Noncommutative ergodic theory of lattices in higher rank simple algebraic groups

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Introduction and motivation

Simple algebraic groups

Let k be any local field i.e. k is a nondiscrete locally compact field.

Definition

Let $\mathbb{G}$ be any (affine) connected algebraic k-group. We say that

1. $\mathbb{G}$ is **semisimple** if its radical (i.e. maximal connected algebraic solvable normal subgroup) is trivial.
2. $\mathbb{G}$ is **absolutely almost simple** (resp. **almost k-simple**) if $\mathbb{G}$ is semisimple and if the only proper normal algebraic (resp. k-closed) subgroups are finite.

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Example

For every $n \geq 2$, SL_n is an absolutely almost simple connected algebraic k-group. Moreover, we have $\mathrm{rk}_k(\mathrm{SL}_n) = n - 1$.

Higher rank lattices

Let $\mathbb{G}$ be any almost k-simple connected algebraic k-group with $\mathrm{rk}_k(\mathbb{G}) \geq 2$.

Denote by $G = \mathbb{G}(k)$ the locally compact group of its k-points.

Let $\Gamma < G$ be any **lattice** i.e. $\Gamma < G$ is a discrete subgroup with finite covolume.

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Examples (Borel–Harish-Chandra, Behr, Harder)

Let $n \geq 3$ and $\mathbb{G} = \mathrm{SL}_n$.

- $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$
- $\mathrm{SL}_n(\mathbb{Z}[i]) < \mathrm{SL}_n(\mathbb{C})$
- $\mathrm{SL}_n\left(\mathbb{F}_q[t^{-1}]\right) < \mathrm{SL}_n\left(\mathbb{F}_q((t))\right)$ where $q = p^r$ with $p \in \mathcal{P}$ a prime and $r \geq 1$

In this talk, we simply say that $\Gamma < G$ is a **higher rank lattice**.

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Margulis' strategy: assuming that $N \lhd \Gamma$ is an infinite normal subgroup, to prove that Γ/N is a finite group, one shows that

1. Γ/N has **property (T)** (Kazhdan). Indeed, G has property (T) and property (T) is inherited by lattices and quotients.
2. Γ/N is **amenable** (Margulis). This follows from:

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Margulis' Factor Theorem (1978)

Let $\mathbb{P} < \mathbb{G}$ be any minimal parabolic k-subgroup and set $P = \mathbb{P}(k)$. Then any measurable Γ-factor of G/P is Γ-isomorphic to G/Q for some unique parabolic k-subgroup $\mathbb{P} < \mathbb{Q} < \mathbb{G}$ where $Q = \mathbb{Q}(k)$.

Goals

In this talk, we present a new framework to study **higher rank lattices** using operator algebras.

Main Problem

Given a **higher rank lattice** $\Gamma < G$, we want to understand:

1. Point stabilizers for ergodic/minimal actions $\Gamma \curvearrowright X$
2. Structure of group C^*-algebras $\mathrm{C}^*_\pi(\Gamma)$ where $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$
3. Dynamical properties of the affine action $\Gamma \curvearrowright \mathscr{P}(\Gamma)$
4. Rigidity aspects of the group von Neumann algebra $\mathrm{L}(\Gamma)$

Bibliography

The present talk is based on two joint works:

[BH19] R. Boutonnet, C. Houdayer, *Stationary characters on lattices of semisimple Lie groups.* Publications mathématiques de l'IHÉS **133** (2021), 1-46. arXiv:1908.07812

[BBH21] U. Bader, R. Boutonnet, C. Houdayer, *Charmenability of higher rank arithmetic groups.* arXiv:2112.01337

The noncommutative Nevo–Zimmer theorem

Structure theory of G/P

Let $\mathbb{G}$ be any almost k-simple connected algebraic k-group with $\mathrm{rk}_k(\mathbb{G}) \geq 2$ and set $G = \mathbb{G}(k)$. Let $\mathbb{P} < \mathbb{G}$ be any minimal parabolic k-subgroup and set $P = \mathbb{P}(k)$. Then $G/P = (\mathbb{G}/\mathbb{P})(k)$.

Example

If $\mathbb{G} = \mathrm{SL}_n$, take $\mathbb{P} < \mathbb{G}$ the subgroup of upper triangular matrices.

