The volume of a compact hyperbolic antiprism

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Introduction

Calculating volumes of polyhedra is a classical problem, that has been well known since Euclid and remains relevant nowadays. This is partly due to the fact that the volume of a fundamental polyhedron is one of the main geometrical invariants for a 3-dimensional manifold.

Every 3-manifold can be presented by a fundamental polyhedron. That means we can pair-wise identify the faces of some polyhedron to construct a 3-manifold. Thus the volume of 3-manifold is the volume of its fundamental polyhedron.

Theorem (Thurston, Jørgensen)

The volumes of 3-dimensional hyperbolic manifolds form a closed non-discrete set on the real line. This set is well ordered. There are only finitely many manifolds with a given volume.

Some motivation to find exact volume formulas

It is difficult problem to find the exact volume formulas for hyperbolic polyhedra of prescribed combinatorial type. It was done for hyperbolic tetrahedron of general type, but even for general hyperbolic octahedron it is an open problem.

Nevertheless, if we know that a polyhedron has a symmetry, then the volume calculation is essentially simplified. Firstly this effect was shown by Lobachevsky. He found the volume of an ideal tetrahedron, which is symmetric by definition.

First examples of hyperbolic 3-manifolds

1914 Gieseking found first example of hyperbolic manifold (non-compact, non-orientable)

1929 Klein wrote in his book «Non-Euclidean Geometry» that examples of compact hyperbolic 3-manifolds are unknown

1931 Löbell presented the example of compact orientable hyperbolic 3-manifold

1933 Weber and Seifert constructed compact orientable «dodecahedral hyperbolic space»

Examples of geom. structures on knots complements in $\mathbb{S}^3$

1975 R. Riley found first examples of hyperbolic structures on seven *excellent* knots and links in $\mathbb{S}^3$.

1977 W. Thurston showed that a complement of any prime knot admits a hyperbolic structure if this knot is not toric or satellite one.

1980 W. Thurston constructed a hyperbolic 3-manifold homeomorphic to the complement of knot 4_1 in $\mathbb{S}^3$ by gluing faces of two regular ideal tetrahedra. This manifold has a complete hyperbolic structure.

1982 J. Minkus suggested a general topological construction for the orbifold whose singular set is a two-bridge knot in $\mathbb{S}^3$.

2004 H. Hilden, J. Montesinos, D. Tejada, M. Toro considered more general topological construction known as *butterfly*.

1998/2006 A. Mednykh, A. Rasskazov found a geometrical realisation of the Minkus construction in $\mathbb{H}^3, \mathbb{S}^3, \mathbb{E}^3$.

2009 E. Molnár, J. Szirmai, A. Vesnin realised the figure-eight knot cone-manifold in the five exotic Thurston's geometries.

Upper half-space model of hyperbolic 3-space

Denote by $\mathbb{H}^3$ a 3-dim *hyperbolic space* (*Lobachevsky–Boljai–Gauss space*). $\mathbb{H}^3$ can be modelled in $\mathbb{R}^3_+ = \{(x, y, t) \ : \ x, y, t \in \mathbb{R}, t > 0\}$ with metric s given by expression $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$.
The boundary $\partial\mathbb{H}^3 = \{(x, y, 0) \ : \ x, y \in \mathbb{R}\}$ caled *absolute* and consist of points at infinity.

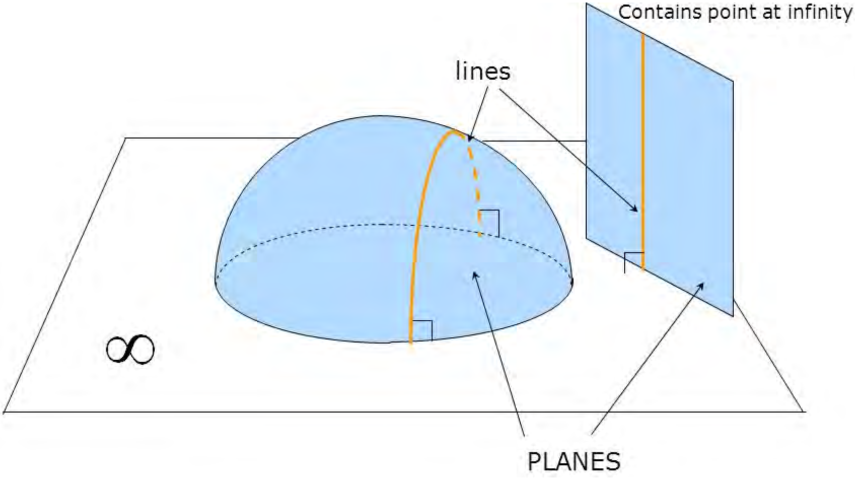
Isometry group $\mathrm{Isom}(\mathbb{H}^3)$ is a group of all actions on $\mathbb{H}^3$ preserving the metric s. Denote by $\mathrm{Isom}^+(\mathbb{H}^3)$ the group of orientation preserving isometries.

