McShane identities for Higher Teichmuller theory and the Goncharov-Shen potential

Zhe Sun, University of Luxembourg

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Objects that we interest

- Riemann surface $S = S_{g,m}$ of genus g with m holes. ($\mathbb{Z}$ for number theory)
- Simple/primitive closed curves on S. (primes in $\mathbb{Z}$)
- Geometry: Horocycle length $\leftrightarrow$ representation theory: Goncharov–Shen potential.
- Identities encode all the simple closed curves.

Table of contents

McShane's identity

- (McShane) Punctured surface $S_{g,m}$ has a hyperbolic structure ρ, $\ell(\gamma) = \log\left|\frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}\right|$

$$\sum_{Y \in \mathcal{P}_p} \frac{1}{1 + e^{\frac{1}{2}(\ell(\beta)+\ell(\gamma))}} = \frac{1}{2}. \tag{1}$$

For $(g, m) = (1, 1)$

$$\sum_{\gamma \in \mathcal{C}_{1,1}} \frac{1}{1 + e^{\ell(\gamma)}} = \frac{1}{2}. \tag{2}$$

McShane's identity

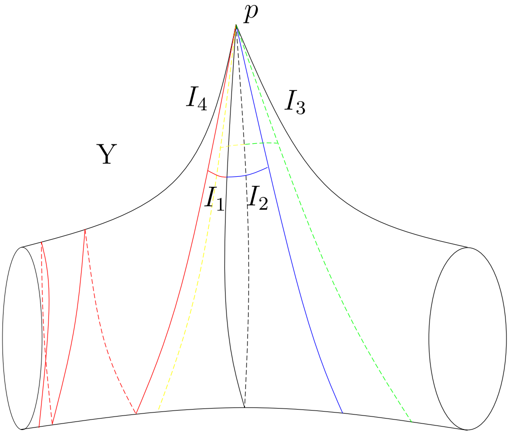

Figure: Gap term for an embedded pair of pants

McShane–Mirzakhani identity

- (Mirzakhani) Surface $S_{g,m}$ has a hyperbolic structure ρ with holes

$$\sum_{[Y]\in\overrightarrow{\mathcal{P}}_\alpha} \mathcal{D}(\ell(\alpha),\ell(\beta),\ell(\gamma)) + \sum_{[Y]\in\mathcal{S}_\alpha} \mathcal{R}(\ell(\alpha),\ell(\gamma),\ell(\beta)) = \ell(\alpha) \tag{3}$$

Where

$$\mathcal{D}(x,y,z) = 2\log\left(\frac{e^{\frac{x}{2}}+e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}}+e^{\frac{y+z}{2}}}\right) \tag{4}$$

$$\mathcal{R}(x,y,z) = x - \log\left(\frac{\cosh(\frac{y}{2})+\cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2})+\cosh(\frac{x-z}{2})}\right) \tag{5}$$

McShane–Mirzakhani identity

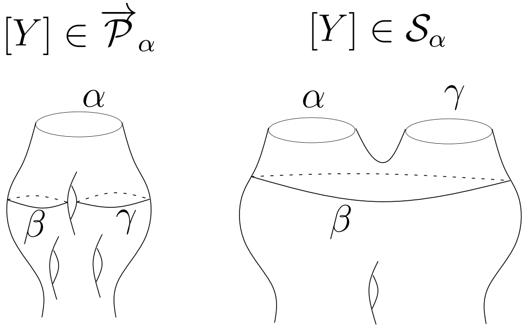

Figure: $\mathcal{D}(\ell(\alpha), \ell(\beta), \ell(\gamma))$ and $\mathcal{R}(\ell(\alpha), \ell(\gamma), \ell(\beta))$

McShane–Mirzakhani identity

- Witten–Kontsevich theorem: two different models of 2-dimensional quantum gravity should have the same partition function.
- First: intersection numbers on the moduli stack of algebraic curves.
- Second: the logarithm of the τ-function of the KdV hierarchy.
- Maryam Mirzakhani: integrates the above identities over $\mathcal{T}(S)/Mod(S)$, then obtain a recursive formula which satisfies Virasoro constraint. Thus again prove Witten–Kontsevich theorem.
- geometric recursion $\rightarrow$ topological recursion.

