On some partial orders on $\mathcal{B}(\mathcal{H})$

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1. Partial orders on $\mathcal{B}(\mathcal{H})$

**Example 1.1**

Let $\mathbb{C}^n$ be the $n$ dimensional complex (Hilbert) Euclid space and let $M_n$ be the algebra of all complex $n \times n$ matrices. Let $M_i$ (resp. $N_i$) be the $m_i$ (resp. $n_i$) dimensional subspace of $\mathbb{C}^n$ for $i = 1, 2$ such that $n = m_1 + m_2 = n_1 + n_2$ and

$$\mathbb{C}^n = M_1 \oplus M_2 = N_1 \oplus N_2.$$  (1)
1. Partial orders on $\mathcal{B}(\mathcal{H})$

Example 1.1

Let $\mathbb{C}^n$ be the $n$ dimensional complex (Hilbert) Euclid space and let $M_n$ be the algebra of all complex $n \times n$ matrices. Let $M_i$ (resp. $N_i$) be the $m_i$ (resp. $n_i$) dimensional subspace of $\mathbb{C}^n$ for $i = 1, 2$ such that $n = m_1 + m_2 = n_1 + n_2$ and

$$\mathbb{C}^n = M_1 \oplus M_2 = N_1 \oplus N_2. \quad (1)$$

Let $A, B \in M_n$. If there are $n_i \times m_i$ matrices $B_i (i = 1, 2)$ such that

$$A = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

with respect to the decomposition (1), then we say that $A \leq B$. 
1. Partial orders on $\mathcal{B}(\mathcal{H})$

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We now may consider this partial order in the algebra $\mathcal{B}(\mathcal{H})$ (or a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$) of all bounded linear operators on a complex Hilbert space $\mathcal{H}$.
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We now may consider this partial order in the algebra $\mathcal{B}(\mathcal{H})$ (or a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$) of all bounded linear operators on a complex Hilbert space $\mathcal{H}$.

**Definition 1.2**

Let $A, B \in \mathcal{M}$. If $A^*A = A^*B, AA^* = BA^*$, then we say that $A \preceq B$.

The order $\preceq$ is called the star partial order in $\mathcal{M}$. 
1. Partial orders on $\mathcal{B}(\mathcal{H})$

Let $M_i$ and $N_i (i = 1, 2)$ be closed subspaces of $\mathcal{H}$ such that

$$\mathcal{H} = M_1 \oplus M_2 = N_1 \oplus N_2. \quad (2)$$

For any $T \in \mathcal{B}(\mathcal{H})$, there are $T_{ji} \in \mathcal{B}(M_i, N_j) (i, j = 1, 2)$ such that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad (3)$$

with respect to the orthogonal decompositions (2).
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Let $M_i$ and $N_i(i = 1, 2)$ be closed subspaces of $\mathcal{H}$ such that

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$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$  \hspace{1cm} (3)

with respect to the orthogonal decompositions (2).

**Proposition 1.3**

Let $A, B \in B(\mathcal{H})$. Then $A \preceq B$ if and only if there is an orthogonal direct decomposition (2) of $\mathcal{H}$ such that

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} A_{11} & 0 \\ 0 & B_{22} \end{pmatrix}.$$  \hspace{1cm} (4)
1. Partial orders on $\mathcal{B}(\mathcal{H})$

We next recall another partial order on $\mathcal{B}(\mathcal{H})$. 
We next recall another partial order on \( \mathcal{B}(\mathcal{H}) \). Let \( A \in \mathcal{B}(\mathcal{H}) \). Denote by \( R(A) \) and \( N(A) \) the range and the kernel of \( A \) respectively.
1. Partial orders on $\mathcal{B}(\mathcal{H})$

We next recall another partial order on $\mathcal{B}(\mathcal{H})$. Let $A \in \mathcal{B}(\mathcal{H})$. Denote by $R(A)$ and $N(A)$ the range and the kernel of $A$ respectively. For a closed subspace $M \subseteq \mathcal{H}$, $P_M$ is the orthogonal projection on $M$. 