$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathbf{PSL}(2, \mathbb{C})$ (Pozitive Special Lorentz group). An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$ by the rule

$$g \ : \ (z, t) \mapsto \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2\, t^2}, \frac{t}{|cz + d|^2 + |c|^2\, t^2} \right),$$

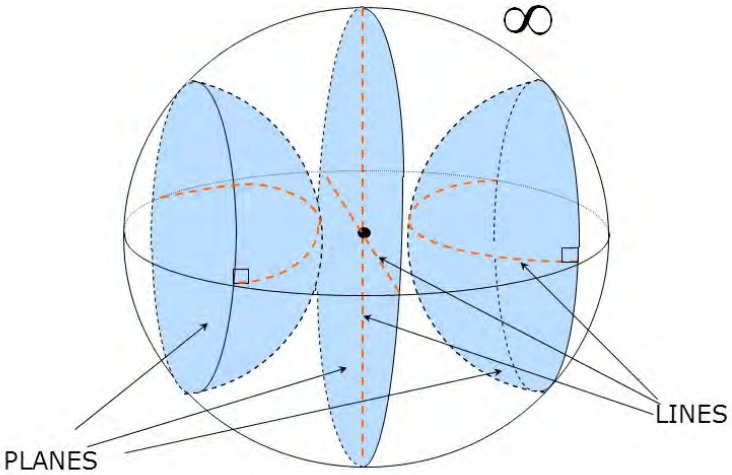
where $z = x + i\, y$.

Geodesic lines and planes in half-space model of $\mathbb{H}^3$



$\mathrm{Isom}(\mathbb{H}^3)$ is generated by reflections with respect to geodesic planes.

Geodesic lines and planes in ball model of $\mathbb{H}^3$



$\mathrm{Isom}(\mathbb{H}^3)$ is generated by reflections with respect to geodesic planes.

Caley-Klein model of hyperbolic 3-space

Consider *Minkowski space* R^4_1 with *Lorentz scalar product*

$$\langle X, Y \rangle = -x_1\, y_1 - x_2\, y_2 - x_3\, y_3 + x_4\, y_4. \qquad (1)$$

The *Caley-Klein model* of hyperbolic space is the set of vectors $K = \{(x_1, x_2, x_3, 1) \; : \; x_1^2 + x_2^2 + x_3^2 < 1\}$ forming the unit 3-ball in the hyperplane $x_4 = 1$. The lines and planes in K are just the intersections of ball K with Euclidean lines and planes in the hyperplane $x_4 = 1$.

The *distance* between vectors V and W is defined as

$$\cosh \rho\,(V, W) = \frac{\langle V, W \rangle}{\sqrt{\langle V, V \rangle \, \langle W, W \rangle}}. \qquad (2)$$

A plane in K is a set $\mathcal{P} = \{V \in K \; : \; \langle V, N \rangle = 0\}$, where N is a normal vector to the plane $\mathcal{P}$.

Every of four *dihedral angles* between the planes $\mathcal{P}, \mathcal{Q}$ with normal vectors N, M are defined by relation

$$\cos \widehat{(\mathcal{P}, \mathcal{Q})} = \pm \frac{\langle N, M \rangle}{\sqrt{\langle N, N \rangle \, \langle M, M \rangle}}. \qquad (3)$$

Constructing manifolds from polyhedra

Consider a *right-angled polyhedron* P (i.e. all the dihedral and planar angles of P are $\pi/2$). In Euclidean space we can take a cube. In the spherical space there is a right-angled tetrahedron ($1/8$ part of $\mathbb{S}^3$). In the hyperbolic space there are infinitely many right-angled polyhedra.

The class of polyhedra that can be realised in hyperbolic geometry with right angles is referred as *Pogorelov polyhedra*.

It follows from Andreev theorem (1968) that any polyhedron which has no triangle and quadrilateral faces and such that any its vertex is of valency 3, can be realised as right-angled polyhedron in $\mathbb{H}^3$.

Example

- n-gonal Löbell prism $R(n), n > 4$;
- all combinatorial fullerenes (including known in chemistry $C_{60}, C_{70}, C_{78}, C_{84}, C_{200}$ etc.)

