Kuroda's theorem for n-tuples in semifinite von Neumann algebras

Hongyin Zhao (joint work with Alexey Ber, Fedor Sukochev, Dmitriy Zanin)

School of Mathematics and Statistics of UNSW

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Weyl-von Neumann theorem

Let H be a separable Hilbert space. An operator d on H is called diagonal if $\exists\{\xi_k\}_{k\geq 1}$ C.O.N.B of H s.t. $d\xi_k = \lambda_k\xi_k, \lambda_k \in \mathbb{C}, k \geq 1$.

Theorem (Weyl-von Neumann, 1909, 1935)

For every self-adjoint operator b on H and every $\varepsilon > 0$, there is a diagonal operator d on H such that $\|b - d\|_{L_2(B(H))} < \varepsilon$. Here $L_2(B(H))$ is the Hilbert-Schmidt class in $B(H)$.

Classical Kuroda's theorem

Theorem (Kuroda, 1958)

Suppose $\mathcal{J}$ is a Banach ideal in $B(H)$ such that $\mathcal{J} \neq L_1(B(H))$. For every self-adjoint operator b on H and every $\varepsilon > 0$, there is a diagonal operator d on H such that $\|b - d\|_{\mathcal{J}} < \varepsilon$. Here, $L_1(B(H))$ is the trace class in $B(H)$.

Voiculescu's method

Let $a \in B(H), \alpha = (\alpha(j))_{j=1}^n \in (B(H))^n$, denote

$$[a, \alpha] = ([a, \alpha(j)])_{j=1}^n, \quad \|\alpha\|_{\mathcal{J}} = \max_{1 \le j \le n} \|\alpha(j)\|_{\mathcal{J}}.$$

Theorem (Voiculescu, 1979)

Let $\alpha \in (B(H))^n$ be a commuting self-adjoint n-tuple. Suppose $\mathcal{J}$ is a Banach ideal in $B(H)$. The following statements are equivalent:
(i) *For every $\varepsilon > 0$, there is a commuting diagonal n-tuple $(\delta(j))_{j=1}^n$ such that $\|\alpha(j) - \delta(j)\|_{\mathcal{J}} < \varepsilon$ for every $1 \le j \le n$;*
(ii) *There is a sequence $\{r_k\}_{k \ge 1} \subset \mathcal{F}_1^+$ such that $r_k \uparrow \mathbf{1}$ and*

$$\|[r_k, \alpha]\|_{\mathcal{J}} \to 0, \quad (k \to \infty)$$

Here $\mathcal{F}_1^+ = \{r \in B(H) : 0 \le r \le \mathbf{1}, \operatorname{rank}(r) < \infty\}$.

The sequence $\{r_k\}_{k \ge 1}$ satisfying the conditions in (ii) is called a quasicentral approximate unit of α relative to $\mathcal{J}$.

Quasicentral modulus

The condition (ii) in the previous slide is equivalent to say that $k_{\mathcal{J}}(\alpha) = 0$. Here, $k_{\mathcal{J}}(\alpha)$ is defined as follows

$$k_{\mathcal{J}}(\alpha) = \sup_{a \in \mathcal{F}_1^+} \inf_{\substack{r \geq a \\ r \in \mathcal{F}_1^+}} \|[r, \alpha]\|_{\mathcal{J}}.$$

Hence, α is diagonal modulo $\mathcal{J}$

$\Leftrightarrow$ the existence of a quasicentral approximate unit of α relative to $\mathcal{J}$

$\Leftrightarrow k_{\mathcal{J}}(\alpha) = 0$.