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Theorem (Furstenberg 1962, Bader–Shalom 2004)

*For every admissible measure $\mu \in \mathrm{Prob}(G)$, there exists a unique μ-stationary measure $\nu \in \mathrm{Prob}(G/P)$ such that $(G/P, \nu)$ is the (G, μ)-**Poisson boundary** i.e.*

$$\mathrm{L}^\infty(G/P, \nu) \underset{G\text{-equiv.}}{\cong} \mathrm{Har}^\infty(G, \mu)$$

Recall that ν is μ-**stationary** if $\nu = \mu * \nu = \int_G g_* \nu \, \mathrm{d}\mu(g)$.

Boundary structures on von Neumann algebras

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This notion is well adapted to induction. Indeed, to any Γ-boundary structure $\Theta : M \to \mathrm{L}^\infty(G/P)$, one can naturally define the **induced** G-boundary structure $\widehat{\Theta} : \mathrm{Ind}_\Gamma^G(M) \to \mathrm{L}^\infty(G/P)$.

Recall that $\Gamma \curvearrowright M$ is **ergodic** if

$$M^\Gamma := \{x \in M \mid \forall \gamma \in \Gamma,\ \sigma_\gamma(x) = x\} = \mathbb{C}1$$

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Since $\Gamma \curvearrowright G/P$ is amenable, there exists a measurable Γ-map $\beta : G/P \to \mathfrak{S}(A) : b \mapsto \beta_b$. By duality, we obtain a ucp Γ-map $\mathrm{E} : A \to \mathrm{L}^\infty(G/P)$ defined by $\mathrm{E}(\cdot)(b) = \beta_b$ for a.e. $b \in G/P$.

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Consider the normal extension $\mathrm{E}^{**} : A^{**} \to \mathrm{L}^\infty(G/P)$ and denote by $z \in \mathscr{Z}(A^{**})$ its central support. Letting $M = A^{**}z$, $\Theta = \mathrm{E}^{**}|_M : M \to \mathrm{L}^\infty(G/P)$ is a Γ-boundary structure.

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We apply the above construction to the following situations:

1. $A = \mathrm{C}(X)$ where X is a compact metrizable Γ-space.
2. $A = \mathrm{C}^*_\pi(\Gamma)$ where $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ is a unitary representation and $\Gamma \curvearrowright \mathrm{C}^*_\pi(\Gamma)$ is the conjugation action.

The noncommutative Nevo–Zimmer theorem

Theorem (BH19, BBH21)

Let $\Gamma < G$ *be any* **higher rank lattice**. *Let* M *be any von Neumann algebra,* $\Gamma \curvearrowright M$ *any ergodic action and* $\Theta : M \to \mathrm{L}^\infty(G/P)$ *any* Γ*-boundary structure.*

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Then the following dichotomy holds:

- *Either $\Theta(M) = \mathbb{C}1$.*
- *Or there are a proper parabolic k-subgroup $\mathbb{P} < \mathbb{Q} < \mathbb{G}$ and a Γ-equivariant normal embedding $\iota : \mathrm{L}^\infty(G/Q) \hookrightarrow M$ such that $\Theta \circ \iota : \mathrm{L}^\infty(G/Q) \hookrightarrow \mathrm{L}^\infty(G/P)$ is the canonical embedding.*

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In case $M = \mathrm{L}^\infty(X)$ and $k = \mathbb{R}$, and considering G-actions instead of Γ-actions, the above theorem was proven by Nevo–Zimmer (2000).

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$\rightsquigarrow$ In [BBH21], we further generalized NZ theorem to deal with lattices in simple algebraic groups defined over an arbitrary local field k.

About the proof of the nc Nevo–Zimmer theorem

Using induction, we prove the theorem for ergodic G-von Neumann algebras $\mathscr{M}$ and G-boundary structures $\Theta : \mathscr{M} \to \mathrm{L}^\infty(G/P)$.

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Exploiting the tension between algebraic geometry and ergodic theory, we show that there exist a proper parabolic k-subgroup $\mathbb{P} < \mathbb{Q} < \mathbb{G}$ and a measurable G-factor map $Z \to G/Q$.
This yields a G-equivariant embedding $\mathrm{L}^\infty(G/Q) \hookrightarrow \mathscr{Z} \subset \mathscr{M}$.

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This part is a generalization of Nevo–Zimmer's Gauss map trick.