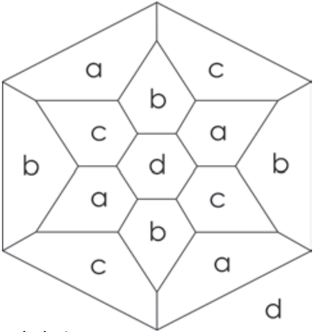
$R(5) = C_{12} = \textit{dodecahedron}.$

Constructing manifolds from polyhedra

To construct hyperbolic manifolds one can follow the algorithm:

1. take a compact right-angled hyperbolic polyhedron P;
2. set a regular colouring of the faces of P (the incidental faces should have different colours; the number of colours will be from 3 to 7);
3. pairwise identify the faces of same colour of several copies of P.

This approach was originally used by Löbell for $R(6)$ (1931) to construct the first example of a compact hyperbolic manifold. M. Takahashi (1985) do this for regular right-angled dodecahedron (or $R(5)$). A. Vesnin (1987) generalised this construction for any compact hyperbolic right-angled polyhedron P. All the manifolds constructed by colourings in 4 colours are orientable. If one use $5, 6$ or 7 colours then non-orientable hyperbolic manifolds can be produced.



Consider a Löbell prism $R(6)$ having 12 pentagonal lateral faces. Let us set the colouring with 4 colours a, b, c, d. We take 8 copies of this coloured $R(6)$. Then identify faces of this 8 copies using the rule:

$$a : (15)(26)(37)(48)$$

$$b : (16)(25)(38)(47)$$

$$c : (17)(28)(35)(46)$$

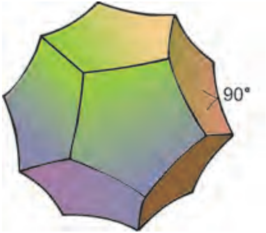
$$d : (18)(27)(36)(45)$$

Constructing manifolds from polyhedra

Denote by D the regular right-angled hyperbolic dodecahedron.

Theorem (Montesinos, 1987)

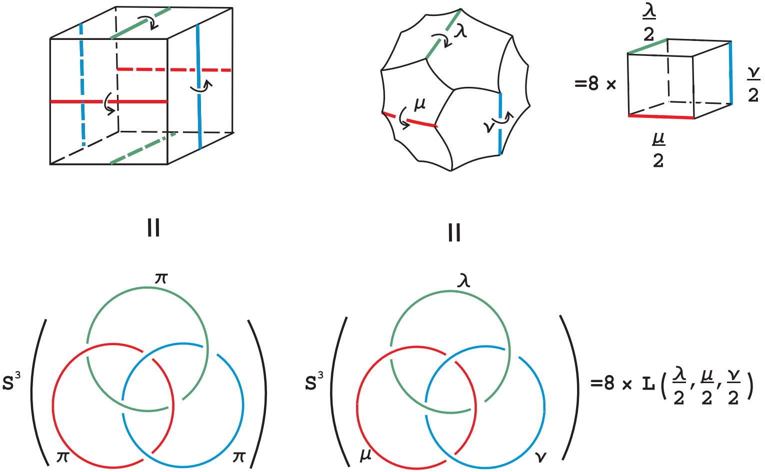
Any closed hyperbolic 3-manifold can be constructed from a finite number of copies of D by some toplogical construction involving pairwise identifying of their faces and coverings.



After gluing in this theorem, each edge will be surrounded by either 4 or 2 dodecahedra or just by 1 dodecahedron. In the first case there is no singularity along this edge (the sum of angles is 2π). If the second or the third case happens then an orbifold arises.

- Borromean Rings cone-manifold and Lambert cube

This construction done by W. Thurston, D. Sullivan and J.M. Montesinos.



From polyhedra to knots and links

From the above consideration we get

$$\mathrm{Vol}\, B(\lambda, \mu, \nu) = 8 \cdot \mathrm{Vol}\, L\left(\frac{\lambda}{2}, \frac{\mu}{2}, \frac{\nu}{2}\right).$$

Recall that $B(\lambda, \mu, \nu)$ is

i) hyperbolic iff $0 < \lambda, \mu, \nu < \pi$ **(E.M. Andreev)**

ii) Euclidean iff $\lambda = \mu = \nu = \pi$

iii) spherical iff $\pi < \lambda, \mu, \nu < 3\pi, \quad \lambda, \mu, \nu \neq 2\pi$
(R. Diaz, D. Derevnin, A. Mednykh)

From polyhedra to knots and links

- Volume calculation for $L(\alpha,\beta,\gamma)$. The main idea.