Bercovici-Voiculescu's theorem

Theorem (Bercovici-Voiculescu, 1989)

Suppose $\mathcal{J}$ *is a Banach ideal in* $B(H)$ *such that* $\mathcal{J} \not\subset L_{n,1}(B(H))$. *For every commuting self-adjoint* n*-tuple* $\alpha = (\alpha(j))_{j=1}^n \in (B(H))^n$ *and for every* $\varepsilon > 0$, *there is a commuting* n*-tuple of diagonal operators* $\delta = (\delta(j))_{j=1}^n \in (B(H))^n$ *such that*

$$\|\alpha(j) - \delta(j)\|_{\mathcal{J}} < \varepsilon, 1 \le j \le n.$$

Here, $L_{n,1}(B(H))$ *is the Lorentz-*$(n,1)$ *ideal in* $B(H)$.

Semifinite setting

We concern about commuting n-tuple in von Neumann algebra. A $*$-algebra of $B(H)$ is called a von Neumann algebra if $\mathcal{M} = \mathcal{M}''$ where $\mathcal{M}''$ is the bicommutant of $\mathcal{M}$. It is called *semifinite* if there exists a faithful normal semifinite trace τ on $\mathcal{M}$. Let $S(\tau)$ denote the set of all τ-measurable operators affiliated with $\mathcal{M}$.

For $x \in S(\tau)$, the *distribution function* of x is defined by

$$d(s;x) = \tau(e^{|x|}(s,\infty)), \quad s \geq 0.$$

The *singular value function* $\mu(x) : t \mapsto \mu(t;x)$ of the operator x, is

$$\mu(t;x) = \inf\{s \geq 0 : d(s;x) \leq t\}, \quad t \geq 0.$$

The function $t \mapsto \mu(t;x)$ is also written as $\mu(x)$.

Definition (Symmetric function space)

A symmetric function space $(E, \|\cdot\|_E)$ is a Banach space of real-valued Lebesgue measurable functions on $(0,\infty)$ such that: If $y \in E, x$ is a measurable function and $\mu(x) \leq \mu(y)$, then $x \in E$, $\|x\|_E \leq \|y\|_E$.

Symmetric space associated with a semifinite von Neumann algebra

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ. Let E be a symmetric function space on $(0,\infty)$ with norm $\|\cdot\|_E$. Define $E(\mathcal{M}) = \{a \in S(\tau) : \mu(a) \in E\}$, and define

$$\|a\|_{E(\mathcal{M})} = \|\mu(a)\|_E, \quad a \in E(\mathcal{M}).$$

From [Kalton-Sukochev, 2008], $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$ is a Banach space and is called a *symmetric space*. In particular, for $1 \le p < \infty$, if $E = L_p$ is the standard Lebesgue L_p function space on $(0,\infty)$, we obtain the classical noncommutative L_p spaces $L_p(\mathcal{M})$. For convenience, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the uniform norm $\|\cdot\|_{\mathcal{M}}$. If $E = L_{p,1}$ is the standard Lorentz-$(p,1)$ space on $(0,\infty)$, we obtain the noncommutative Lorentz-$(p,1)$ space $L_{p,1}(\mathcal{M})$.

Question in semifinite setting

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ. Let $\alpha = (\alpha(j))_{j=1}^n, \beta = (\beta(j))_{j=1}^n \in \mathcal{M}^n$, we write $\alpha \pm \beta = (\alpha(j) \pm \beta(j))_{j=1}^n$.

Question

Let $\alpha \in \mathcal{M}^n$ be a commuting self-adjoint n-tuple. Suppose E is a symmetric function space on $(0, \infty)$. When does there exist a commuting diagonal n-tuple $\delta = (\delta(j))_{j=1}^n \in \mathcal{M}^n$ such that $\|\alpha - \delta\|_{E(\mathcal{M})} < \varepsilon$?

Development in semifinite factor

Zsidó, L., *The Weyl-von Neumann theorem in semifinite factors.* J. Functional Analysis **18** (1975), 60–72.

Kaftal, V., *On the theory of compact operators in von Neumann algebras.II*, Pacific Journal of Mathematics **79**, no.1 (1978), 129–137.