McShane–Mirzakhani identity

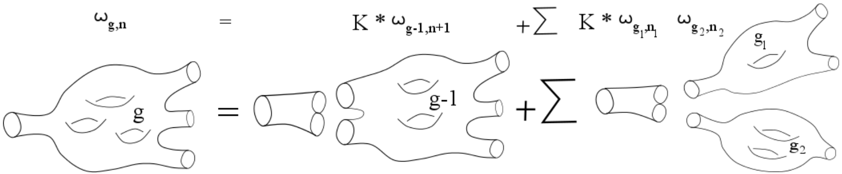

Figure: From Wikipedia: Topological recursion

Punctured case

- (Y. Huang–S.) Theorem: Given the positive representation $\rho \in p(\mathcal{A}_{\mathrm{SL}_n,S_{g,m}}(\mathbb{R}_{>0}))$ with parabolic boundary monodromy, for any puncture p and any $i = 1, \cdots, n-1$

$$1 = \sum_{[Y]\in\overrightarrow{\mathcal{P}}_p} \frac{1}{1 + \frac{\cosh\frac{d_2}{2}}{\cosh\frac{d_1}{2}} \cdot e^{\frac{1}{2}(\kappa_i(p,\gamma)+\ell_i(\gamma)+\kappa_i(p,\beta)+\ell_i(\beta))}}. \tag{6}$$

- For $(g, m) = (1, 1)$

$$1 = \sum_{[\gamma]\in\overrightarrow{\mathcal{C}}_{1,1}} \frac{1}{1 + e^{\kappa_i(p,\gamma)+\ell_i(\gamma)}}. \tag{7}$$

- In Fuchsian case, $\frac{\cosh\frac{d_2}{2}}{\cosh\frac{d_1}{2}} = 1$, $\kappa_i(p,\gamma) = 0$, $\ell_i = \ell$.

Boundary case

- (Y. Huang–S.) Theorem: Given the Hitchin (positive) representation $\rho \in H_n(S_{g,m})$ with loxodromic boundary monodromy, for any oriented boundary curve α and any $i = 1, \cdots, n-1$

$$\ell_i(\alpha) = \sum_{[Y]\in\overrightarrow{\mathcal{P}}_\alpha} \log \frac{e^{\frac{\ell_i(\alpha)}{2}} + \frac{\cosh\frac{d_2}{2}}{\cosh\frac{d_1}{2}} \cdot e^{\frac{1}{2}\left(\kappa_i(\alpha^-,\gamma)+\ell_i(\gamma)+\kappa_i(\alpha^-,\beta)+\ell_i(\beta)\right)}}{e^{-\frac{\ell_i(\alpha)}{2}} + \frac{\cosh\frac{d_2}{2}}{\cosh\frac{d_1}{2}} \cdot e^{\frac{1}{2}\left(\kappa_i(\alpha^-,\gamma)+\ell_i(\gamma)+\kappa_i(\alpha^-,\beta)+\ell_i(\beta)\right)}} + \sum_{[Y]\in\mathcal{S}_\alpha} \cdots . \tag{8}$$

- Fuchsian case recovers McShane–Mirzakhani identity.
- $\kappa_i(\alpha^-,\gamma)$: an invariant associates to an ideal triangle.
- $\frac{\cosh\frac{d_2}{2}}{\cosh\frac{d_1}{2}}$: an invariant associates to a pair of pants Y.