0. Existence

$$L(\alpha,\beta,\gamma):\begin{cases} 0<\alpha,\beta,\gamma<\pi/2, & \mathbb{H}^3 \\ \alpha=\beta=\gamma=\pi/2, & \mathbb{E}^3 \\ \pi/2<\alpha,\beta,\gamma<\pi, & \mathbb{S}^3. \end{cases}$$

1. Schläfli formula for $V = Vol\, L(\alpha,\beta,\gamma)$

$$k\,\mathrm{d}V=\frac{1}{2}(\ell_\alpha d\alpha+\ell_\beta d\beta+\ell_\gamma d\gamma), \quad k=\pm 1, 0\ (curvature)$$

In particular in hyperbolic case:

$$\begin{cases} \dfrac{\partial V}{\partial \alpha}=-\dfrac{\ell_\alpha}{2}, \quad \dfrac{\partial V}{\partial \beta}=-\dfrac{\ell_\beta}{2}, \quad \dfrac{\partial V}{\partial \gamma}=-\dfrac{\ell_\gamma}{2} & (*) \\ \mathrm{Vol}\ L\left(\dfrac{\pi}{2},\dfrac{\pi}{2},\dfrac{\pi}{2}\right)=0. & (**) \end{cases}$$

2. Relations between lengths and angles

(i) Tangent Rule

$$\frac{\tan\alpha}{\tanh\ell_\alpha} = \frac{\tan\beta}{\tanh\ell_\beta} = \frac{\tan\gamma}{\tanh\ell_\gamma} =: T \quad (\textbf{Kellerhals})$$

(ii) Sine-Cosine Rule (3 different cases)

$$\frac{\sin\alpha}{\sinh\ell_\alpha}\frac{\sin\beta}{\sinh\ell_\beta}\frac{\cos\gamma}{\cosh\ell_\gamma} = 1 \quad (\textbf{Derevnin}, \textbf{Mednykh})$$

(iii) Tangent Rule

$$\frac{T^2 - A^2}{1 + A^2}\frac{T^2 - B^2}{1 + B^2}\frac{T^2 - C^2}{1 + C^2}\frac{1}{T^2} = 1, \quad (\textbf{Hilden}, \textbf{Lozano}, \textbf{Montesinos})$$

where $A = \tan\alpha, B = \tan\beta, C = \tan\gamma$. Equivalently,

$$(T^2 + 1)(T^4 - (A^2 + B^2 + C^2 + 1)T^2 + A^2B^2C^2) = 0.$$

Remark. (ii) ⇒(i) and (i) & (ii) ⇒ (iii).

3. Integral formula for volume

Hyperbolic volume of $L(\alpha, \beta, \gamma)$ is given by

$$W = \frac{1}{4} \int\limits_{T}^{\infty} \log \left(\frac{t^2 - A^2}{1 + A^2} \, \frac{t^2 - B^2}{1 + B^2} \, \frac{t^2 - C^2}{1 + C^2} \, \frac{1}{t^2} \right) \frac{\mathrm{d}t}{1 + t^2},$$

where T is a positive root of the integrant equation (iii).

Proof. By direct calculation and Tangent Rule (i) we have:

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A} \frac{\partial A}{\partial \alpha} = -\frac{1}{2} \arctan \frac{A}{T} = -\frac{\ell_\alpha}{2}.$$

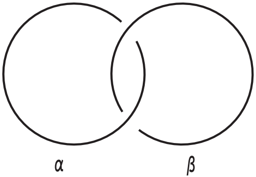
In a similar way

$$\frac{\partial W}{\partial \beta} = -\frac{\ell_\beta}{2} \quad \text{and} \quad \frac{\partial \mathrm{W}}{\partial \gamma} = -\frac{\ell_\gamma}{2}.$$

By convergence of the integral $W(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0$. Hence, $W = V = Vol\, L(\alpha, \beta, \gamma)$.

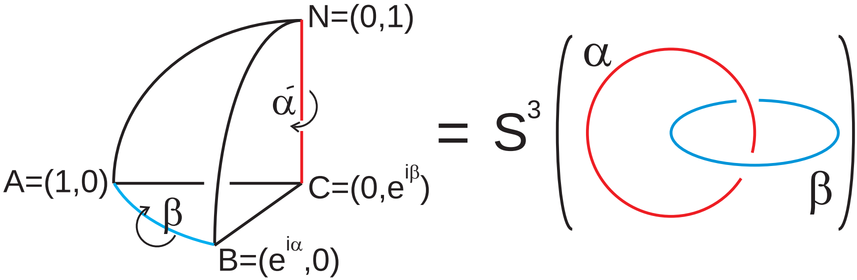
- The Hopf link

The Hopf link 2_1^2 is the simplest two component link.