Definition 1.4 (J. K. Baksalary, J. Hauke, 1990)

For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \leq \bigtriangledown B$ if $R(A) \subseteq R(B)$, $R(A^*) \subseteq R(B^*)$ and $AA^* = AB^*$. It is known that $\leq \bigtriangledown$ is a partial order on $\mathcal{B}(\mathcal{H})$ and is called the diamond partial order.
We next recall another partial order on $\mathcal{B}(\mathcal{H})$. Let $A \in \mathcal{B}(\mathcal{H})$. Denote by $R(A)$ and $N(A)$ the range and the kernel of $A$ respectively. For a closed subspace $M \subseteq \mathcal{H}$, $P_M$ is the orthogonal projection on $M$.

**Definition 1.4** (J. K. Baksalary, J. Hauke, 1990)

For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \preceq B$ if $\overline{R(A)} \subseteq \overline{R(B)}$, $\overline{R(A^*)} \subseteq \overline{R(B^*)}$ and $AA^*A = AB^*A$.

It is known that $\preceq$ is a partial order on $\mathcal{B}(\mathcal{H})$ and is called the diamond partial order.
1. Partial orders on $\mathcal{B}(\mathcal{H})$

**Proposition 1.5**

Let $A, B \in \mathcal{B}(\mathcal{H})$ and $U \in \mathcal{B}(\mathcal{H})$ is unitary.

1. If $A^* \leq B$ (resp. $A \leq B^\Diamond$), then $UA^* \leq UB$ (resp. $UA \leq B^\Diamond U$) and $AU \leq BU$ (resp. $AU \leq B^\Diamond U$).

2. $A^* \leq B \iff A = P_{R(A)}B = BP_{R(A^*)}$.

3. $A \leq B^\Diamond \iff A = P_{R(A)}BP_{R(A^*)} = P_{R(B)}AP_{R(B^*)}$.

4. If $A^* \leq B$, then $A \leq B^\Diamond$. 


2. Partial order-hereditary subspaces

Definition 2.1
Let \( \leq \) be a partial order on a von Neumann algebra \( \mathcal{M} \) and \( \mathcal{A} \subseteq \mathcal{M} \) a subspace. For any \( A \in \mathcal{M} \) and \( B \in \mathcal{A} \), if \( A \in \mathcal{A} \) whenever \( A \leq B \), then we say that \( \mathcal{A} \) is a hereditary subspace with respect to the partial order \( \leq \).
2. Partial order-hereditary subspaces

**Definition 2.1**

Let ” ≤ ” be a partial order on a von Neumann algebra $\mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{M}$ a subspace. For any $A \in \mathcal{M}$ and $B \in \mathcal{A}$, if $A \in \mathcal{A}$ whenever $A \leq B$, then we say that $\mathcal{A}$ is a hereditary subspace with respect to the partial order ” ≤ ”. If $\mathcal{A}$ is a hereditary subspace with respect to the star(resp. diamond) partial order, then we say that $\mathcal{A}$ is a star(resp. diamond) partial order hereditary subspace of $\mathcal{M}$. 
2. Partial order-hereditary subspaces

Remark

1. $\mathcal{A}$: star (resp. diamond) partial order hereditary $\implies \mathcal{A}^* = \{X^* : X \in \mathcal{A}\}$ is.

2. $\mathcal{A}$: diamond partial order hereditary $\implies \mathcal{A}$: star partial order hereditary.

3. $I \subseteq \mathcal{M}$: left (resp. right) ideal $\implies I$: diamond (star) partial order hereditary.
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Remark

1. \( \mathcal{A} \): star\( (\text{resp. diamond}) \) partial order hereditary \( \implies \) \( \mathcal{A}^* = \{X^* : X \in \mathcal{A}\} \) is.

2. \( \mathcal{A} \): diamond partial order hereditary \( \implies \) \( \mathcal{A} \): star partial order hereditary.

3. \( I \subseteq \mathcal{M} \): left\( (\text{resp. right}) \) ideal \( \implies \) \( I \): diamond\( (\text{star}) \) partial order hereditary.