Theorem (Li-Shen-Shi, 2020)

Suppose $n \geq 2$. Let $\alpha \subset \mathcal{M}^n$ be a commuting self-adjoint n-tuple. For every $\varepsilon > 0$, there exists a commuting diagonal n-tuple $\delta = (\delta(j))_{j=1}^n \in \mathcal{M}^n$ such that $\|\alpha - \delta\|_{L_n(\mathcal{M})} < \varepsilon$.

Li-Shen-Shi obtained a version of Kuroda's theorem under the assumption that $\mathcal{M}$ is properly infinite vNa and $a \in \mathcal{M}$ is bounded.

Theorem (Li-Shen-Shi, 2020)

Let $a \in \mathcal{M}$ be a self-adjoint operator. Suppose E is a symmetric function space on $(0, \infty)$ such that $E \not\subset L_1$. For every $\varepsilon > 0$, there exists a diagonal $d \in \mathcal{M}$ such that $\|a - d\|_{E(\mathcal{M})} < \varepsilon$.

Extension of Voiculescu's result in semifinite vNa

Let $\mathcal{F}_1^+(\mathcal{M}) = \{x \in \mathcal{M} : 0 \le x \le \mathbf{1}, \tau(\mathfrak{l}(x)) < \infty\}$, where $\mathfrak{l}(x)$ is the left support projection of x, i.e. $\mathfrak{l}(x)$ is the projection onto $\overline{x(H)}$. Define

$$k_{E(\mathcal{M})}(\alpha) = \sup_{a \in \mathcal{F}_1^+(\mathcal{M})} \inf_{\substack{r \ge a \\ r \in \mathcal{F}_1^+(\mathcal{M})}} \|[r, \alpha]\|_{E(\mathcal{M})}.$$

Theorem 1 (Ber-Sukochev-Zanin-Zhao, 2023 (factor), 2024 (vNa))

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ. Let $\alpha \in \mathcal{M}^n$ be a commuting self-adjoint n-tuple. Suppose E is a symmetric function space on $(0, \infty)$. T.F.A.E.
$\mathrm{(i)}$ $k_{E(\mathcal{M})}(\alpha) = 0$;
$\mathrm{(ii)}$ *There exists a commuting diagonal n-tuple $\delta \in \mathcal{M}^n$ such that $\|\alpha - \delta\|_{E(\mathcal{M})} < \varepsilon$.*

Remark. If the Hilbert space H is non-separable, but $\mathcal{M}$ is σ-finite, i.e. each orthogonal family of non-zero projections in $\mathcal{M}$ is countable, then Theorem 1 still holds.

Let $\mathcal{M}$ be a σ-finite von Neumann algebra with a faithful normal semifinite trace τ. Let $\alpha \in \mathcal{M}^n$ be a commuting self-adjoint n-tuple.

Corollary 1. When $n \geq 2$, it can be proved that $k_{L_n(\mathcal{M})}(\alpha) = 0$. Thus, by Theorem 1, α is diagonal modulo $L_n(\mathcal{M})$.

Corollary 2. Every normal operator $b \in \mathcal{M}$ is diagonal modulo $L_2(\mathcal{M})$. This extends the Weyl-von Neumann theorem to normal operators in semifinite von Neumann algebras.

non-commutative Weyl-von Neumann theorem

Let $\mathcal{A}$ be a unital C^*-algebra. Suppose $\psi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms. We say that ψ and ρ are *approximately equivalent in* $\mathcal{M}$ if there exists a sequence $\{u_m\}_{m\geq 1}$ such that

$$\|\psi(a) - u_m^*\rho(a)u_m\|_{\mathcal{M}} \to 0, \quad (m \to \infty),\ a \in \mathcal{A},$$

and write $\psi \sim_{\mathcal{M}} \rho$. In the case when $\mathcal{M} = B(H)$, Voiculescu obtained the following non-commutative Weyl-von Neumann theorem.