The fundamental group $\pi_1(\mathbb{S}^3 \backslash 2_1^2) = \mathbb{Z}^2$ is a free Abelian group of rank 2. It makes us sure that **any finite covering of $\mathbb{S}^3 \backslash 2_1^2$ is homeomorphic to $\mathbb{S}^3 \backslash 2_1^2$ again**.
The orbifold $2_1^2(\pi, \pi)$ arises as a factor space by $\mathbb{Z}_2$-action on the projective space $\mathbb{P}^3$. That is, $\mathbb{P}^3$ **is a two-fold covering of the sphere $\mathbb{S}^3$ branched over the Hopf link**. It turns that **the sphere $\mathbb{S}^3$ is a two-fold unbranched covering of the projective space $\mathbb{P}^3$**.

- The Hopf link (Construction by Abr. and Mednykh)



Fundamental tetrahedron

$$\mathcal{T}(\alpha,\beta) = \mathcal{T}\left(\alpha,\beta,\tfrac{\pi}{2},\tfrac{\pi}{2},\tfrac{\pi}{2},\tfrac{\pi}{2}\right) \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{C}\times\mathbb{C}$$

for the cone-manifold $2^2_1(\alpha,\beta)$.

Relations between lengths and angles: $\ell_\alpha = \beta,\ \ell_\beta = \alpha.$

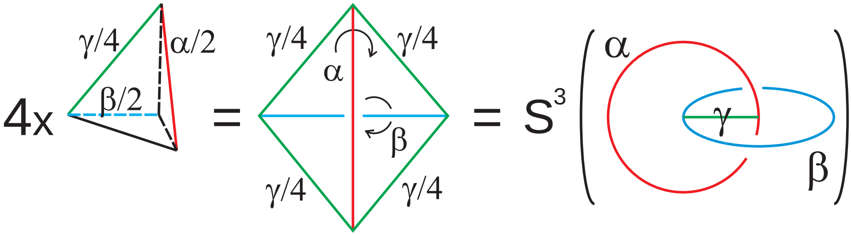
Theorem (Abr., Mednykh)

The Hopf link cone-manifold $2^2_1(\alpha, \beta)$ is spherical for all positive α and β. The spherical volume is given by the formula

$$\mathrm{Vol}\; 2^2_1(\alpha, \beta) = \frac{\alpha\,\beta}{2}.$$

Proof. Let $0 < \alpha, \beta \leqslant \pi$. Consider a spherical tetrahedron $\mathcal{T}(\alpha, \beta)$ with dihedral angles α and β prescribed to the opposite edges and with right angles prescribed to the remained ones. To obtain a cone-manifold $2^2_1(\alpha, \beta)$ we identify the faces of tetrahedron by α- and β-rotations in the respective edges. Hence, $2^2_1(\alpha, \beta)$ is spherical and $\mathrm{Vol}\; 2^2_1(\alpha, \beta) = \mathrm{Vol}\, \mathcal{T}(\alpha,\, \beta) = \frac{\alpha\,\beta}{2}$. We note that $\mathcal{T}(\alpha,\, \beta)$ is a union of n^2 tetrahedra $\mathcal{T}(\frac{\alpha}{n},\, \frac{\beta}{n})$. Hence, for large positive α and β we also obtain $\mathrm{Vol}\; 2^2_1(\alpha, \beta) = n^2 \cdot \mathrm{Vol}\, \mathcal{T}(\frac{\alpha}{n},\, \frac{\beta}{n}) = \frac{\alpha\,\beta}{2}$.

- The Hopf link with bridge (Construction by Abr. and Mednykh)



Fundamental tetrahedron $\mathcal{T}\left(\alpha, \beta, \frac{\gamma}{4}, \frac{\gamma}{4}, \frac{\gamma}{4}, \frac{\gamma}{4}\right)$

for the Hopf link with bridge cone-manifold $\mathcal{H}(\alpha, \beta; \gamma)$.

Geometry of two bridge knots and links

- The Hopf link with bridge

Relations between lengths and angles:

Tangent Rule (Abr., Mednykh)

$$\tan\frac{\alpha}{2}\,\tanh\frac{\ell_\alpha}{2} = \frac{\tanh \ell_\gamma}{\tan\frac{\gamma}{4}} = \tan\frac{\beta}{2}\,\tanh\frac{\ell_\beta}{2}$$

Sine-Cosine Rule (Abr., Mednykh)

$$\frac{\cos\frac{\gamma}{4}}{\cosh \ell_\gamma} = \frac{\sin\frac{\alpha}{2}}{\cosh\frac{\ell_\alpha}{2}} \cdot \frac{\sin\frac{\beta}{2}}{\cosh\frac{\ell_\beta}{2}}$$

Given α, β, γ these theorems are sufficient to determine $\ell_\alpha, \ell_\beta, \ell_\gamma$. This allows us to use Schläfli equation: we are able to solve the system of PDE's to get the volume formula.

