Intermediate von Neumann subalgebras arising from multiple transitive actions

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Overview

(Some results are based on joint work with Adam Skalski)

I: Setup and a general question

Setup and notations

- G: a countable infinite group.
- X: an infinite space.
- $G \curvearrowright X$: an action, i.e. a group homomorphism $G \to Perm(X)$.
- H: the stabilizer subgroup of any point $x \in X$, i.e.

$$H = \{s \in G : sx = x\}.$$

- $L(G)$: the group von Neumann algebra, i.e.

$$L(G) = \overline{span\{\lambda_g : g \in G\}}^{SOT},$$

  where $\lambda_g \in \mathcal{U}(\ell^2(G))$ is defined by $\lambda_g(\delta_s) = \delta_{gs}$ for all $s \in G$.
- $L(G)$ is a (II_1) factor, i.e. $\mathcal{Z}(L(G)) = \mathbb{C}$ iff G is I.C.C., i.e. $\sharp\{sgs^{-1} : s \in G\} = \infty,\ \forall\, g \neq e$.

Note that $H < G$ induces $L(H) < L(G)$.

Transitive actions

Definition ((Sharply) n-transitive actions)

An action $G \curvearrowright X$ is *faithful* if $G \to Perm(X)$ is injective.
It is *n-transitive* if for every n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in X^n with distinct entries, there exists some $g \in G$ s.t. $gx_i = y_i$ for all $i = 1, \dots, n$.
It is *highly transitive if it is n-transitive for all $n \geq 2$.*
It is *sharply n-transitive* if it is n-transitive and the element g above is unique.

Examples of n-transitive actions

(1) 1-transitive=transitive, i.e. a left translation $G \curvearrowright G/H$.

(2) Examples of 2-transitive actions:

  (i) Let $G \curvearrowright A$ be an action. Consider the affine action $A \rtimes G \curvearrowright A$, i.e. $(ag).x = a + g.x$. One can check this affine action is 2-transitve iff $G \curvearrowright A \setminus \{0_A\}$ is transitive. E.g. $\mathbb{Q}^2 \rtimes SL_2(\mathbb{Q}) \curvearrowright \mathbb{Q}^2$ is 2-transitive.

  (ii) Left-right multiplication $G \times G \curvearrowright G$: $(s,t).g = sgt^{-1}$ is 2-transitive iff G has exactly 2 conjugacy classes.

(3) Examples of 3-transitive actions:

  (i) $PSL_2(\mathbb{Q}) \curvearrowright P^1(\mathbb{Q})$, where $P^1(\mathbb{Q}) := \mathbb{Q}^2/\sim$ and $(x,y) \sim (-x,-y)$.

  (ii) the affine action $A \rtimes G \curvearrowright A$, where $A = \oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ and $G = Aut_f(A)$.

(4) Examples of n-transitive actions: $S_\infty \curvearrowright \mathbb{N}$.

On highly transitive actions

The 1st explicit example of faithful, highly transitive action of free groups are constructed by McDonough in 1976, which relies on the lemma below.

Lemma (McDonough '76)

Let $X = \mathbb{Z}$, and let $s \in Perm(X)$ be defined by $s(i) = i + 1$, $\forall\ i \in X$. Suppose t is an infinite cycle (i.e. $\langle t \rangle \curvearrowright X \setminus Fix(t)$ is transitive and $\sharp(X \setminus Fix(t)) = \infty$) satisfying the conditions

(i) *If $t(i) \neq i$, then $t(j) \neq j$, $\forall\ j > i$;*

(ii) *$Fix(t) \neq \emptyset$.*

Then $\langle s, t \rangle$ is highly transitive on X.

Remark: Many group theorists studied the question of characterizing which groups admit faithful highly transitive actions. Huge classes of groups are known to admit faithful highly transitive actions.

Some properties of n-transitive actions

(1) $(n+1)$-transitivity $\Rightarrow$ n-transitivity.