Theorem (Voiculescu, 1976)

If $\psi : \mathcal{A} \to B(H)$ and $\rho : \mathcal{A} \to B(H)$ are $*$-monomorphisms such that $\psi(\mathcal{A}) \cap \mathcal{K} = \rho(\mathcal{A}) \cap \mathcal{K} = \{0\}$, then $\psi \sim_{B(H)} \rho$. Here, $\mathcal{K}$ is the ideal of compact operators.

Theorem (Hadwin, 1981)

If $\psi : \mathcal{A} \to B(H)$ and $\rho : \mathcal{A} \to B(H)$ are $*$-homomorphisms, then $\psi \sim_{B(H)} \rho$ iff $\mathrm{rank}(\psi(a)) = \mathrm{rank}(\rho(a)), \forall a \in \mathcal{A}$.

Theorem (Ciuperca-Giordano-Ng-Niu, 2013)

Suppose $\mathcal{A}$ is a separable unital C^*-algebra and $\mathcal{M}$ is a infinite factor acting on H. If $\psi : \mathcal{A} \to \mathcal{M}$ and $\rho : \mathcal{A} \to \mathcal{M}$ are $*$-monomorphisms such that
$\psi(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}) = \rho(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}) = \{0\}$,
then $\psi \sim_{\mathcal{M}} \rho$. Here, $\mathcal{K}(\mathcal{M})$ is the closure of all finite-supported operators in $\mathcal{M}$.

The above theorem is no longer true when $\mathcal{M}$ is a non-factor.

When $\mathcal{M}$ is a general von Neumann algebra and when $\mathcal{A}$ is commutative, the following result holds.

Theorem (Ding-Hadwin, 2005)

Suppose $\mathcal{A}$ is a separable commutative unital C^*-algebra and $\mathcal{M}$ is a von Neumann algebra acting on H. We have $\psi \sim_{\mathcal{M}} \rho$ iff $\mathfrak{l}(\psi(a)) \sim \mathfrak{l}(\rho(a))$.

Here $\mathfrak{l}(x)$ is the left support projection of x, i.e. $\mathfrak{l}(x)$ is the projection onto $\overline{x(H)}$.

Intermediate step for the proof of Theorem 1

For a subset $A \subset \mathcal{M}$, let $C^*(A)$ (resp. $W^*(A)$) denote the C^*-subalgebra (resp. von Neumann subalgebra) generated by A and $\mathbf{1}$. Let $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$, i.e. the centre of $\mathcal{M}$. Let $WZ^*(A)$ be the von Neumann subalgebra generated by $W^*(A)$ and $\mathcal{Z}(\mathcal{M})$. Let $\mathcal{K}(\mathcal{M}, \tau)$ denote the closure of all τ-finitely supported operators in $\mathcal{M}$.

Theorem 2 (Ber-Sukochev-Zanin-Zhao, 2024)

Let $\psi : C^*(\alpha) \to \mathcal{M}$ *be a* $*$*-monomorphism. Suppose*
(i) $WZ^*(\alpha) \cap \mathcal{K}(\mathcal{M}, \tau) = WZ^*(\psi(\alpha)) \cap \mathcal{K}(\mathcal{M}, \tau) = \{0\}$;
(ii) $\psi \sim_{\mathcal{M}} \mathrm{Id}_{C^*(\alpha)}$;
(iii) $k_{E(\mathcal{M})}(\alpha) = k_{E(\mathcal{M})}(\psi(\alpha)) = 0$.
We have $\psi \sim_{E(\mathcal{M})} \mathrm{Id}_{C^*(\alpha)}$, *i.e. there is a sequence of unitaries* $\{u_m\}_{m \geq 1}$ *in* $\mathcal{M}$ *such that* $\|\psi(a) - u_m^* a u_m\|_{E(\mathcal{M})} \to 0$ *as* $m \to \infty$, *for all* $a \in C^*(\alpha)$.

