Rigidity of quantum group actions on compact connected spaces

Simeng Wang

Institute for Advanced Study in Mathematics, Harbin Institute of Technology

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WHAT IS A QUANTUM GROUP?

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Berkeley, California, USA, 1986

Quantum Groups

V. G. DRINFEL'D

This is a report on recent works on Hopf algebras (or quantum groups, which is more or less the same) motivated by the quantum inverse scattering method (QISM), a method for constructing and studying integrable quantum systems, which was developed mostly by L. D. Faddeev and his collaborators. Most of the definitions, constructions, examples, and theorems in this paper are inspired by the QISM. Nevertheless I will begin with these definitions, constructions, etc. and then explain their relation to the QISM. Thus I reverse the history of the subject, hoping to make its logic clearer.

1. What is a quantum group? Recall that both in classical and in quan-

- algebra of functions on a group G: $F(G) = \{f : G \to \mathbb{C}\}$
  - group multiplication $G \times G \to G \rightsquigarrow \Delta : F(G) \to F(G) \otimes F(G)$
  - inverse $G \to G \rightsquigarrow S : F(G) \to F(G)$
  - unit $e \in G \rightsquigarrow$ counit $\epsilon(f) = f(e)$

  $\Rightarrow$ Hopf algebra structure $(F(G), \Delta, \epsilon, S)$
- group ring $\mathbb{C}[G]$: $\Delta(g) = g \otimes g$, $\epsilon(g) = \delta_{g=e}$, $S(g) = g^{-1}$.

  $\Rightarrow$ Hopf algebra structure $(\mathbb{C}[G], \Delta, \epsilon, S)$

ORIGIN: MATHEMATICAL PHYSICS

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- Quantum groups from mathematical physics:
  Hopf algebras that provides solutions to the Quantum Yang-Baxter Equation (Faddeev et al)
- Drinfeld-Jimbo 80s': a large family of quantum groups by deformations of univesal Lie algebras

- Fourier transform on locally compact abelian groups

$$\mathcal{F} : L_1(\mathbb{T}) \to c_0(\mathbb{Z}), \qquad \mathcal{F} : L_1(\mathbb{R}) \to C_0(\mathbb{R}).$$

More generally

$$\mathcal{F} : L_1(G) \to C_0(\hat{G}),$$

where $\hat{G} = \{\gamma : G \to \mathbb{T} \text{homomorphisms}\}$ Pontryagin dual group. We have $\hat{\hat{G}} = G$

Origin: noncommutative Fourier analysis

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- For non-abelian groups, $\hat{G}$ is not a group.
  Natural question: find a large category of quantum objects $G, \hat{G}$ which fit with $\mathcal{F} : L_1(G) \to C_0(\hat{G})$.
  Answer by Enock, Schwartz, Kac, Vajnermann 70s': Kac algebra theory in terms of operator algebras

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- Drinfeld-Jimbo's quantum group goes beyond Kac algebras $\rightsquigarrow$ more general analytic theory of locally compact quantum groups (Woronowicz, Vaes,.. 80s'-00s')

Origin: noncommutative geometry à la Connes

- (locally compact) topological space $\Omega \leftrightarrow$ commutative C*-algebra $C_0(\Omega) \subset B(\ell_2(\Omega))$
  "quantum topological space" $\leftrightarrow$ noncommutative C*-algebra
- symmetries on topological spaces $\leftrightarrow$ topological groups
  "quantum symmetries on classical/quantum topological spaces"
  $\leftrightarrow$ topological quantum groups

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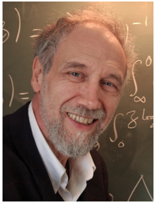

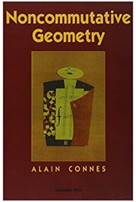

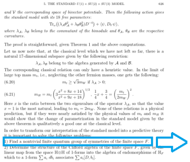

5. THE STANDARD $U(1) \times SU(2) \times SU(3)$ MODEL 626

and $\mathcal{V}$ the corresponding space of bivector potentials. Then the following action gives the standard model with its 18 free parameters:

$$\mathrm{Tr}_\omega\left((\lambda_A \theta_A^2 + \lambda_B \theta_B^2) D^{-4}\right) + \langle \psi, D_V \psi \rangle,$$

where λ_A, λ_B belong to the commutant of the bimodule and θ_A, θ_B are the respective curvatures.

The proof is straightforward, given Theorem 1 and the above computations.

