \otimes A$ (minimal tensor product) satisfying:

- (Associativity)$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$;
- (Cancellation)$\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

If $A$ is commutative, then $A = C(G)$ for a compact group $G$.

# Examples



1. $C(G)$ for a compact group $G$ with $\Delta(f)(x, y) = f(xy)$ for $f$ in $C(G)$ and $x, y$ in $G$;
2. $C^*(\Gamma)$ for a discrete group $\Gamma$ with $\Delta(s) = s \otimes s$ for every $s$ in $\Gamma$.
3. For $q$ in $[-1, 1]$, define $C(SU_q(2))$ (the twisted $SU(2)$) to be the universal unital $C^*$-algebra generated by $a$ and $b$ such that the matrix $\begin{pmatrix} a & -qb^* \\ b & a^* \end{pmatrix}$ is unitary. Here $\Delta(a) = a \otimes a - qb^* \otimes b$ and $\Delta(b) = b \otimes a + a^* \otimes b$.
4. The universal unital $C^*$-algebra $C^*(p, q)$ generated by two projections $p$ and $q$ with $\Delta(p) = p \otimes p + (1 - p) \otimes (1 - p)$ and $\Delta(q) = q \otimes q + (1 - q) \otimes (1 - q)$.

There exists a unique state $h$ (**the Haar measure** of $\mathbb{G}$) on $A = C(\mathbb{G})$ such that

$$(h \otimes id)\Delta(a) = (id \otimes h)\Delta(a) = h(a)1$$

for every $a$ in $A$, and a unique functional $\varepsilon$ (**the counit** of $A$) defined on a dense $*$-subalgebra $\mathcal{A}$ of $A$ such that

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id.$$

A nondegenerate (unitary) **representation** $U$ of a compact quantum group $\mathbb{G}$ is an invertible (unitary) element in $M(K(H) \otimes A)$ for some Hilbert space $H$ satisfying that $U_{12}U_{13} = (id \otimes \Delta)U$.

$K(H)$-compact operators on $H$;
$M(K(H) \otimes A)$-the multiplier $C^*$-algebra of $K(H) \otimes A$.

For two representations $U_1$ and $U_2$ with the carrier Hilbert spaces $H_1$ and $H_2$ respectively, the set of **intertwiners** between $U_1$ and $U_2$, $\mathrm{Mor}(U_1, U_2)$, is defined by

$$\mathrm{Mor}(U_1, U_2) = \{T \in B(H_1, H_2) | (T \otimes 1)U_1 = U_2(T \otimes 1)\}.$$

Two representations $U_1$ and $U_2$ are equivalent if there exists a bijection $T$ in $\mathrm{Mor}(U_1, U_2)$. A representation $U$ is called **irreducible** if $\mathrm{Mor}(U, U) \cong \mathbb{C}$.

$\mathbb{G}$-CQG

Every irreducible representation of a CQG is finite dimensional.

$\widehat{\mathbb{G}}$- the set of equivalence classes of irreducible unitary representations of $\mathbb{G}$;
$d_\alpha$-the dimension of the carrier Hilbert space of $\alpha \in \widehat{\mathbb{G}}$;
$\chi(\alpha) = \sum_{i=1}^{d_\alpha} u_{ii}^\alpha$ is the **character** of $\alpha$ in $\widehat{\mathbb{G}}$ where $(u_{ij}^\alpha)_{1 \leq i,j \leq d_\alpha}$ is a representative of $\alpha$.

Take a representative $U^\alpha$ as $(u_{ij})_{1\leq i,j\leq d_\alpha}$ with $u_{ij} \in A$. The matrix $\overline{U^\alpha}$ is still an irreducible representation (not necessarily unitary) called the **conjugate** representation of $U^\alpha$ and the equivalence class of $\overline{U^\alpha}$ is denoted by $\bar{\alpha}$.