We recall that if \( I \) is a weak* closed left, right or two-sided ideal in \( \mathcal{M} \) respectively, then there are projections \( E, F \in \mathcal{M} \) or a central projection \( P \in Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' \) such that \( I = ME, I = FM \) or \( I = PM \).
2. Partial order-hereditary subspaces

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1. \( \mathcal{A} \): star (resp. diamond) partial order hereditary \( \Rightarrow \) \( \mathcal{A}^* = \{ X^* : X \in \mathcal{A} \} \) is.

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We recall that if \( I \) is a weak* closed left, right or two-sided ideal in \( \mathcal{M} \) respectively, then there are projections \( E, F \in \mathcal{M} \) or a central projection \( P \in Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' \) such that \( I = \mathcal{M}E, I = FM \) or \( I = PM \).

In general, let \( E, F \in \mathcal{M} \) be two projections. Then \( EMF \) is weak* closed diamond as well as star partial order-hereditary.
2. Partial order-hereditary subspaces

What is a diamond or star partial order hereditary subspace?
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What is a diamond or star partial order hereditary subspace?

We next consider star partial order hereditary subspaces. Let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators in $\mathcal{B}(\mathcal{H})$. 

**Theorem 2.2**

Let $A$ be a nonzero norm closed star partial order hereditary subspace in $\mathcal{B}(\mathcal{H})$. Then there exists a unique pair of projections $E, F \in \mathcal{B}(\mathcal{H})$ such that $A \cap \mathcal{K}(\mathcal{H}) = E\mathcal{K}(\mathcal{H})F$ and $A_{w^*} = E\mathcal{B}(\mathcal{H})F$, where $A_{w^*}$ is the weak* closure of $A$. That is, $E\mathcal{K}(\mathcal{H})F \subseteq A \subseteq E\mathcal{B}(\mathcal{H})F$. 
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We next consider star partial order hereditary subspaces. Let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators in $\mathcal{B}(\mathcal{H})$.

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Let $\mathfrak{A}$ be a nonzero norm closed star partial order hereditary subspace in $\mathcal{B}(\mathcal{H})$. Then there exists a unique pair of projections $E, F \in \mathcal{B}(\mathcal{H})$ such that

$$\mathfrak{A} \cap \mathcal{K}(\mathcal{H}) = E\mathcal{K}(\mathcal{H})F$$

and

$$\overline{\mathfrak{A}}^{w*} = E\mathcal{B}(\mathcal{H})F,$$

where $\overline{\mathfrak{A}}^{w*}$ is the weak* closure of $\mathfrak{A}$. That is,

$$E\mathcal{K}(\mathcal{H})F \subseteq \mathfrak{A} \subseteq E\mathcal{B}(\mathcal{H})F.$$
2. Partial order-hereditary subspaces

Example 2.3

Let $\mathcal{N}$ be an infinite dimensional Hilbert space and $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}$. Put

$$\mathfrak{A} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : X_{11}, X_{12} \in \mathcal{B}(\mathcal{N}), X_{21}, X_{22} \in \mathcal{K}(\mathcal{N}) \right\}.$$

Then $\mathfrak{A}$ is a norm closed star partial order hereditary subspace such that $\mathcal{K}(\mathcal{H}) \nsubseteq \mathfrak{A} \nsubseteq \mathcal{B}(\mathcal{H})$. 
2. Partial order-hereditary subspaces

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Let $\mathcal{N}$ be an infinite dimensional Hilbert space and $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}$. Put

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Then $\mathfrak{A}$ is a norm closed star partial order hereditary subspace such that $\mathcal{K}(\mathcal{H}) \nsubseteq \mathfrak{A} \nsubseteq \mathcal{B}(\mathcal{H})$.

It is also known that both $\mathfrak{A}^*$ and $\mathfrak{A} \cap \mathfrak{A}^*$ are also star partial order hereditary subspaces containing $\mathcal{K}(\mathcal{H})$. However $\mathfrak{A} \vee \mathfrak{A}^*$ is not star partial order hereditary.
2. Partial order-hereditary subspaces

Let $\mathcal{M}$ be a von Neumann algebra and $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ the center of $\mathcal{M}$. Let $A \in \mathcal{M}$.

$$C_A = \inf \{ E \in Z(\mathcal{M}) : E \text{ is a projection such that } EA = A \}$$