Hitchin component

- $S = S_g$, n-**Fuchsian representation**
  $\rho : \pi_1(S) \xrightarrow{d.f.} \mathrm{PSL}(2,\mathbb{R}) \xrightarrow{irr.} \mathrm{PGL}(n,\mathbb{R})$
- **Hitchin component** $H_n(S)$ is a connected component of $Hom(S, \mathrm{PGL}(n,\mathbb{R}))//\,\mathrm{PGL}(n,\mathbb{R})$ that contains n-Fuchsian representations.
- (Hitchin) Theorem: $S = S_{g,0}$ closed. $H_n(S)$ as a section of Hitchin fibration is topologically a $(n^2-1)(2g-2)$ dimensional cell.
- (Labourie, Fock–Goncharov $S = S_{g,m}$) Theorem: For every $\rho \in H_n(S)$, we have $\rho(\pi_1(S))$ is discrete faithful in $\mathrm{PGL}(n,\mathbb{R})$. There is a lift into $\mathrm{SL}(n,\mathbb{R})$, for any non-trivial simple closed geodesic (non-homotopy to hole or cusp) $\gamma \in \pi_1(S)$ and its eigenvalue $\lambda_1(\gamma) > \cdots > \lambda_n(\gamma) > 0$.

Limit curve and geometric structures

- (Labourie–Guichard) Theorem: $\rho \in H_n(S)$, $\exists! \xi_\rho^1 : \partial_\infty \pi_1(S) \cong S^1 \to \mathbb{RP}^{n-1}$ which is ρ-equivariant and hyperconvex.
- (Benoist–Sambarino) Theorem: If $\rho \in H_n(S)$ is not n-Fuchsian, then $\xi_\rho^1(S^1)$ is C^1 not C^2. If it is C^2 then ρ is n-Fuchsian, then $\xi_\rho^1(S^1)$ is conic.
- When $n = 3$, $\xi_\rho(S^1)$ bounds a domain Ω_ρ, $\Omega_\rho/(\pi_1(S)) \cong S$, $(\mathrm{PSL}(3,\mathbb{R}), \mathbb{RP}^2)$ structure—**strictly convex real projective structure** on S.
- Equipped with a metric: $\phi(\widetilde{S}) = \Omega \subset \mathbb{RP}^2$ is equipped with Hilbert metric, then $S \cong \Omega/\pi_1(S)$ is equipped with an induced metric.

Fock–Goncharov $(\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}, \mathcal{A}_{\mathrm{SL}_n,\hat{S}})$ moduli spaces

- Let $\hat{S} = (S, m_b)$, where S is a connected oriented Riemann surface of $\chi(S) < 0$ with at least one hole, marked points $m_b \subset \partial S$ finite, considered modulo homotopy.
- E is n dimensional vector space. Flag variety

$$\mathcal{B} := \{F_0 \subset F_1 \subset \cdots \subset F_n = E \mid \dim F_i = i\}$$

equipped with fixed volume form Ω in E.
- An affine flag is $F \in \mathcal{B}$ and a choice of $\overline{f}_i \neq 0 \in F_i/F_{i-1}$ for $i = 1, \cdots, n-1$. Let $\mathcal{A}$ be the collection of affine flags.

Fock–Goncharov $(\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}, \mathcal{A}_{\mathrm{SL}_n,\hat{S}})$ moduli spaces

- A framed G-local system $(\rho, \xi) \in \mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ is a G-local system ρ and monodromy invariant map $\xi : m_b \cup m_p \to \mathcal{B}$.
- A decorated G-local system $(\rho, \xi) \in \mathcal{A}_{\mathrm{SL}_n,\hat{S}}$ is a twisted G-local system ρ with parabolic boundary monodromy and monodromy invariant map $\xi : m_b \cup m_p \to \mathcal{A}$.
- (Labourie–Mcshane) Theorem: When $m_b = \emptyset$, $\mathcal{X}_{\mathrm{PGL}_n,S}(\mathbb{R}_{>0})$ is a finite cover of $H_n(S)$, related by Weyl group actions on the flags for boundary components.
- (Fock, Goncharov) Cluster ensemble structure generalizes cluster algebra introduced by Fomin and Zelevinsky. Positive structure, parametrization following Lusztig, tropicalization and compactification. Duality conjecture (solved by GHKK) and mirror symmetry.