(2) If $G \curvearrowright X$ is 2-transitive, then H is a maximal subgroup in G. Indeed, it is an exercise to check $G = H \sqcup HsH$ for any $s \in G \setminus H$.

(3) If $G \curvearrowright X$ is faithful and 2-transitive, then G is I.C.C., i.e. $\sharp\{tgt^{-1} : t \in G\} = \infty$ for all $g \neq e$.

Proof.

Let $e \neq g \in G$, say $y := gx \neq x$ (by faithfulness). Let $H = Stab(x)$. Take infinitely many distinct pts $z_i \in X \setminus \{x\}$, and define $h_i(x, y) = (x, z_i)$, $\forall\ i$. Then

$$h_i g h_i^{-1} \neq h_j g h_j^{-1},\ \forall\ i \neq j.$$

Indeed, $h_j^{-1} h_i g (h_j^{-1} h_i)^{-1} x = h_j^{-1} h_i y \neq y = gx$. □

A general question

Question

Let $n \geq 2$. Assume $G \curvearrowright X$ is a faithful and n-transitive action and H is the stabilizer subgroup of any point $x \in X$, can we describe all intermediate von Neumann algebras between $L(H)$ and $L(G)$?

RK: in the above context, the following hold.

- H is a maximal subgroup in G (with infinite index);
- H does not contain non-trivial normal subgroups of G.

Proof.

Assume $K \lhd G$ and $K \subseteq H$. Then $g^{-1}Kg = K$ fixes x, thus, K fixes $gx, \forall g \in G$. Transitivity implies $X = Gx$, thus $K \subseteq Ker(G \to Perm(X)) = \{e\}$. □

Some typical results on studying intermediate vN algs

Who? When?	Setting	Assumptions
Nakamura-Takeda, N.Suzuki '60	$N^G \subseteq N$	N: a II_1 factor; G: finite; $(N^G)' \cap N = \mathbb{C}$
Choda '78	$N \subseteq N \rtimes G$	N: a II_1 factor; $G \curvearrowright N$ outer
Packer '85	$L^\infty(Y) \rtimes G \subseteq L^\infty(X) \rtimes G$	$G \curvearrowright X \twoheadrightarrow Y$ p.m.p. free ergodic
Ge-Kadison '96	$M \subseteq M \bar{\otimes} N$	M: a factor
Izumi-Longo-Popa '98	$N^G \subseteq N$	N: a factor with sep.predual; G: cpt; $(N^G)' \cap N = \mathbb{C}$
Y.Suzuki '19	$L^\infty(Y) \rtimes G \subseteq L^\infty(X) \rtimes G$	$G \curvearrowright X \twoheadrightarrow Y$ non-singular free
Chifan-Das '19	$L(G) \subseteq L^\infty(X) \rtimes G$	$G \curvearrowright X$: compact action
Chifan-Das '19	$L(H) \subseteq L(G)$	$H \triangleleft G$, $L(H)' \cap L(G) = \mathbb{C}$

II: Motivation

Motivation I

Definition

A subfactor/subalgebra is *maximal* if it is not contained in any proper subalgebra other than itself.

Question (Ge, '03)

- *Can a non-hyperfinite factor of type II_1 have a hyperfinite subfactor as its maximal subfactor?*
- *Can a maximal subfactor of the hyperfinite factor of type II_1 have an infinite index Jones index?*
- *Can LF_∞ be embedded in LF_2 as a maximal subfactor?*

A natural approach: If H is a maximal subgroup in G (with infinite index), then perhaps we may show $L(H)$ is maximal in $L(G)$.

Bad news: Maximal subgroup does not yield maximal subalgebra

In general, H is maximal in $G \not\Rightarrow L(H)$ is also maximal in $L(G)$.

An example:

Consider $G = K \times K \curvearrowright X := K$ defined by $(s,t).k = skt^{-1}$.