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**Theorem 2.4**

Let $\mathcal{A} \subseteq \mathcal{M}$ be a weak* closed star partial order hereditary subspace. Then there exists a unique pair of projections $E, F \in \mathcal{M}$ with the same central carriers $C_E = C_F$ such that $\mathcal{A} = EMF$. 
Corollary 2.5

Let $\mathcal{A} \subseteq \mathcal{M}$ be a weak* closed diamond partial order hereditary subspace. Then there exists a unique pair of projections $E, F \in \mathcal{M}$ with the same central carriers $C_E = C_F$ such that $\mathcal{A} = E\mathcal{M}F$. 
Corollary 2.5

Let $\mathcal{A} \subseteq \mathcal{M}$ be a weak$^*$ closed diamond partial order hereditary subspace. Then there exists a unique pair of projections $E, F \in \mathcal{M}$ with the same central carriers $C_E = C_F$ such that $\mathcal{A} = E \mathcal{M} F$.

Remark

$\mathcal{A} \subseteq \mathcal{M}$: norm closed star partial order hereditary subspace. $\Rightarrow \mathcal{A}$: diamond partial order hereditary?
2. Partial order-hereditary subspaces

Corollary 2.5
Let $\mathcal{A} \subseteq \mathcal{M}$ be a weak* closed diamond partial order hereditary subspace. Then there exists a unique pair of projections $E, F \in \mathcal{M}$ with the same central carriers $C_E = C_F$ such that $\mathcal{A} = E\mathcal{M}F$.

Remark
$\mathcal{A} \subseteq \mathcal{M}$: norm closed star partial order hereditary subspace.
$\implies \mathcal{A}$: diamond partial order hereditary?

If it is also weak* closed, then it is.
3. Order automorphisms on the unit interval

We consider the unit interval with respect to a partial order.
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We consider the unit interval with respect to a partial order.

Put $\Sigma = \{T \in \mathcal{B}(\mathcal{H}) : 0 \preceq T \preceq I\}$, the unit interval with respect to the star partial order. Then
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\[
\Sigma = \{ E \in \mathcal{B}(\mathcal{H}) : E \text{ is a projection} \}
\]

and for any \( E, F \in \Sigma \), \( E \preceq F \iff E \leq F \).

In this case, \( \Sigma \) is just the lattice of subspaces in \( \mathcal{H} \).
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\[
\Sigma = \{E \in \mathcal{B}(\mathcal{H}) : E \text{ is a projection}\}
\]

and for any \( E, F \in \Sigma \), \( E^* \leq F \iff E \leq F \).

In this case, \( \Sigma \) is just the lattice of subspaces in \( \mathcal{H} \).

**Theorem 3.1 (A basic theorem)**

Assume \( \dim \mathcal{H} = n \geq 3 \), that is, \( \mathcal{H} = \mathbb{C}^n \). Let \( \varphi \) be a lattice automorphism on \( \Sigma \). Then there are a ring automorphism \( \tau \) on \( \mathbb{C} \) and a \( \tau \) linear bijection \( S(S(ax + by) = \tau(a)x + \tau(b)y, \forall a, b \in \mathbb{C}, x, y \in \mathbb{C}^n) \) on \( \mathbb{C}^n \) such that

\[
\varphi(E) = P_R(SES^{-1}) = P_R(SE), \quad \forall E \in \Sigma.
\]
3. Order automorphisms on the unit interval

Theorem 3.2 (Fillmore and Longstaff)

Assume \( \dim \mathcal{H} = \infty \). Let \( \varphi \) be a lattice automorphism on \( \Sigma \). Then there is a bounded invertible linear or conjugate linear operator \( S \) on \( \mathcal{H} \) such that

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\varphi(E) = P_R(SES^{-1}) = P_R(SE), \quad \forall E \in \Sigma.
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3. Order automorphisms on the unit interval

**Theorem 3.2 (Fillmore and Longstaff)**

Assume $\dim \mathcal{H} = \infty$. Let $\varphi$ be a lattice automorphism on $\Sigma$. Then there is a bounded invertible linear or conjugate linear operator $S$ on $\mathcal{H}$ such that

$$\varphi(E) = P_R(SES^{-1}) = P_R(SE), \quad \forall E \in \Sigma.$$ 

Theorems 3.1 and 3.2 give a complete description of order automorphism on $\Sigma$. 
3. Order automorphisms on the unit interval