Fock–Goncharov coordinates for $(\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}, \mathcal{A}_{\mathrm{SL}_n,\hat{S}})$

- Given an ideal triangulation $\mathcal{T}$ of S, each vertex decorated by a flag invariant by monodromy.

$$a_i = \pm\Delta\left(x^m \wedge y^l \wedge z^p\right) \tag{9}$$

$$X_i = \prod_j a_j^{\epsilon_{ij}} \tag{10}$$

- $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ ($\mathcal{A}_{\mathrm{SL}_n,\hat{S}}$ resp.) is birational to a variety obtained by taking split tori parametrized by the Fock–Goncharov $\mathcal{X}$ ($\mathcal{A}$ resp.) coordinates in one fundamental domain, and gluing them with **subtraction free** transition maps given by cluster transformations.

Quiver for $n = 3$

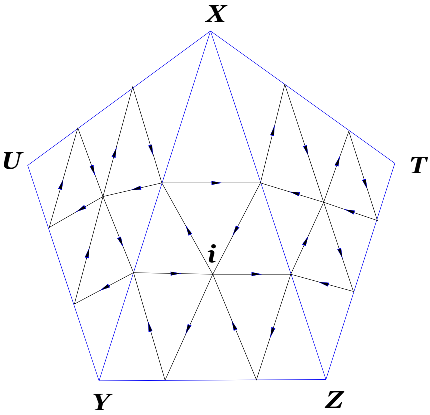

Figure: $\epsilon_{ij} = \#\{\textit{arrow } i \textit{ to } j\} - \#\{\textit{arrow } j \textit{ to } i\}$

Fock–Goncharov(Goldman) Poisson bracket $\{X_i, X_j\} = \epsilon_{ij} X_i X_j$

Triple ratio in $\mathbb{RP}^2$

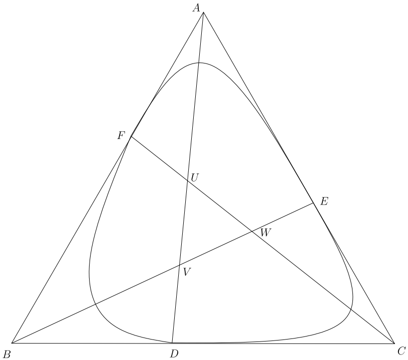

Figure: $F_1 = (D, \overline{BC})$, $F_2 = (E, \overline{AC})$, $F_3 = (F, \overline{AB})$, **triple ratio**
$T(F_1, F_2, F_3) = \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} \cdot \frac{|AF|}{|FB|} = \frac{|AU|}{|AV|} \frac{|DV|}{|DU|} = \frac{|BV|}{|BW|} \frac{|EW|}{|EV|} = \frac{|CW|}{|CU|} \frac{|FU|}{|FW|}$.
$T(F_1, F_3, F_2) = \frac{1}{T(F_1,F_2,F_3)} > 1$.

Definition

- For any $(F,G,H)\in\mathcal{A}^3$ in generic position, let $\pi:\mathcal{A}\to\mathcal{B}$ be the natural projection. There is a unique unipotent matrix $u\in U$ such that $(F,\pi(H))\cdot u=(F,\pi(G))$.
- The $(n-i,n-i+1)$ entry of u is additive when F is fixed, called *i-th character*. Denoted by $P_i(F;G,H)$, $P_i(f;g,h)$ and $P_i(\theta)$ depending on each case.
- The ratio of two i-th character with same F is *i-th ratio*.
- Given $(\rho,\xi)\in\mathcal{A}_{\mathrm{SL}_n,S_{g,m}}$. For $p\in m_p$ and $i=1,\cdots,n-1$, Goncharov–Shen potential is

$$P_i^p=\sum_{\theta \text{ marked triangle around cusp } p} P_i(\theta). \tag{11}$$