Fix $x = e_K \in X$, then $H = Stab(x) = \Delta(G) = \{(k,k) : k \in K\}$ is maximal iff K is simple.

Notice that $L(H) \subsetneq Fix(\phi) \subsetneq L(G)$, where $\phi \in Aut(L(G))$ is induced by the flip automorphism $\phi \in Aut(G)$, i.e. $\phi(s,t) = (t,s)$.

Indeed, $u_{(e_K,s)} + u_{(s,e_K)} \in Fix(\phi) \setminus L(H)$ for all $e_K \neq s \in K$.

$u_{(e_K,s)} \in L(G) \setminus Fix(\phi)$ for all $e_K \neq s \in K$.

Good news: Many known works on the existence of maximal subgroups with infinite index

A nice survey on the state of the art can be found in "Maximal subgroups of countable groups, a survey" arXiv: 1909.09361. A pioneering result is

Theorem (Margulis, Soĭfer,'77-81)

For $n \geq 3$, there exists a maximal subgroup of $SL_n(\mathbb{Z})$ of infinite index.

However, the proof relies on Zorn's lemma, thus it is hard to put hands on the algebraic properties of maximal subgroups above.

Motivation II

We initiated the study of maximal Haagerup von Neumann subalgebras together with Adam Skalski in 2019.

Definition (Maximal Haagerup von Neumann algebras)

Let $N \subseteq M$ be an inclusion of von Neumann algebras. We say N is a *maximal Haagerup von Neumann subalgebra* if N has Haagerup property and for every P s.t. $N \subsetneq P \subseteq M$, P does not have Haagerup property.

Question

Is $L(SL_2(\mathbf{k}))$ a maximal Haagerup von Neumann subalgebra in $L(\mathbf{k}^2 \rtimes SL_2(\mathbf{k}))$, where $\mathbf{k} = \mathbb{Z}$ or $\mathbb{Q}$?

Recall that the affine action $G := \mathbb{Q}^2 \rtimes SL_2(\mathbb{Q}) \curvearrowright \mathbb{Q}^2 := X$ is 2-transitive and $H := Stab(\binom{0}{0}) = SL_2(\mathbb{Q})$.

III: Known results

Known results: $n \geq 4$, $n = 3$

Recall that we want to study the question:

Question

Let $n \geq 2$. Assume $G \curvearrowright X$ is a faithful and n-transitive action and H is the stabilizer subgroup of any point $x \in X$, can we describe all intermediate von Neumann algebras between $L(H)$ and $L(G)$?

Theorem (J., 2019)

If $n \geq 4$, then $L(H)$ is a maximal von Neumann subalgebra in $L(G)$, i.e. if $L(H) \subseteq P \subseteq L(G)$, then $P = L(H)$ or $L(G)$.

Proposition (J., 2019)

Consider the sharply 3-transitive action:

$$G := PSL_2(\mathbb{Q}) \curvearrowright X := P^1(\mathbb{Q}),\ x = [\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)] \in X.$$

Then $L(H)$ is maximal in $L(G)$. Note that $H = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq G$.

Corollary

Corollary (J.-Skalski 2019, J. 2019)

Ge's question mentioned before has affirmative answers.

More precisely, we have:

(1) The hyperfinite subfactor $L(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix})$ is maximal in the non-hyperfinite factor $L(PSL_2(\mathbb{Q}))$.
$[PSL_2(\mathbb{Q}) \curvearrowright P^1(\mathbb{Q})]$

(2) $L(Fix(\{1\}))$ is maximal in $L(S_\infty)$ with finite Jones index.
$[S_\infty \curvearrowright \mathbb{N}]$

(3) LF_∞ can be embedded into $L(F_2)$ as a maximal subfactor.
$[F_2 \curvearrowright X]$

N.B. It is still open whether the hyperfinite II_1 factor R can be embedded into $L(F_2)$ as a maximal subfactor.