Put $\Lambda = \{T \in \mathcal{B}(\mathcal{H}) : 0 \leq \diamond T \leq \diamond I\}$. Then $\Lambda$ is a poset but not a lattice.
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**Theorem 3.3**

$\Lambda = \{ EF : E, F \in \Sigma \}$. 
3. Order automorphisms on the unit interval

Put \( \Lambda = \{ T \in \mathcal{B}(\mathcal{H}) : 0 \leq T \leq I \} \). Then \( \Lambda \) is a poset but not a lattice.

**Theorem 3.3**
\[
\Lambda = \{ EF : E, F \in \Sigma \}.
\]

Let \( T \in \Lambda \). Then

\[
T = P_{R(T)}P_{N(T)}\perp = P_{R(T)}P_{R(T^*)} = P_{N(T^*)}\perp P_{N(T)}\perp.
\]

This factorization is called the canonical factorization of \( T \).
3. Order automorphisms on the unit interval

\[ \varphi : \Lambda \rightarrow \Lambda : \text{diamond order automorphism}. \quad \varphi = ? \]
3. Order automorphisms on the unit interval

φ : Λ → Λ: diamond order automorphism. φ = ?

Let Δ be the set of all semi-linear bijections on H if the dimension of H is finite, and let Δ be the set of all bounded invertible linear or conjugate linear operators on H if the dimension of H is infinite.
3. Order automorphisms on the unit interval

ϕ : Λ → Λ: diamond order automorphism. ϕ = ?

Let Δ be the set of all semi-linear bijections on \( \mathcal{H} \) if the dimension of \( \mathcal{H} \) is finite, and let Δ be the set of all bounded invertible linear or conjugate linear operators on \( \mathcal{H} \) if the dimension of \( \mathcal{H} \) is infinite.

For \( A \in \Delta \), we define \( \delta_A(T) = ATA^{-1}, \forall T \in \mathcal{B}(\mathcal{H}) \). We now define two canonical maps on Λ.

\[
\varphi^1_A(T) = P_{R(\delta_A(T))}P_{R(\delta_A(T)^*)}, \forall T \in \Lambda;
\]

\[
\varphi^2_A(T) = P_{R(\delta_A(T^*))}P_{R((\delta_A(T^*))^*)}, \forall T \in \Lambda.
\]
3. Order automorphisms on the unit interval

Proposition 3.4

Let $A \in \Delta$. Then $\varphi_A^1$ and $\varphi_A^2$ defined as above are automorphisms of the poset $(\Lambda, \leq^\circ)$. 
3. Order automorphisms on the unit interval

**Proposition 3.4**
Let \( A \in \Delta \). Then \( \varphi_A^1 \) and \( \varphi_A^2 \) defined as above are automorphisms of the poset \( (\Lambda, \leq^\Diamond) \).

**Theorem 3.5**
Let \( \varphi : \Lambda \rightarrow \Lambda \) be a map. Then \( \varphi \) is an automorphism of the poset \( (\Lambda, \leq^\Diamond) \) if and only if there is some \( A \in \Delta \) such that either \( \varphi = \varphi_A^1 \) or \( \varphi = \varphi_A^2 \).
3. Order automorphisms on the unit interval

**Remark**

Let $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$: star(diamond) partial order automorphism. $\varphi =$?
3. Order automorphisms on the unit interval

**Remark**

Let $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}):$ star(diamond) partial order automorphism. $\varphi =$?

**Theorem 3.6**

Let $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}):$ star partial order automorphism. If $\varphi$ is additive, then there are a nonzero constant $\alpha \in \mathbb{C}$ and both unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ or both anti-unitary operators $U$ and $V$ on $\mathcal{H}$ such that $\varphi(A) = \alpha UAV$, $\forall A \in \mathcal{B}(\mathcal{H})$ or $\varphi(A) = \alpha U A^*V$, $\forall A \in \mathcal{B}(\mathcal{H})$. 
3. Order automorphisms on the unit interval

Remark
Let $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$: star(diamond) partial order automorphism. $\varphi =$?