Let us now note that, at the classical level which we have not left so far, there is a natural 17-dimensional subspace given by the following restriction:

$$\lambda_A, \lambda_B \text{ belong to the algebra generated by } \mathcal{A} \text{ and } \mathcal{B}.$$

The corresponding classical relation can only have a heuristic value. In the limit of large top mass m_t, i.e., neglecting the other fermion masses, one gets the following:

$$m_t \geq \sqrt{3} m_W \text{ if } \lambda_A > 0; \tag{6.20}$$

$$m_H = m_t \left(3 \frac{x^2 + 8x + 14}{x^2 + 9x + 18}\right)^{1/2}, \quad \frac{1}{4}x + \frac{3}{4} = \left(\frac{m_t}{2m_W}\right)^2. \tag{6.21}$$

Here x is the ratio between the two eigenvalues of the operator λ_A, so that the value $x = 1$ is the most natural, leading to $m_t = 2m_W$. None of these relations is a physical prediction, but if they were nearly satisfied by the physical values of m_t and m_H it would show that the change of parametrization in the standard model given by the above theorem is qualitatively a good one at the classical level.

In order to transform our interpretation of the standard model into a predictive theory it is important to solve the following problems:

1) Find a nontrivial finite quantum group of symmetries of the finite space F.

2) Determine the structure of the Clifford algebra of the finite space F, given by the linear map from the space $H \oplus H$ of 1-forms into the algebra of endomorphisms of H_F which to a 1-form $\sum a_i\, db_i$ associates $\sum a_i [D, b_i]$.

"... it is important to solve the following problems:
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– Alain Connes

- A (classical) permutation matrix $C = [c_{ij}]_{1 \le i,j \le N} \in S_N$ is such that

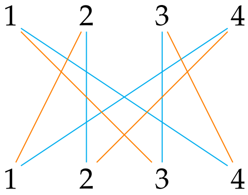
$$c_{ij} \in \{0, 1\}, \quad CC^t = C^tC = I.$$

The algebra $C(S_N)$ of functions on S_N is generated by the functions $C \mapsto c_{ij}$.

- Quantum permutations (Shuzhou Wang): Consider the universal C*-algebra A generated by operators $(u_{ij})_{1 \le i,j \le N}$ s.t. for the matrix $U = [u_{ij}]_{1 \le i,j \le N}$,

$$u_{ij} = u_{ij}^* = u_{ij}^2, \quad UU^t = U^tU = I.$$

Intuitive notation: quantum permutation group $S_N^+ = (A, U)$ and $A =$"$C(S_N^+)$".



$$\begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$$

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- Interpretation in language of quantum physics: u_{ij}'s = observables measurement on a quantum state ξ $\to$ random permutation

$$\mathbb{P}(i \to j) = \langle \xi | u_{ij} | \xi \rangle$$

Atserias,Lupini,Mancinska,Roberson,..., 19′-20′: quantum permutations much better than the classicals one when contructing strategies in non-local games on graphs.

General framework: Compact matrix quantum groups

- Compact linear group $G \subset U_N(\mathbb{C})$
  - $u_{ij} : G \to \mathbb{R}, g \mapsto g_{ij}$ coordinate function, $C(G) = C^*(u_{ij} \mid i,j = 1, \ldots, N)$
  - $U := [u_{ij}]_{i,j} \in \mathbb{M}_N(C(G))$ unitary.
  - Group multiplication $\leftrightarrow \Delta(u_{ij})(g,h) := u_{ij}(gh) = \sum_k u_{ik}(g)u_{kj}(h)$.
- Woronowicz: compact matrix quantum group $\mathbb{G} = (A, U)$ such that
  - $A = C^*(u_{ij} \mid i,j = 1, \ldots, N)$
  - $U := [u_{ij}]_{i,j} \in \mathbb{M}_N(A)$ and $\bar{U} := [u^*_{ij}]_{i,j} \in \mathbb{M}_N(A)$ are unitary;
  - $*$-homomorphism $\Delta : A \to A \otimes A, \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$

Intuitively, write $A = C(\mathbb{G})$.

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- More general framework of locally compact quantum groups (not necessarily matricial): Woronowicz, Vaes, etc..