Known results: $n = 3$

Another partial answer:

Theorem (Zhou, 2020)

If $n = 3$ and further assume $G \curvearrowright X$ is sharply 3-transitive or $sHs^{-1} \cap H$ is I.C.C. for all $s \in G$, then $L(H)$ is maximal in $L(G)$.

Examples of transitive actions such that $sHs^{-1} \cap H$ is I.C.C.:

(1) the affine action $A \rtimes G \curvearrowright A$, where $A = \oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ and $G = Aut_f(A)$.

(2) Let $G \curvearrowright S^1$ be a minimal, proximal, and not topologically free action. Then consider $G \curvearrowright X = G \cdot p$ for any $p \in S^1$.

Here, for a minimal action $G \curvearrowright S^1$, it is called *proximal* if for all open intervals $I, J \subsetneq S^1$, $J \neq \emptyset$, there exists $g \in G$ such that $g(I) \subseteq J$.

[Le Boudec A, Matte Bon N., 19] proved for the above G, every faithful, 3-transitive action of G on a set Ω, there exists a G-orbit $\mathcal{O} \subseteq S^1$ such that the action of G on Ω is conjugate to the action on $\mathcal{O}$.

Known results: $n = 2$

Building on the known works (due to Park '92, Witte '94) on classification of all quotient actions of $SL_2(\mathbb{Z}) \curvearrowright \widehat{\mathbb{Z}^2}$, we prove

Theorem (J., 2019)

Let $SL_2(\mathbf{k}) \curvearrowright Y$ be the quotient action of $SL_2(\mathbf{k}) \curvearrowright \widehat{\mathbf{k}^2}$ defined by modding out the relation $\phi \sim \phi'$, where $\phi'(x,y) := \phi(-x,-y)$ for all $(x,y) \in \mathbf{k}^2$. If $L(SL_2(\mathbf{k})) \subsetneq P \subsetneq L^\infty(Y) \rtimes SL_2(\mathbf{k})$, then

$$P = \begin{cases} q[L(SL_2(\mathbf{k}))] \oplus (1-q)[L^\infty(Y) \rtimes SL_2(\mathbf{k})], & \text{if } \mathbf{k} = \mathbb{Q} \\ \quad q[(L^\infty(Y) \cap L^\infty(\widehat{m_1\mathbb{Z}^2})) \rtimes SL_2(\mathbf{k})] \oplus & \\ (1-q)[(L^\infty(Y) \cap L^\infty(\widehat{m_2\mathbb{Z}^2})) \rtimes SL_2(\mathbf{k})], & \text{if } \mathbf{k} = \mathbb{Z} \end{cases}$$

where $q \in \{\frac{u_{id}+u_{-id}}{2}, \frac{u_{id}-u_{-id}}{2}\}$ is a central projection in $L^\infty(Y) \rtimes SL_2(\mathbf{k})$.

Remark: this theorem basically says up to the central elements q and $1-q$, every intermediate vN alg comes from a quotient action.

Corollary

Using the previous result and other works (due to Jones and Xu '04, Ioana '10), we could show

Corollary (J., 2019)

$L(SL_2(\mathbf{k}))$ is a maximal Haagerup von Neumann subalgebra in $L(\mathbf{k}^2 \rtimes SL_2(k))$ for $\mathbf{k} = \mathbb{Q}, \mathbb{Z}$.

N.B. It is open whether $L(\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix})$ is maximal Haagerup in $L(SL_3(\mathbb{Z}))$.

IV: Proofs and remaining questions

A general strategy for the proof

To determine P such that $L(H) \subseteq P \subseteq L(G)$, it suffices to determine $\{E(u_g) : g \in G\}$, where $E : L(G) \to P$ is the C.E.

(Idea) Treat it as a set of unknowns, find sufficiently many equations, e.g.

(1) $\phi(E(u_g)) = E(\phi(u_g))$, $\forall \phi \in Aut(L(G), P)$, e.g. $\phi = Ad(u_h)$, $\forall h \in H$;

(2) $E(E(u_s)u_t) = E(u_s)E(u_t)$, $\forall s, t \in G$.