- The Hopf link with bridge

Theorem (Abr., Mednykh)

The Hopf link with bridge cone manifold $\mathcal{H}(\alpha,\beta;\gamma)$ *is hyperbolic for any* $\alpha,\beta\in(0,\pi)$ *if and only if*

$$\begin{cases}\gamma > 2(\pi-\alpha)\\ \gamma > 2(\pi-\beta)\\ \gamma < 2\pi\end{cases}$$

The hyperbolic volume is given by the formula

$\mathrm{Vol}\,\mathcal{H}(\alpha,\beta;\gamma) = i\cdot S\left(\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{4}\right)$, *where* $S\left(\frac{\pi}{2}-x,y,\frac{\pi}{2}-z\right) =$

$$\tilde{S}(x,y,z) = \sum_{m=1}^{\infty}\left(\frac{D-\sin x\sin z}{D+\sin x\sin z}\right)^m\cdot\frac{\cos 2mx-\cos 2my+\cos 2mz-1}{m^2} - x^2+y^2-z^2$$

is the Schläfli function.

Antiprism $\mathcal{A}_n$

Definition

An *antiprism* $\mathcal{A}_n$ is a convex polyhedron with two equal regular n-gons as the top and the bottom and $2n$ equal triangles as the lateral faces.

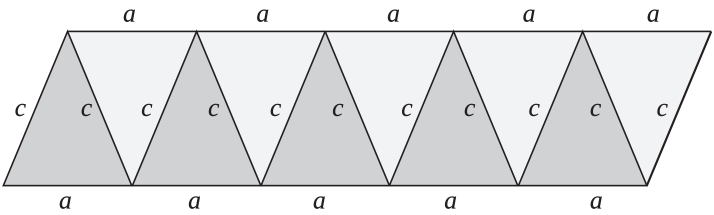


Fig.: The lateral faces of antiprism $\mathcal{A}_5$

The antiprism can be regarded as a drum with triangular sides (see Fig. where for $n = 5$ the lateral boundary is shown).

Antiprism $\mathcal{A}_n$

An antiprism $\mathcal{A}_n$ with $2n$ vertices admits a symmetry group S_{2n} generated by a mirror-rotational symmetry of order $2n$ denoted by $C_{2n\,h}$ (in Shönflies notation). The element $C_{2n\,h}$ is a composition of a rotation by the angle of π/n about an axis passing through the centres of the top and the bottom faces and reflection with respect to a plane perpendicular to this axis and passing through the middles of the lateral edges

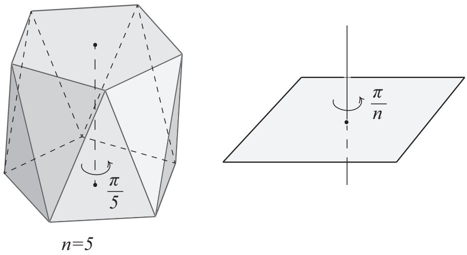


Fig.: The symmetry of an antiprism

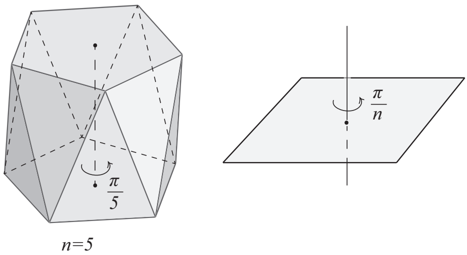


Fig.: The symmetry of an antiprism

The above definitions of an antiprism $\mathcal{A}_n$ and its symmetry group S_{2n} take place either for Euclidean or the hyperbolic space. By definition, $\mathcal{A}_n$ has two types of edges. Denote by a the length of those edges that form top and bottom n-gonal faces. Set c for the length of the lateral edges. Denote the dihedral angles by A, C respectively. Thus, we will designate as $\mathcal{A}_n(a, c)$ for the antiprism $\mathcal{A}_n$ given by its edge lengths a, c.