Key ingredient in the proof of Theorem 2

Theorem 3

Let $\mathcal{M}$ be a properly infinite σ-finite von Neumann algebra with a faithful normal semifinite trace τ. Let $\alpha \in \mathcal{M}^n$ be a commuting self-adjoint n-tuple such that

$$WZ^*(\alpha) \cap \mathcal{K}(\mathcal{M}, \tau) = \{0\}. \tag{1}$$

There exists a sequence of isometries $\{v_j\}_{j\geq 0} \subset \mathcal{M}$ such that
(i) $v_{j_1}^* v_{j_2} = \delta_{j_1,j_2}\mathbf{1}, \quad j_1, j_2 \geq 0;$
(ii) $[v_j, \alpha(i)] \in \mathcal{K}(\mathcal{M}, \tau)$ *for* $j \geq 0$, $1 \leq i \leq n$;
(iii) $\|[v_j, \alpha]\|_{\mathcal{M}} \to 0$ *as* $j \to \infty$.

Sketch of the proof of Theorem 1 when $\mathcal{M}$ is a factor

Suppose $\mathcal{M}$ is an infinite factor. Assume $W^*(\alpha) \cap \mathcal{K}(\mathcal{M}, \tau) = \{0\}$. Suppose $k_{E(\mathcal{M})}(\alpha) = 0$.

Choose a separating family $\{\psi_k\}_{k \in \mathbb{N}}$ of characters of $C^*(\alpha)$ (such a family exists since the spectrum of $C^*(\alpha)$ is compact and metrizable, hence, separable). Choose a sequence $\{p_k\}_{k \geq 1}$ of pairwise orthogonal projections in $\mathcal{M}$ such that $p_k \sim \mathbf{1}$ for each $k \geq 1$. Set

$$\psi(a) = \sum_{k \geq 1} \psi_k(a) p_k, \quad a \in C^*(\alpha).$$

Clearly, $\psi : C^*(\alpha) \to \mathcal{M}$ is a faithful $*$-homomorphism. Thus, $\psi \sim_{\mathcal{M}} \mathrm{Id}_{C^*(\alpha)}$ by the theorem of [Ciuperca-Giorrdano-Ng-Niu, 2013]. Hence, by Theorem 2, $\psi \sim_{E(\mathcal{M})} \mathrm{Id}_{C^*(\alpha)}$. Then

$$\|\psi(\alpha) - u_m^* \alpha u_m\|_{E(\mathcal{M})} \to 0, \quad (m \to \infty).$$

Set $\delta = \psi(\alpha)$, which is a commuting diagonal n-tuple.

Sketch of the proof when $\mathcal{M}$ is a non-factor

Suppose $\mathcal{M}$ is a *properly infinite*. That is, every central projection z is either infinite or zero. Let $\alpha \in \mathcal{M}^n$ be a commuting self-adjoint n-tuple such that $W^*(\alpha) \cap \mathcal{K}(\mathcal{M}, \tau) = \{0\}$ and $k_{E(\mathcal{M})}(\alpha) = 0$.

We can still construct a $*$-monomorphism $\psi : C^*(\alpha) \to \mathcal{M}$ as in the previous slide. That is,

$$\psi(a) = \sum_{k \geq 1} \psi_k(a) p_k, \quad a \in C^*(\alpha).$$

However, we no longer have $\psi \sim_{\mathcal{M}} \mathrm{Id}_{C^*(\alpha)}$ since $\mathcal{M}$ is not a factor.

Here comes the key idea: Instead of considering a separating family of characters $\{\psi_k\}_{k \geq 1}$ of $C^*(\alpha)$, how about considering a separating family of centre-valued homomorphisms $\{\psi_k : C^*(\alpha) \to \mathcal{Z}(\mathcal{M})\}_{k \geq 1}$ which "roughly resembles" a separating family of characters?