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Let $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$: star partial order automorphism. If $\varphi$ is additive, then there are a nonzero constant $\alpha \in \mathbb{C}$ and both unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ or both anti-unitary operators $U$ and $V$ on $\mathcal{H}$ such that $\varphi(A) = \alpha UAV$, $\forall A \in \mathcal{B}(\mathcal{H})$ or $\varphi(A) = \alpha U A^* V$, $\forall A \in \mathcal{B}(\mathcal{H})$.

In general, unknown.
4. Maximal lower bounds and minimal upper bounds

**Definition 4.1**

Let $\leq$ be a partial order on a von Neumann algebra $\mathcal{M}$ and let $S \subseteq \mathcal{M}$ be a subset.

1. If there is a $C \in \mathcal{M}$ such that $S \leq C$, $\forall S \in S$, then we say that $C$ is an upper bound of $S$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then we abbreviate as $\bigvee S$ if there exists the supremum.

We similarly may define maximal lower bounds and the infimum of $S$ and denoted by $\bigwedge S$.
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2. Let $C$ be an upper bound of $S$. If there are not any upper bound $D$ of $S$ such that $D \leq C$, then we say that $C$ is a minimal upper bound of $S$. If for any upper bound $D$ of $S$, $C \leq D$, then we say that $C$ is the supremum of $S$ and denoted by $\bigvee_{\mathcal{M}} S$. 

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then we abbreviate as $\bigvee_{\mathcal{M}} S$ if there exists the supremum. We similarly may define maximal lower bounds and the infimum of $S$ and denoted by $\bigwedge_{\mathcal{M}} S$. 

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2. Let \( C \) be an upper bound of \( S \). If there are not any upper bound \( D \) of \( S \) such that \( D \leq C \), then we say that \( C \) is a minimal upper bound of \( S \). If for any upper bound \( D \) of \( S \), \( C \leq D \), then we say that \( C \) is the supremum of \( S \) and denoted by \( \bigvee_{\mathcal{M}} S \).

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Let $\leq$ be a partial order on a von Neumann algebra $\mathcal{M}$ and let $S \subseteq \mathcal{M}$ be a subset.

1. If there is a $C \in \mathcal{M}$ such that $S \leq C$, $\forall S \in S$, then we say that $C$ is an upper bound of $S$.

2. Let $C$ be an upper bound of $S$. If there are not any upper bound $D$ of $S$ such that $D \leq C$, then we say that $C$ is a minimal upper bound of $S$. If for any upper bound $D$ of $S$, $C \leq D$, then we say that $C$ is the supremum of $S$ and denoted by $\bigvee_{\mathcal{M}} S$.

3. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then we abbreviate as $\bigvee S$ if there exists the supremum.

We similarly may define maximal lower bounds and the infimum of $S$ and denoted by $\bigwedge_{\mathcal{M}} S$ ($\bigwedge S$) the infimum of $S$. 
4. Maximal lower bounds and minimal upper bounds

We firstly consider the star partial order.
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We firstly consider the star partial order.

**Lemma 4.2**

Let $\mathcal{H} = H_1 \oplus H_2$ and $A, B \in \mathcal{B}(\mathcal{H})$ such that

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}. $$

Then

1. $A \land B = \begin{pmatrix} A_{11} \land B_{11} & 0 \\ 0 & A_{22} \land B_{22} \end{pmatrix}$. 

2. If there is a upper bound for $A$ and $B$, then $A \lor B = \begin{pmatrix} A_{11} \lor B_{11} & 0 \\ 0 & A_{22} \lor B_{22} \end{pmatrix}$.
4. Maximal lower bounds and minimal upper bounds

Theorem 4.3
Let $S \subseteq M$ be a subset of $M$.

$\bigwedge_M S = \bigwedge S.$
4. Maximal lower bounds and minimal upper bounds

Theorem 4.3

Let $S \subseteq \mathcal{M}$ be a subset of $\mathcal{M}$.