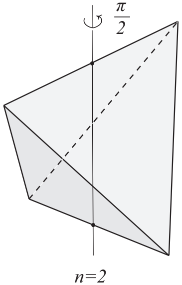
Antiprism: ideal or compact

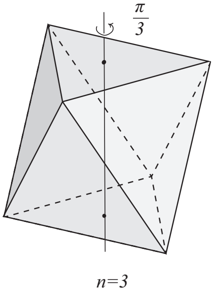
The ideal antiprism in $\mathbb{H}^3$ with all vertices at infinity was studied by A. Vesnin and A. Mednykh (1995–1996). A particular case of ideal regular antiprism is due to W. Thurston (1980).

In the case of ideal antiprism the dihedral angles are related by a condition $2A + 2C = 2\pi$ while in a compact case the inequality $2A + 2C > 2\pi$ holds.

Antiprism $\mathcal{A}_n$: cases n=2 and n=3

For $n = 2$ the n-gons at the top and the bottom of antiprism $\mathcal{A}_n$ degenerate to corresponding two skew edges. Thus we obtain a tetrahedron with symmetry group S_4 (see Fig.). The volume a compact hyperbolic tetrahedron of this type was given by Abr. and Vuong (2017).





For $n = 3$ the antiprism $\mathcal{A}_n$ is an octahedron with symmetry group S_6 (see Fig.). The volume of a compact hyperbolic octahedron with this type of symmetry was found by Abr., Kudina and Mednykh (2015).

Since the symmetry group S_{2n} acts transitively on the set of vertices of $\mathcal{A}_n$, it suffices to point out the coordinates of one of the vertices. Without loss of generality, we can assume that a vertex v_1 has coordinates $(r, 0, h/2)$ with some positive real numbers r and h. The orbit of v_1 under the action of $C_{2n\,h}$ consists of all vertices of the antiprism $\mathcal{A}_n$.

$$C_{2n\,h} = \begin{pmatrix} \cos\pi/n & -\sin\pi/n & 0 \\ \sin\pi/n & \cos\pi/n & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_{2n\,h} : v_i \longmapsto v_{i+1},$$

where the indices are taken modulo $2n$.
The coordinates of the vertices of the antiprism $\mathcal{A}_n$ in $\mathbb{E}^3$

$$\begin{aligned} v_{2k+1} &= \left(r\,\cos\frac{2k\pi}{n}, r\,\sin\frac{2k\pi}{n}, \frac{h}{2}\right), \\ v_{2k+2} &= \left(r\,\cos\frac{(2k+1)\pi}{n}, r\,\sin\frac{(2k+1)\pi}{n}, -\frac{h}{2}\right), \end{aligned} \tag{4}$$

where $k = 0, \ldots, n-1$. Odd vertices form the top n-gonal face and even vertices form the bottom n-gonal face of $\mathcal{A}_n$.

Caley-Klein model of hyperbolic 3-space

Consider *Minkowski space* R^4_1 with *Lorentz scalar product*

$$\langle X, Y \rangle = -x_1\, y_1 - x_2\, y_2 - x_3\, y_3 + x_4\, y_4. \tag{5}$$

The *Caley-Klein model* of the hyperbolic space is the set of vectors $K = \{(x_1, x_2, x_3, 1) \ : \ x_1^2 + x_2^2 + x_3^2 < 1\}$ forming the unit 3-ball in the hyperplane $x_4 = 1$. The lines and planes in K are just the intersections of ball K with Euclidean lines and planes in the hyperplane $x_4 = 1$.
The *distance* between vectors V and W is defined as

$$\cosh \rho\,(V, W) = \frac{\langle V, W \rangle}{\sqrt{\langle V, V \rangle \langle W, W \rangle}}. \tag{6}$$

A plane in K is a set $\mathcal{P} = \{V \in K \ : \ \langle V, N \rangle = 0\}$, where N is a normal vector to the plane $\mathcal{P}$.
Every of four *dihedral angles* between the planes $\mathcal{P}, \mathcal{Q}$ with normal vectors N, M are defined by relation

$$\cos\,\widehat{(\mathcal{P}, \mathcal{Q})} = \pm \frac{\langle N, M \rangle}{\sqrt{\langle N, N \rangle \langle M, M \rangle}}. \tag{7}$$

The coordinates of the vertices of the antiprism $\mathcal{A}_n$ in the Caley-Klein model of $\mathbb{H}^3$

$$
\begin{aligned}
V_{2k+1} &= \left(r\, \cos \frac{2k\pi}{n}, r\, \sin \frac{2k\pi}{n}, \frac{h}{2}, 1 \right), \\
V_{2k+2} &= \left(r\, \cos \frac{(2k+1)\pi}{n}, r\, \sin \frac{(2k+1)\pi}{n}, -\frac{h}{2}, 1 \right),
\end{aligned}
\tag{8}
$$

where $k = 0, \ldots, n-1$. Odd vertices form the top n-gonal face and even vertices form the bottom n-gonal face of $\mathcal{A}_n$.