Dichotomy for topological dynamics

We derive the following topological analogue of Stuck–Zimmer's stabilizer rigidity theorem (1992).

Theorem (BH19, BBH21)

Let $\Gamma < G$ *be any* **higher rank lattice**. *Assume that* $\mathscr{Z}(\mathbb{G}) = \{e\}$. *Let* $\Gamma \curvearrowright X$ *be any minimal action on a compact metrizable space.*

The following dichotomy holds:

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The following dichotomy holds:

- *Either* X *is finite.*
- *Or* $\Gamma \curvearrowright X$ *is* **topologically free** *i.e. for every* $\gamma \in \Gamma \setminus \{e\}$, $\mathrm{Fix}(\gamma) := \{x \in X \mid \gamma x = x\}$ *has empty interior.*

This solves a question raised by Glasner–Weiss (2014).

Dynamics of positive definite functions and character rigidity

Dynamics of $\Lambda \curvearrowright \mathscr{P}(\Lambda)$

For any countable discrete group Λ, set

$$\mathscr{P}(\Lambda) := \{\varphi : \Lambda \to \mathbb{C} \mid \text{normalized positive definite function}\}$$

Then $\mathscr{P}(\Lambda) \subset \ell^\infty(\Lambda)$ is a weak-$*$ compact convex set.

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To any $\varphi \in \mathscr{P}(\Lambda)$, one associates the GNS triple $(\pi_\varphi, \mathscr{H}_\varphi, \xi_\varphi)$:

$$\forall \gamma \in \Lambda, \quad \varphi(\gamma) = \langle \pi_\varphi(\gamma)\xi_\varphi, \xi_\varphi \rangle$$

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Definition

A **character** $\varphi \in \mathscr{P}(\Lambda)$ is a fixed point for $\Lambda \curvearrowright \mathscr{P}(\Lambda)$.

Denote by $\mathrm{Char}(\Lambda) \subset \mathscr{P}(\Lambda)$ the convex subset of all characters.

Examples of characters

Denote by $\mathsf{Sub}(\Lambda)$ the compact metrizable space of all subgroups of Λ endowed with the conjugation action $\gamma \cdot H = \gamma H \gamma^{-1}$.

Consider the Λ-equivariant continuous map

$$\mathsf{Sub}(\Lambda) \to \mathscr{P}(\Lambda) : H \mapsto 1_H$$

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1. If $N \lhd \Lambda$ is a normal subgroup, then $\varphi = 1_N \in \mathrm{Char}(\Lambda)$.
   Its GNS unirep $\pi_\varphi = \lambda_{\Lambda/N}$ is the quasi-regular representation.
   - When $N = \Lambda$, then 1_Λ is the trivial character.
   - When $N = \{e\}$, then $1_{\{e\}}$ is the regular character.

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   - When $N = \Lambda$, then 1_Λ is the trivial character.
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2. If $\Lambda \curvearrowright (X, \nu)$ is pmp, then $\varphi_\nu : \gamma \mapsto \nu(\mathrm{Fix}(\gamma)) \in \mathrm{Char}(\Lambda)$.
   $\rightsquigarrow$ When $\varphi_\nu = 1_{\{e\}}$, the action $\Lambda \curvearrowright (X, \nu)$ is essentially free.

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   $\rightsquigarrow$ When $\varphi_\nu = 1_{\{e\}}$, the action $\Lambda \curvearrowright (X, \nu)$ is essentially free.
3. If $\pi : \Lambda \to \mathscr{U}(n)$ is a finite dim unirep, then $\mathrm{tr}_n \circ \pi \in \mathrm{Char}(\Lambda)$.

Charmenability

Our noncommutative Nevo–Zimmer theorem yields new applications regarding **existence** and **classification** of characters.

Theorem (BH19, BBH21)

Let $\Gamma < G$ *be any* **higher rank lattice**. *Then*

1. *Any nonempty* Γ*-invariant weak-*$*$ *compact convex subset* $\mathscr{C} \subset \mathscr{P}(\Gamma)$ *contains a character.*
2. Γ *is* **character rigid** *i.e. any extremal character* $\varphi \in \mathrm{Char}(\Gamma)$ *is either supported on* $\mathscr{Z}(\Gamma)$ *or* π_φ *is finite dimensional.*

Our theorem strengthens results by Margulis (1978), Stuck–Zimmer (1992), Bekka (2006), Creutz–Peterson (2013), Peterson (2014).