Actions of compact matrix quantum groups

Recall the quantum permutation group $S_N^+ = (A, U), A = C^*(u_{ij} \mid i, j = 1, \ldots, N)$ acts on the C*-algebra $\mathbb{C}^N = C(\{1, \ldots, N\})$ by usual matrix operations:

$$\delta_i \mapsto \sum_k \delta_k \otimes u_{ki}, \quad \forall i \in \{1, \ldots, N\}.$$

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Definition

An action of a compact matrix quantum group $\mathbb{G}$ on a unital C*-algebra B is an $*$-homomorphism $\alpha : B \to B \otimes C(\mathbb{G})$ with

- (coaction property) $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$
- (nondegeneracy condition) $\overline{\mathrm{span}}\{(1 \otimes C(\mathbb{G}))\alpha(B)\} = B \otimes C(\mathbb{G})$

α is called ergodic if the fixed point space

$$B^{\mathbb{G}} := \{b \in B \mid \alpha(b) = b \otimes 1\} = \mathbb{C}1.$$

Partitions & rigidity of actions

- Goswami 11′:
  - "there has not been any example of faithful action of a genuine compact quantum group on $C(X)$ when X is connected."
  - "conjecture that the quantum permutations are the only possible actions of genuine compact quantum groups on classical spaces".
- Huang 13′:
  - construct faithful non-ergodic $S_N^+ \curvearrowright$ compact connected X
  - reformulated question: faithful ergodic genuine quantum group actions on compact connected spaces?
- Goswami-Joardar GAFA 18′, Goswami Adv 20′:
  no genuine quantum actions on compact connected smooth manifolds.

Theorem (Freslon-Taipe-W.)

S_N^+ (and many other "easy" quantum groups) cannot act ergodically on a compact connected X unless X is a point.

In the sequel consider the case $u_{ij} = u_{ij}^*$.

- $u^{\otimes k} := (u_{i_1 j_1} \cdots u_{i_k j_k}) \in \mathbb{M}_N^{\otimes k} \otimes A$ with intertwiners

$$\mathrm{Mor}_{\mathbb{G}}(k,l) := \{T : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l} T\},\ k,l \in \mathbb{N}$$

- A collection of vector spaces of operators

$$\mathcal{R}_{\mathbb{G}} := \cup_{k,l} \mathrm{Mor}_{\mathbb{G}}(k,l) \subset \cup_{k,l} B((\mathbb{C}^N)^{\otimes k}, (\mathbb{C}^N)^{\otimes l})$$

  - contains $\mathrm{id}_{\mathbb{C}^N}$ and $1 \mapsto \sum_k e_k \otimes e_k$
  - stable under $\circ, \otimes, *$

- Tannaka-Krein reconstruction (Woronowicz 88′):

$$\mathbb{G} \leftrightarrow \mathcal{R}_{\mathbb{G}} \quad \text{one-to-one correspondence}$$

PARTITIONS & QUANTUM GROUPS

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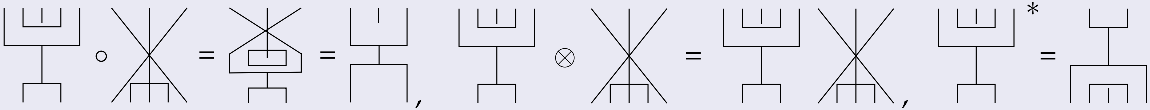
Category of partitions

A collection of partitions of two lines of points, such that:

- contains $|$ and $\sqcap$ - stable under $\circ, \otimes, *$



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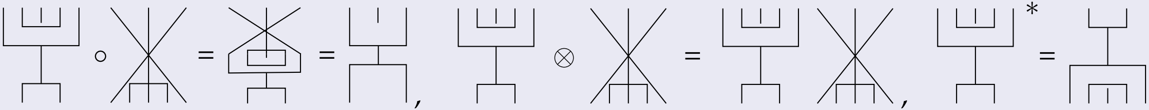

- Banica-Speicher:

$$\text{any category of partitions} \to \mathcal{R}_{\mathbb{G}} \text{ for some } \mathbb{G}$$

Such a $\mathbb{G}$ is called an "easy quantum group".

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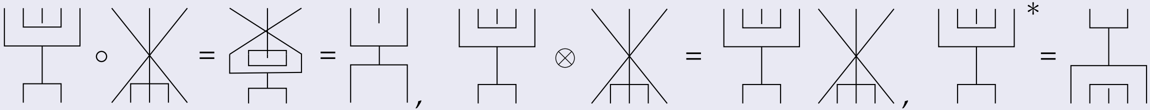

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category of all partitions $\to \mathcal{R}_{S_N}$,   category of all noncrossing partitions $\to \mathcal{R}_{S_N^+}$

B unital C*-algebra

An action of $\mathbb{G}$ on B is an $*$-homomorphism $\alpha : B \to B \otimes C(\mathbb{G})$ with

- (coaction property) $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$
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α is called ergodic if the fixed point space

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Particular example: $B = C(\mathbb{G})$ and $\alpha = \Delta : u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$