Compact Hyperbolic Antiprism

Theorem 1 (Abr., Vuong)

The dihedral angles A, C and the edge lengths a, c of a compact hyperbolic antiprism with $2n$ vertices are related by the equalities

$$\cos A = \frac{-\sqrt{\cosh a - 1}\,\left(1 + \cosh a - 2\cosh c\ \cos\frac{\pi}{n}\right)}{\sqrt{2(1 + \cosh a - 2\cosh^2 c)(\cos\frac{2\pi}{n} - \cosh a)}},$$

$$\cos C = \frac{\cosh c - \cosh a \cosh c + 2(\cosh^2 c - 1)\cos\frac{\pi}{n}}{1 + \cosh a - 2\cosh^2 c}.$$

Compact Hyperbolic Antiprism

Theorem 2 (Abr., Vuong)

A compact hyperbolic antiprism $\mathcal{A}_n(a, c)$ with the symmetry group S_{2n} is exist if and only if $1 + \cosh a - 2\cosh c + 2\,(1 - \cosh c)\cos\frac{\pi}{n} < 0$.

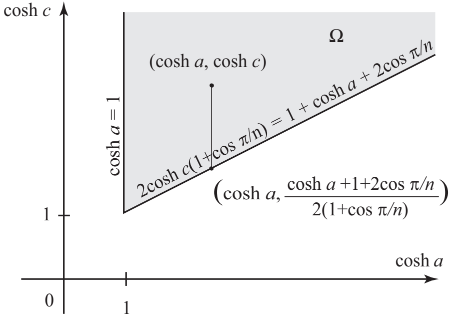


Fig.: Existence domain of a compact hyperbolic antiprism $\mathcal{A}_n(a, c)$

Proposition (Schläfli differential equation)

Let $\mathbb{H}^3$ be a hyperbolic 3-dimensional space of constant curvature -1. Consider a family of convex polyhedra P in $\mathbb{H}^3$ depending on one or more parameters in a differential manner and keeping the same combinatorial type. Then the differential of the volume $V = V(P)$ satisfies the equation

$$dV = -\frac{1}{2}\sum_{\theta} \ell_{\theta}\, d\theta$$

where the sum is taken over all edges of P, ℓ_{θ} denotes the edge length and θ is the interior dihedral angle along it.

Compact Hyperbolic Antiprism

Theorem 3 (Abr., Vuong)

The volume of a compact hyperbolic antiprism with $2n$ vertices and edge lengths a, c is given by the formula

$$V = n\int_{c_0}^{c} \frac{a\,G + t\,H}{(2\cosh^2 t - 1 - \cosh a)\sqrt{R}}\,dt, \ \textit{where}$$

$$G = 2\left(\cosh t - \cos\frac{\pi}{n}\right)\sinh a\ \sinh t,$$

$$H = -(\cosh a - 1)\left(1 + \cosh a + 2\cosh^2 t - 4\cosh t\cos\frac{\pi}{n}\right),$$

$$R = 2 - \cosh a(2 + \cosh a) + \cosh 2t + 4(\cosh a - 1)\cosh t\cos\frac{\pi}{n}$$

$$- 2\sinh^2 t\cos\frac{2\pi}{n} \qquad \textit{and } c_0 \textit{ is the root of the equation}$$

$$2\cosh c\left(1 + \cos\frac{\pi}{n}\right) = 1 + \cosh a + 2\cos\frac{\pi}{n}.$$

References

W. P. Thurston, The geometry and topology of 3-manifolds, Lecture Notes, Princeton University (1980).

A. Yu. Vesnin and A. D. Mednykh, Hyperbolic volumes of Fibonacci manifolds, *Siberian Mathematical Journal* **36**(2) (1995) 235–245.

N. V. Abrosimov, E. S. Kudina and A. D. Mednykh, On the volume of a hyperbolic octahedron with $\bar{3}$-symmetry, *Proceedings of the Steklov Institute of Mathematics* **288** (2015) 1–9.

N. V. Abrosimov and Vuong Huu Bao, The volume of a hyperbolic tetrahedron with symmetry group S_4, *Trudy Instituta Matematiki i Mekhaniki UrO RAN* **23**(4) (2017) 7–17.

Nikolay Abrosimov, Bao Vuong, The volume of a compact hyperbolic antiprism // Journal of Knot Theory and Its Ramifications, accepted, https://doi.org/10.1142/S0218216518420105 arXiv:1807.08297 [math.MG]

Thank you for your attention!