Set $\phi = Ad(u_{gsg^{-1}})$, $\forall s \in g^{-1}Hg \cap H$ in (1), check (1) becomes

$$u_g^* E(u_g) \in L(g^{-1}Hg \cap H)' \cap L(G).$$

Remark: (1) has been used in many works; (2) has not attracted enough attention.

Setting 1

Setting 1: Let $G \curvearrowright X$ be a 4 or 3-transitive action.

Goal: show 4-transitivity or certain 3-transitivity implies $L(H)$ is maximal in $L(G)$.

Observation:

- Assume H is maximal in G, then $L(H)$ is maximal in $L(G)$ iff $u_g^* E(u_g) \in \mathbb{C}$ for all $g \in G$.

Proof.

" $\Leftarrow$ ": Let $K = \{g \in G : E(u_g) \neq 0\}$. Show $P = L(K)$ and $H < K$. □

- $\forall_{g \in G \setminus H} E(u_g) = 0$ (resp. $\forall_{g \in G \setminus H} E(u_g) = u_g$) $\Rightarrow P = L(H)$ (resp. $P = L(G)$).

Recall

$$u_g^* E(u_g) \in L(g^{-1} H g \cap H)' \cap L(G).$$

(Key pt) 4-transitivity (resp. 3-transitivity) implies $L(g^{-1} H g \cap H)' \cap L(G)$ is small. Then apply (2) (in last slide) to suitable s, t to determine $E(u_g)$.

From 3(or 4)-transitivity to small relative commutant

Q: Why 3 (or 4)-transitivity implies $L(g^{-1}Hg \cap H)' \cap L(G)$ is small?

(Exc) Let $K < G$ be groups and $x \in L(K)' \cap L(G)$. Then $\sharp\{ktk^{-1} : k \in K\} = \infty$ implies $t \notin supp(x)$.

Proof.

Let $x = \sum_{t \in G} \lambda_t t \in L(K)' \cap L(G)$. Thus, $\lambda_t = \lambda_{ktk^{-1}}$, $\forall\ k \in K, t \in G$. Fix any $e \neq t$. Let $C := \{ktk^{-1} : k \in K\}$. Note that

$$\infty > \sum_{s \in G} |\lambda_s|^2 \geq \sum_{c \in C} |\lambda_c|^2 = \sum_{c \in C} |\lambda_t|^2 = |\lambda_t|^2 \sharp C,$$

we deduce $\lambda_t = 0$ if $\sharp C = \infty$. □

Thus $L(K)' \cap L(G)$ is small if we can find sufficiently many t whose K-conjugacy orbit has infinite size.

Let $K = g^{-1}Hg \cap H = Stab(x) \cap Stab(g^{-1}x)$.

(Key pt) 3(or 4)-transitivity tells us we have freedom to construct many k.

Setting 2

Setting 2: the affine 2-transitive action $G := \mathbb{Q}^2 \rtimes SL_2(\mathbb{Q}) \curvearrowright X := \mathbb{Q}^2$.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{Q}^2$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $H := Stab(\overrightarrow{0}) = SL_2(\mathbb{Q})$. Let $L(SL_2(\mathbb{Q})) \subseteq P \subseteq L(\mathbb{Q}^2 \rtimes SL_2(\mathbb{Q}))$.
Then $c_v := u_v^* E(u_v) \in L(v^{-1}Hv \cap H)' \cap L(G)$, $\forall\ v \in \mathbb{Q}^2$; moreover, for $v = e_1$, $c_{e_1} \in L(\begin{pmatrix} \mathbb{Q} \\ 0 \end{pmatrix} \rtimes \pm \begin{pmatrix} 1 & \mathbb{Q} \\ 0 & 1 \end{pmatrix}))$.
We may write

$$c_{e_1} = \sum_{x,y \in \mathbb{Q}} \lambda_{x,y} \begin{pmatrix} x \\ 0 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} + \sum_{x,y \in \mathbb{Q}} \mu_{x,y} \begin{pmatrix} x \\ 0 \end{pmatrix} \begin{pmatrix} -1 & y \\ 0 & -1 \end{pmatrix}.$$