Proposition 4

Let $\mathcal{M}$ be a von Neumann algebra. Suppose $G \subset \mathcal{P}(\mathcal{M})$ is a countable commuting family of projections, and $Z \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ is a countable family of central projections in $\mathcal{M}$ such that $\mathbf{1} \in Z$. Let $\mathcal{A} = C^*(G \cup Z)$ and $\mathcal{B} = C^*(Z)$. There exists a family $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathrm{Hom}(\mathcal{A}, \mathcal{Z}(\mathcal{M}))$ such that
(i) $\psi_k|_{\mathcal{B}} = \mathrm{Id}_{\mathcal{B}}$;
(ii) *for every* $0 \neq a \in \mathcal{A}$ *there exists* $k \in \mathbb{N}$ *such that* $\psi_k(a) \neq 0$;
(iii) $p \leq \vee_{k\in\mathbb{N}}\psi_k(p)$ *for every* $p \in G$.

The above proposition can be transformed into a purely topological one, which we will now show.

Topological lemma

The lemma below follows from the fact that a continuous mapping of a compact metrizable space has a Borel inverse mapping.

Fact

Let X be a compact metric space, let Y be a Hausdorff topological space, and let $\pi : X \to Y$ be a continuous surjective mapping. $\exists$ Borel set $B \subset X$ such that π is injective on B. In addition, $f^{-1} : Y \to B$ is Borel.

Lemma 5 (Ber-Sukochev-Zanin-Zhao, 2024)

Let X be a totally disconnected compact metrizable space and let Y be a Hausdorff topological space. Let $\pi : X \to Y$ be a continuous surjective map. There exists a family of Borel mappings $\{\pi_k : Y \to X\}_{k\in\mathbb{N}}$ such that
(i) $\pi \circ \pi_k = \mathrm{Id}_Y$;
(ii) *for every open* $A \subset X$, *we have* $A \subset \cup_{k\in\mathbb{N}}(\pi_k \circ \pi)^{-1}(A)$;
(iii) *for every* $0 \le f \in C(X)$, *we have* $f \le \sup_{k\in\mathbb{N}} f \circ \pi_k \circ \pi$.

Construction

Let $\mathcal{M}$ be a properly infinite von Neumann algebra. Let $\alpha \in \mathcal{M}^n$ be a commuting self-adjoint n-tuple.
For every $m \in \mathbb{Z}_+$, let At_m be the collection of all cubes

$$\Big[\frac{k_1}{2^m}, \frac{k_1+1}{2^m}\Big) \times \cdots \times \Big[\frac{k_n}{2^m}, \frac{k_n+1}{2^m}\Big), \quad (k_1, \ldots, k_n) \in \mathbb{Z}^n.$$

Set

$$G = \cup_{m \in \mathbb{Z}_+} \{e^{\alpha}(U) : U \in \mathrm{At}_m\}, \quad Z = \{c(p) : p \in G\}, \quad \mathcal{A} = C^*(G \cup Z).$$

Let the sequence $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathrm{Hom}(\mathcal{A}, \mathcal{Z}(\mathcal{M}))$ be given by Proposition 4. Let $\{p_k\}_{k \in \mathbb{N}}$ be a sequence of pairwise orthogonal projections in $\mathcal{M}$ such that $p_k \sim \mathbf{1}$ for each $k \in \mathbb{N}$ and such that $\sum_{k \in \mathbb{N}} p_k = \mathbf{1}$. Define the $*$-homomorphism $\psi : \mathcal{A} \to \mathcal{M}$ by the formula

$$\psi(a) = \sum_{k \in \mathbb{N}} \psi_k(a) p_k, \quad a \in \mathcal{A},$$

Completion of the proof of Theorem 1

Theorem 6

Suppose $WZ^*(\alpha) \cap \mathcal{K}(\mathcal{M}, \tau) = \{0\}$*. The* $*$*-homomorphism* $\psi : \mathcal{A} \to \mathcal{M}$ *constructed in the previous slide is faithful, and satisfies* $\psi \sim_{\mathcal{M}} \mathrm{Id}_{C^*(\alpha)}$.