Given two finite dimensional representations $\alpha, \beta$ of $\mathbb{G}$, fix orthonormal bases for $\alpha$ and $\beta$ and write $\alpha, \beta$ as $U^\alpha, U^\beta$ in matrix forms respectively. The **direct sum** $\alpha + \beta$ is the equivalence class of unitary representations of dimension $d_\alpha + d_\beta$ given by $\left(\begin{smallmatrix} U^\alpha & 0 \\ 0 & U^\beta \end{smallmatrix}\right)$,

The **tensor product** $\alpha\beta$, is an equivalence class of unitary representations of dimension $d_\alpha d_\beta$ whose matrix form is given by $U^{\alpha\beta} = U^\alpha_{13} U^\beta_{23}$.

$\mathbb{G}$-CQG

Every unitary representation of $\mathbb{G}$ is a direct sum of irreducible representations.

The tensor product $\alpha\beta$ of two irreducible representations $\alpha$ and $\beta$ is a direct sum of irreducible representations:

$$\alpha\beta = \sum_{\gamma\in\widehat{\mathbb{G}}} N_{\alpha,\beta}^{\gamma}\gamma.$$

$R(\widehat{\mathbb{G}})$ is a fusion algebra with $N_{\alpha,\beta}^{\gamma}$, $\bar{\alpha}$ and $d_{\alpha}$ given above.

For

$$c_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha\in\widehat{\mathbb{G}}} B(H_\alpha) = \bigoplus_{\alpha\in\widehat{\mathbb{G}}} M_{d_\alpha}(\mathbb{C}),$$

define $\Delta_{\widehat{\mathbb{G}}} : \mathcal{M}(c_0(\widehat{\mathbb{G}})) \to \mathcal{M}(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}))$ by $\Delta_{\widehat{\mathbb{G}}}(e^{\alpha}_{ij}) = \sum_{k=1}^{d_\alpha} e^{\alpha}_{ik} \otimes e^{\alpha}_{kj}$, and $\widehat{\varepsilon} : \mathcal{M}(c_0(\widehat{\mathbb{G}})) \to \mathbb{C}$ by $\widehat{\varepsilon}(e^{\alpha}_{ij}) = 1$ for $\alpha$ being the trivial representation of $\mathbb{G}$, otherwise 0.

# Discrete quantum group actions

## Definition

A (left) action of a discrete quantum group $\widehat{\mathbb{G}}$ on a unital $C^*$-algebra $A$ is a non-degenerate $*$-homomorphism $\rho : A \to \mathcal{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$ such that

(1) $(id \otimes \rho)\rho = (\Delta_{\widehat{\mathbb{G}}} \otimes id)\rho$;

(2) $(\widehat{\varepsilon} \otimes id)\rho = id$.

Denote this action by $\rho : \widehat{\mathbb{G}} \curvearrowright A$.

A discrete quantum group action $\rho : \widehat{\mathbb{G}} \curvearrowright A$ gives a fusion algebraic action of $R(\widehat{\mathbb{G}})$ on $A$ by

$$\sigma^{\rho}_{\gamma} : a \to \frac{\chi(\gamma)}{d_{\gamma}} \cdot a$$

for all $a$ in $A$ and $\gamma$ in $\mathrm{SG}_{\widehat{\mathbb{G}}}$ with $\chi_{\gamma} \cdot a = (\chi_{\gamma} \otimes id)\rho(a)$.

For any action $\rho : \mathbb{G} \curvearrowright A$, a state $\varphi$ on $A$ is called **FA-invariant** under $\rho$ if $\varphi(\chi(\alpha) \cdot a) = d_\alpha \varphi(a)$, for every $\alpha$ in $\widehat{\mathbb{G}}$ and $a$ in $A$.

When $h$ is faithful and $\varepsilon$ can be extended to $A = C(\mathbb{G})$, we say that $\widehat{\mathbb{G}}$ is an **amenable discrete quantum group**.

Examples of amenable discrete quantum groups include amenable discrete groups, dual of compact groups and $\widehat{SU_q(2)}$.

### Theorem (Chen-Goswami-H.)

*Let $\rho : \widehat{\mathbb{G}} \curvearrowright A$ be an action of a discrete quantum group $\widehat{\mathbb{G}}$ on a compact quantum space $A$. Then the following are equivalent:*

1. *The discrete quantum group $\widehat{\mathbb{G}}$ is amenable.*
2. *The action $\rho$ is FA-amenable, and there exists an FA-invariant state on $A$.*

Thank you.