From $c_{g.v} = \sigma_g(c_v)$, we deduce that

$$c_{e_2} = \sum_{x,y \in \mathbb{Q}} \lambda_{x,y} \begin{pmatrix} 0 \\ x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} + \sum_{x,y \in \mathbb{Q}} \mu_{x,y} \begin{pmatrix} 0 \\ x \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -y & -1 \end{pmatrix}.$$

The goal is to solve for $\{E(u_v) : v \in \mathbb{Q}^2\}$; equivalently, solve for $\{c_v : v \in \mathbb{Q}^2\}$, which boils down to solving for $\lambda_{x,y}$ and $\mu_{x,y}$ for all x, y.

Continue...

Recall that $L(SL_2(\mathbb{Q})) \subseteq P \subseteq L(\mathbb{Q}^2 \rtimes SL_2(\mathbb{Q}))$ and we have

$$E(u_{e_1}) = u_{e_1} c_{e_1} = \sum_{x,y\in\mathbb{Q}} \lambda_{x,y} \left(\begin{smallmatrix} x+1 \\ 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) + \sum_{x,y\in\mathbb{Q}} \mu_{x,y} \left(\begin{smallmatrix} x+1 \\ 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & y \\ 0 & -1 \end{smallmatrix}\right),$$
$$E(u_{e_2}) = u_{e_2} c_{e_2} = \sum_{x,y\in\mathbb{Q}} \lambda_{x,y} \left(\begin{smallmatrix} 0 \\ x+1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -y & 1 \end{smallmatrix}\right) + \sum_{x,y\in\mathbb{Q}} \mu_{x,y} \left(\begin{smallmatrix} 0 \\ x+1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ -y & -1 \end{smallmatrix}\right).$$

We apply $E(E(u_{e_1})u_{e_2}) = E(u_{e_1})E(u_{e_2})$.

(Key pt) we may find Fourier expansion for both sides to get various equations on the unknowns $\lambda_{x,y}, \mu_{x,y}$.

For technical reasons, we assume $P \subseteq L^\infty(Y) \rtimes SL_2(\mathbb{Q})$ to get direct relations between $\lambda_{x,y}$ and $\mu_{x,y}$, which help solving for the unknowns.

The case of $\mathbb{Z}$-coefficient follows a similar strategy.

Open questions

Question

(1) *Is* $A \rtimes \mathbb{Q}^\times$ *the only (non-trivial) intermediate vN alg between* $L(\mathbb{Q}^\times)$ *and* $L(\mathbb{Q} \rtimes \mathbb{Q}^\times)$, *where* $A := \{\sum_{s\in\mathbb{Q}} \lambda_s u_s : \lambda_s = \lambda_{-s}\ \forall\ s \in \mathbb{Q}\} \cap L(\mathbb{Q})$?

(2) *If* H *is a maximal subgroup in* G *and* $[G : H] = \infty$, *is* $L(H)$ *rigid in* $L(G)$, *i.e.* $\forall \phi \in Aut(L(G)), \phi|_{L(H)} = id \Rightarrow \phi = id$?

RK: (1) The affine action $\mathbb{Q} \rtimes \mathbb{Q}^\times \curvearrowright \mathbb{Q}$ is sharply 2-transitive, so $u_g^* E(u_g) \in L(g^{-1}Hg \cap H)' \cap L(G)$ is no longer helpful as $g^{-1}Hg \cap H = \{e\}$.

(2) has positive answers in all known examples and surprisingly, $L(H)$ is also maximal in $L(G)$ for these examples.

Thank you for your attention!