1. \[ ^* \bigwedge_{\mathcal{M}} S = ^* \bigwedge S. \]

2. If there is an upper bound for $S$, then \[ ^* \bigvee_{\mathcal{M}} S = ^* \bigvee S. \]
4. Maximal lower bounds and minimal upper bounds

Theorem 4.3
Let $\mathcal{S} \subseteq \mathcal{M}$ be a subset of $\mathcal{M}$.

1. $^*\bigwedge_{\mathcal{M}} \mathcal{S} = \bigwedge \mathcal{S}$.

2. If there is an upper bound for $\mathcal{S}$, then $^*\bigvee_{\mathcal{M}} \mathcal{S} = \bigvee \mathcal{S}$.
4. Maximal lower bounds and minimal upper bounds

However, there are no supremum and infimum for a bounded subset in $B(H)$ with respect to the diamond partial order in general.

Are there maximal lower bounds and minimal upper bounds for a bounded subset?
4. Maximal lower bounds and minimal upper bounds

However, there are no supremum and infimum for a bounded subset in $\mathcal{B}(\mathcal{H})$ with respect to the diamond partial order in general.

Are there maximal lower bounds and minimal upper bounds for a bounded subset?

**Theorem 4.4**

Let $S \subseteq \mathcal{B}(\mathcal{H})$. If $S$ is bounded with respect to the diamond partial order, then there exists a minimal upper bound for $S$. 
4. Maximal lower bounds and minimal upper bounds

However, there are no supremum and infimum for a bounded subset in $\mathcal{B}(\mathcal{H})$ with respect to the diamond partial order in general.

Are there maximal lower bounds and minimal upper bounds for a bounded subset?

**Theorem 4.4**

Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$. If $\mathcal{S}$ is bounded with respect to the diamond partial order, then there exists a minimal upper bound for $\mathcal{S}$.

Let $B$ be an upper bound of $\mathcal{S}$. Put $P = P \bigvee \{ \overline{P_{R(S)}} : S \in \mathcal{S} \}$, $Q = P \bigvee \{ \overline{P_{R(S^*)}} : S \in \mathcal{S} \}$ and $A = PBQ$. Moreover, put $H_1 = \overline{R(A)}$, $H_2 = P(\mathcal{H}) \ominus H_1$ and $H_3 = \mathcal{H} \ominus P(\mathcal{H})$. $K_1 = \overline{P_{R(A^*)}}$, $K_2 = \mathcal{H} \ominus Q(\mathcal{H})$ and $K_3 = Q(\mathcal{H}) \ominus K_1$. Then

$$\mathcal{H} = K_1 \oplus K_2 \oplus K_3 = H_1 \oplus H_2 \oplus H_3.$$
4. Maximal lower bounds and minimal upper bounds

Theorem 4.5
Let $T \in \mathcal{B}(K_2, H_2)$ and $S \in \mathcal{B}(K_3, H_3)$ such that both $T$ and $S^*$ with dense ranges. Then

$$B_{T,S} = \begin{pmatrix} A & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & S \end{pmatrix}$$

with respect to decomposition (5) of $\mathcal{H}$ is an upper bound of $S$. $B_{T,S}$ is a minimal upper bound if and only if both $T$ and $S^*$ are surjective.
4. Maximal lower bounds and minimal upper bounds

Proposition 4.6

Let \( \{A_{\alpha}\}_{\alpha \in \Lambda} \) be an increasing net in \( \mathcal{B}(\mathcal{H}) \) and bounded from above with respect to the diamond partial order. Then

1. \( A_{\alpha} \rightarrow A(SOT) \) for some \( A \in \mathcal{B}(\mathcal{H}) \) such that \( A \leq^\diamond D \) for any upper bound \( D \) of \( \{A_{\alpha}\}_{\alpha \in \Lambda} \).