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Dual viewpoint: consider an **ergodic** α for the sequel.
Recall $u^{\otimes k} : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes k} \otimes C(\mathbb{G})$ & $\alpha : B \to B \otimes C(\mathbb{G})$
$\Rightarrow \mathbb{G} \curvearrowright (\mathbb{C}^N)^{\otimes k} \otimes B$ by

$$(u^{\otimes k})_{(13)}\alpha_{(23)} : (\mathbb{C}^N)^{\otimes k} \otimes B \to (\mathbb{C}^N)^{\otimes k} \otimes B \otimes C(\mathbb{G}).$$

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- The fixed point space $K_k := ((\mathbb{C}^N)^{\otimes k} \otimes B)^{\mathbb{G}}$ is a Hilbert space.
- $K_k = (\mathbb{C}^N)^{\otimes k}$ if $B = C(\mathbb{G})$ and $\alpha = \Delta$ (comultiplication); $K_k \otimes K_l \subset K_{k+l}$.
- $T \in \mathrm{Mor}_{\mathbb{G}}(k, l) \Rightarrow T \otimes \mathrm{id} \in B(K_k, K_l)$.

Recall Tannaka-Krein reconstruction for quantum groups

$$\mathbb{G} \leftrightarrow \mathcal{R}_{\mathbb{G}} := \cup_{k,l}\mathrm{Mor}_{\mathbb{G}}(k,l)$$

$\Rightarrow$ recognize $\mathbb{G}$ (in particular the action $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$) out of categorical data:

- $u^{\otimes k} \otimes u^{\otimes l} \cong u^{\otimes k+l} \to$ Hilbert spaces $((\mathbb{C}^N)^{\otimes k})_{k\in\mathbb{N}}$ with unitary $(\mathbb{C}^N)^{\otimes k} \otimes (\mathbb{C}^N)^{\otimes l} \cong (\mathbb{C}^N)^{\otimes k+l}$
- $T \in \mathrm{Mor}_{\mathbb{G}}(k,l) \to T \in B((\mathbb{C}^N)^{\otimes k}, (\mathbb{C}^N)^{\otimes l})$, compatible with $\otimes, \circ, *$.

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- $T \in \mathrm{Mor}_{\mathbb{G}}(k,l) \to T \in B((\mathbb{C}^N)^{\otimes k}, (\mathbb{C}^N)^{\otimes l})$, compatible with $\otimes, \circ, *$.

Theorem (Pinzari-Roberts 08', Neshveyev 14', Freslon-Taipe-W.)

Assume that we have

- *Hilbert spaces $(K_k)_{k\in\mathbb{N}}$ with isometric inclusions $\iota : K_k \otimes K_l \subset K_{k+l}$*
- *$T \in \mathrm{Mor}_{\mathbb{G}}(k,l) \to \varphi(T) \in B(K_k, K_l)$, compatible with $\otimes, \circ, *, \iota$.*

Then we may construct $\alpha : B \to B \otimes C(\mathbb{G})$ (with $K_k = ((\mathbb{C}^N)^{\otimes k} \otimes B)^{\mathbb{G}}$).

Partitions & rigidity of actions

Theorem (Freslon-Taipe-W.)

Free easy quantum groups ($\mathbb{G}_N(\mathcal{C})$ for noncrossing $\mathcal{C}$, e.g., $O_N^+, S_N^+, \ldots$) cannot act ergodically on a compact connected X unless X is a point.

idea of the proof via Tannaka-Krein duality:

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- multiplication in $C(X) \rightleftarrows$ decomposition into irreducible components

$$(H_u \otimes C(X))^{\mathbb{G}} \otimes (H_{u'} \otimes C(X))^{\mathbb{G}} \subset \oplus_v (H_v \otimes C(X))^{\mathbb{G}}$$

- commutativity of $C(X) \rightleftarrows$ symmetries in the decomposition

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- commutativity of $C(X) \rightleftarrows$ symmetries in the decomposition
  $\Rightarrow$ existence of symmetric/antisymmetric vectors of $H_v \subset H_u \otimes H_u$ for $u, v \in \mathrm{Irr}(\mathbb{G})$

Theorem (Freslon-Taipe-W.)

Free easy quantum groups ($\mathbb{G}_N(\mathcal{C})$ for noncrossing $\mathcal{C}$, e.g., $O_N^+, S_N^+, \ldots$) cannot act ergodically on a compact connected X unless X is a point.

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Partitions & rigidity of actions

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- For most v, $H_v \subset H_u \otimes H_u$ has no symmetric/antisymmetric vectors
- enforce a nontrivial finite-dimensional C*-subalgebra of $C(X)$.

Thank you very much!