Thus, by Theorem 2, $\psi \sim_{E(\mathcal{M})} \mathrm{Id}_{C^*(\alpha)}$, namely, for every $\varepsilon > 0$, there is unitary $u \in \mathcal{M}$ such that $\|u\psi(\alpha)u^* - \alpha\|_{E(\mathcal{M})} < \varepsilon$.

It can be proved that there exists a commuting diagonal n-tuple $\delta \in \mathcal{M}^n$ such that $\|\psi(\alpha) - \delta\|_{E(\mathcal{M})} < \varepsilon$. Hence,

$$\|\alpha - u\delta u^*\|_{E(\mathcal{M})} < \varepsilon.$$

Note that $u\delta u^*$ is a commuting diagonal n-tuple since u is a unitary. The proof is complete.

Extension of Kuroda-Bercovici-Voiculecu's theorem

Theorem 7 (Ber-Sukochev-Zanin-Zhao, 2024)

Let $\mathcal{M}$ be a σ-finite von Neumann algebra with a faithful normal semifinite trace τ and let $n \in \mathbb{N}$. Let $(E, \|\cdot\|_E)$ be a symmetric function space on $(0, \infty)$. If $E \cap L_\infty \not\subset L_{n,1}$, then for every commuting self-adjoint n-tuple $\alpha \in (\mathrm{Aff}(\mathcal{M}))^n$ and every $\varepsilon > 0$, there exists a commuting diagonal n-tuple $\delta \in (\mathrm{Aff}(\mathcal{M}))^n$ such that

$$\alpha - \delta \in (E^0(\mathcal{M}) \cap \mathcal{M})^n, \quad \|\alpha - \delta\|_{(E \cap L_\infty)(\mathcal{M})} < \varepsilon.$$

A few words for the proof of Theorem 7

Consider the special case when E is a Lorentz space Λ_ψ.

Let ψ be an increasing concave function on $[0, \infty)$ such that $\psi(0) = 0$. The Lorentz space Λ_ψ is

$\Lambda_\psi = \{f \in S(0, \infty) : \int \mu(t; f) d\psi(t) < \infty\}$.

Lemma 8

Let $n \in \mathbb{N}$ and let ψ be an increasing concave function on $[0, \infty)$ such that $\psi(0) = 0$. The following conditions are equivalent.
(1) $\Lambda_\psi \cap L_\infty \not\subset L_{n,1}$; *(2)* $\liminf_{t\to\infty} \frac{\psi(t)}{t^{\frac{1}{n}}} = 0$; *(3)* $\liminf_{m\to\infty} \frac{\psi(2^{mn})}{2^m} = 0$.

The key point is to construct a sequence of τ-finite projections $\{p_m\}_{m\in\mathbb{N}}$ in $\mathcal{M}$ such that

$$\|[\alpha(j), p_m]\|_{\Lambda_\psi(\mathcal{M})} \leq C \frac{\psi(2^{mn})}{2^m} \to 0, \quad (m \to \infty),$$

or equivalently, $k_{\Lambda_\psi(\mathcal{M})}(\alpha) = 0$. Then Theorem 1 yields the proof for this case.

Pass from Λ_ψ to E

We have proved Kuroda's theorem for Lorentz spaces Λ_ψ.

To prove it for general symmetric function spaces E on $(0,\infty)$ with $E \cap L_\infty \not\subset L_{n,1}$, suppose that there is a commuting self-adjoint n-tuple $\alpha \in \mathcal{M}^n$ such that Theorem 7 does not hold. By Theorem 1,

$$k_{E(\mathcal{M})}(\alpha) > 0. \tag{2}$$

It can be proved that, there exists an increasing concave function ψ on $(0,\infty)$ such that $E \subset \Lambda_\psi$, $\psi(0+) = 0$ and

$$k_{\Lambda_\psi}(\alpha) > 0.$$

Then $\Lambda_\psi \cap L_\infty \subset L_{n,1}$. In particular, $E \cap L_\infty \subset L_{n,1}$, which is a contradiction. This proves Theorem 7.