2. \( A \) is the supremum of \( \{A_{\alpha}\}_{\alpha \in \Lambda} \) if and only if \( P_{A_{\alpha}} \rightarrow P_A(SOT) \) and \( Q_{\alpha} \rightarrow Q_A(SOT) \).
4. Maximal lower bounds and minimal upper bounds

Example 4.7

Let $\mathcal{H} = L^2([0, 1]) \oplus L^2([0, 1]) \oplus L^2([0, 1])$. Put $\Delta_s = [0, s]$, $\Omega_s = (s, 1]$ and $f_s(t) = \chi_{\Delta_s}(t) + \frac{1}{2}\chi_{\Omega_s}(t)$, $\forall s \in [0, 1)$. We define

$$P_s = \begin{pmatrix} f_s & \frac{1}{2}\chi_{\Omega_s} & 0 \\ \frac{1}{2}\chi_{\Omega_s} & f_s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_s = \begin{pmatrix} f_s & 0 & \frac{1}{2}\chi_{\Omega_s} \\ 0 & 0 & 0 \\ \frac{1}{2}\chi_{\Omega_s} & 0 & f_s \end{pmatrix}.$$  

Then $P_s$ and $Q_s$ are projections and $A_s = P_sQ_s$ is increasing and with an upper bound $I$. Note that $P = P\bigvee\{R(P_sQ_s) : 0 \leq s < 1\} = I \oplus I \oplus 0$ and $Q = P\bigvee\{R(Q_sP_s) : 0 \leq s < 1\} = I \oplus 0 \oplus I$. However, $A_s = P_sQ_s \to PQ = A = I \oplus 0 \oplus 0 (SOT)$ and $A$ is not an upper bound of $\{A_s : 0 \leq s < 1\}$. By Theorem 2.6, there many minimal upper bounds for $\{A_s : 0 \leq s < 1\}$.
4. Maximal lower bounds and minimal upper bounds

Are there maximal lower bounds for $S$?

**Theorem 4.8**

Let $A$ be a nonempty subset in $B(\mathcal{H})$. If $B \in B(\mathcal{H})$ is an upper bound of $A$ with respect to the diamond partial order, then $EBF$ is a maximal lower bound of $A$. 
Are there maximal lower bounds for $S$?

**Theorem 4.8**

Let $\mathcal{A}$ be a nonempty subset in $\mathcal{B}(\mathcal{H})$. If $B \in \mathcal{B}(\mathcal{H})$ is an upper bound of $\mathcal{A}$ with respect to the diamond partial order, then $EBF$ is a maximal lower bound of $\mathcal{A}$.

**Proposition 4.9**

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a decreasing net in $\mathcal{B}(\mathcal{H})$ respect to the diamond partial order. Then $A_\alpha \rightarrow A(SOT)$ and $A$ is a maximal lower bound for $\{A_\alpha\}_{\alpha \in \Lambda}$. $A$ is the infimum of $\{A_\alpha\}_{\alpha \in \Lambda}$ if and only if $P_{A_\alpha} \rightarrow P_A(SOT)$ and $Q_\alpha \rightarrow Q_A(SOT)$. 
4. Maximal lower bounds and minimal upper bounds

**Example 4.10**

\( \mathcal{K} \): separable infinite dimensional Hilbert space.
\( C, D \in \mathcal{B}(\mathcal{K}) \): injective with dense range.
\( M \subset \mathcal{K} \): linear manifold s.t. \( R(D^*) + M = \mathcal{K} \) and \( \overline{M} = \mathcal{K} \).
\( \{f_1, f_2, \cdots, f_n, \cdots\} \subset M \): basis for \( \mathcal{K} \).
\( F_n \): projection onto \( \bigvee [f_{n+1}, f_{n+2}, \cdots, f_{n+k}, \cdots] \), \( \forall n \in \mathbb{N} \).
\( F_n \downarrow 0(SOT) \).
\( \mathcal{H} = \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K} \), \( B = C \oplus D \oplus D^* \).
\( P_n = I \oplus I \oplus F_n \downarrow P = I \oplus I \oplus 0 \).
\( Q_n = I \oplus F_n \oplus I \downarrow Q = I \oplus 0 \oplus I \).
\( A_n = P_n B Q_n \downarrow A = C \oplus 0 \oplus 0 \).
Put \( B_E = EBQ \), for any projection \( E \leq P \) such that \( EP_A \neq PA_E \).
\( B_E \leq^\circ A_n \) for all \( n \).
\( B_E \): not comparable with \( A \).
On some partial orders on $\mathcal{B}(\mathcal{H})$

Guoxing Ji

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Thank You!