Structure theorem for group C^*-algebras $C^*_\pi(\Gamma)$

When $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ is a unirep, we may regard

$$\mathfrak{S}(C^*_\pi(\Gamma)) \hookrightarrow \mathscr{P}(\Gamma) : \psi \mapsto \psi \circ \pi$$

as a Γ-invariant weak-$*$ compact convex subset. We obtain:

Theorem (BH19, BBH21)

Let $\Gamma < G$ be any **higher rank lattice**. *Let $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ be any unirep. Then $C^*_\pi(\Gamma)$ admits a trace.*

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Theorem (BH19, BBH21)

Let $\Gamma < G$ be any **higher rank lattice***. Let $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ be any unirep. Then $C^*_\pi(\Gamma)$ admits a trace.*

Assume that $\mathscr{Z}(\mathbb{G}) = \{e\}$. If π is weakly mixing, then $\lambda \prec \pi$ i.e. there is a $*$-homomorphism $\Theta : C^*_\pi(\Gamma) \to C^*_\lambda(\Gamma)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Gamma$. Moreover

1. *$\tau_\Gamma \circ \Theta$ is the unique trace on $C^*_\pi(\Gamma)$.*
2. *$\ker(\Theta)$ is the unique maximal proper ideal of $C^*_\pi(\Gamma)$.*

This extends results by Bekka–Cowling–de la Harpe (1994) for $C^*_\lambda(\Gamma)$.

The noncommutative factor theorem and Connes' rigidity conjecture

Connes' rigidity conjecture for higher rank lattices

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In view of Mostow–Margulis' rigidity results, we state the following version of **Connes' rigidity conjecture** for lattices in higher rank simple real Lie groups.

Connes' rigidity conjecture

For $i \in \{1,2\}$, let G_i be any real connected simple Lie group with trivial center and $\mathrm{rk}_{\mathbb{R}}(G_i) \geq 2$, and let $\Gamma_i < G_i$ be any lattice.

$$\begin{aligned} \mathrm{L}(\Gamma_1) \cong \mathrm{L}(\Gamma_2) \quad &\Rightarrow \quad G_1 \cong G_2 \\ &\Rightarrow \quad \mathrm{rk}_{\mathbb{R}}(G_1) = \mathrm{rk}_{\mathbb{R}}(G_2) \end{aligned}$$

The noncommutative factor theorem

Let $\Gamma < G$ be any higher rank lattice. Assume that $\mathscr{Z}(\mathbb{G}) = \{e\}$.

Consider the ergodic action $\Gamma \curvearrowright G/P$ and its associated **group measure space** von Neumann algebra $\mathrm{L}(\Gamma \curvearrowright G/P)$.

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Theorem (BBH21)

For every von Neumann subalgebra $\mathrm{L}(\Gamma) \subset M \subset \mathrm{L}(\Gamma \curvearrowright G/P)$*, there exists a unique parabolic* k*-subgroup* $\mathbb{P} < \mathbb{Q} < \mathbb{G}$ *such that*

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There are exactly $2^{\mathrm{rk}_k(\mathbb{G})}$ parabolic k-subgroups $\mathbb{P} < \mathbb{Q} < \mathbb{G}$.

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The inclusion $\mathrm{L}(\Gamma) \subset \mathrm{L}(\Gamma \curvearrowright G/P)$ retains the k-rank $\mathrm{rk}_k(\mathbb{G})$.

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☺ It gives hope to prove that $\mathrm{L}(\Gamma)$ retains the k-rank $\mathrm{rk}_k(\mathbb{G})$.

About the proof of the nc factor theorem

Set $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright G/P)$. Since $\mathscr{Z}(\mathbb{G}) = \{e\}$, one can show that $\mathrm{L}(\Gamma)' \cap \mathscr{B} = \mathbb{C}1$. Thus, the conjugation action $\Gamma \curvearrowright \mathscr{B}$ is ergodic.

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  Since $\Gamma \curvearrowright G/Q$ is ess. free, by Suzuki's theorem (2018), up to taking a smaller $\mathbb{P} < \mathbb{Q} < \mathbb{G}$, we have $M = \mathrm{L}(\Gamma \curvearrowright G/Q)$.

Thank you for your attention!