(To find the mentioned ψ: (2) provides us with a convex subset in $(E(\mathcal{M}))^n$ that does not intersect the open unit ball in $(E(\mathcal{M}))^n$, the we use Hahn-Banach separation theorem and consider the Köthe dual of E.)

The case not covered by Kuroda's theorem, i.e. $E = L_1$

Let a be a self-adjoint operator on H. Let e^a be the spectral measure of a. There exists a complex measure μ on $\mathbb{R}$ such that $e^a(A) = 0$ iff $\mu(A) = 0$ for any Borel sets $A \subset \mathbb{R}$. Let $\mu = \mu_{ac} + \mu_s$ be the Radon-Nikodym decomposition of μ with respect to the Lebesgue measure. The projection $P_{ac}(a) = e^a(\mathrm{supp}(\mu_{ac}))$ (resp. $P_s(a) = e^a(\mathrm{supp}(\mu_s))$) is called the absolutely continuous part (resp. singular part) of a.

Theorem (Kato-Rosenblum, 1957)

If a, b are self-adjoint operators on H such that $a - b \in L_1(B(H))$, then the following limits exist in the strong operator topology
$W_\pm = \lim_{t\to\pm\infty} e^{itb}e^{-ita}P_{ac}(a)$.
As a result, the absolutely continuous parts of a and b are unitarily equivalent, i.e. $W_\pm^* W_\pm = P_{ac}(a), W_\pm W_\pm^* = P_{ac}(b)$.

Corollary: A self-adjoint operator a is diagonal modulo $L_1(B(H))$ if and only if $P_{ac}(a) = 0$.

The case not covered by Bercovici-Voiculescu's theorem, i.e. $E = L_{n,1}$

Theorem (Voiculescu, 1979)

Let $n \in \mathbb{N}$. *We have*
$(k_{L_{n,1}(B(H))}(\alpha))^n = \gamma_n \int_{\mathbb{R}^n} m(s) d\lambda(s)$
where $0 < \gamma_n < \infty$ *is a constant independent of* α, m *is the multiplicity function of the absolutely continuous part of* α. *For* $n = 1$ *we have* $\gamma_1 = \frac{1}{\pi}$. *In particular, if* $\alpha_{ac} \neq 0$, *then* $k_{L_{n,1}(B(H))}(\alpha) \neq 0$.

Theorem (Voiculescu, 1981)

Suppose $n \geq 2$. *Let* $\alpha, \beta \in (B(H))^n$ *be commuting self-adjoint* n*-tuple. If* $\beta(j) - \alpha(j) \in L_{n,1}(B(H))$ *for all* $1 \leq j \leq n$, *then* $\alpha_{ac} = u^* \beta_{ac} u$ *for some unitary* u.

Corollary

α *is diagonal modulo* $L_{n,1}(B(H))$ *if and only if* $\alpha_{ac} = 0$.

Kato-Rosenblem theorem in semifinite setting

Theorem (Li-Shen-Shi-Wang, 2018)

If A, B are densely-defined self-adjoint operators affiliated with $\mathcal{M}$ such that $A - B \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$, then
$W := s.o.t.\text{-}\lim_{t\to\infty} e^{itB}e^{-itA}P_{ac}^{\infty}(A)$ *exists in $\mathcal{M}$.*
Moreover, $W^*W = P_{ac}^{\infty}(A)$, and $WW^* = P_{ac}^{\infty}(B)$.

Here the definition of $P_{ac}^{\infty}(\cdot)$ is an adaptation of the notion of absolutely continuous part in the von Neumann algebras setting, which equals to $P_{ac}(\cdot)$ when $\mathcal{M} = B(H)$.

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