Explicit matrix for $n = 3$

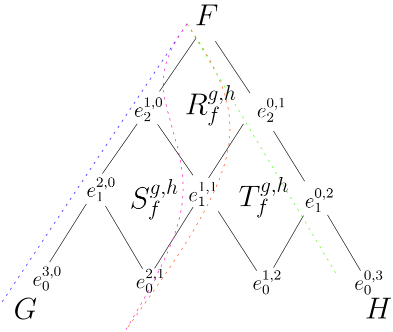

Figure:

$$u = x_1(S_f^{g,h}) \cdot x_2(R_f^{g,h}) \cdot x_1(T_f^{g,h}) = \begin{pmatrix} 1 & S_f^{g,h} + T_f^{g,h} & S_f^{g,h} R_f^{g,h} \\ 0 & 1 & R_f^{g,h} \\ 0 & 0 & 1 \end{pmatrix},$$

Explicit examples

- For $(\rho, \xi) \in \mathcal{A}_{\mathrm{SL}_2, S_{1,1}}$, it provides Markoff equation

$$P_{p,1} = \frac{\lambda_a}{\lambda_b\lambda_c} + \frac{\lambda_b}{\lambda_c\lambda_a} + \frac{\lambda_c}{\lambda_a\lambda_b}. \tag{12}$$

- For $(\rho, \xi) \in \mathcal{A}_{\mathrm{SL}_3, S_{1,1}}$

$$P_1^p = \frac{w}{br} + \frac{w}{ds} + \frac{w}{ac} + \frac{q}{cr} + \frac{q}{bd} + \frac{q}{as}, \tag{13}$$

$$P_2^p = \frac{bc}{aw} + \frac{rd}{ws} + \frac{bs}{wr} + \frac{ad}{wc} + \frac{ar}{bw} + \frac{cs}{dw} + \frac{ar}{sq} + \frac{cb}{dq} + \frac{dr}{cq} + \frac{bs}{aq} + \frac{ad}{bq} + \frac{cs}{rq}. \tag{14}$$

Explicit examples

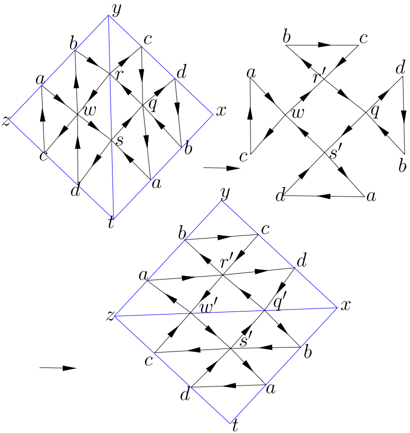

Figure: $\mathcal{A}_{\mathrm{SL}_3,S_{1,1}}$, mutation formula $r' = \frac{bq+cw}{r}$, $s' = \frac{aw+dq}{s}$, $w' = \frac{as'+cr'}{w}$, $q' = \frac{br'+ds'}{q}$.

Mirror symmetry conjecture after Goncharov–Shen

- $\mathcal{W} = \sum_p \sum_i P_i^p$
- $\mathcal{Y}_{\mathcal{W}}^{+}(\mathbb{Z}^t) := \{l \in \mathcal{Y}(\mathbb{Z}^t) \mid \mathcal{W}^t(l) \geq 0\}$.
- (Goncharov–Shen)Theorem:

$$Conf_n^{+}(\mathcal{A})(\mathbb{Z}^t) \sim \{\overline{Conf_n(Gr)_{t.d.}}\}$$

- The latter, using geometric Satake correspondence, parametrizes $\left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}\right)^{G^L}$.
- Generalize

$$Conf_3^{+}(\mathcal{A})(\mathbb{Z}^t) \sim \{Knutson - Tao's\ hives\}$$

Mirror symmetry conjecture after Goncharov–Shen

- (Goncharov–Shen) Conjecture: $(Conf_n^{\times}(\mathcal{A}), \mathcal{W}, \Omega)$ is mirror dual to $(Conf_n(\mathcal{A}_L)_{\mathbf{a}})$

$$\mathcal{FS}_{wr}\left(Conf_n^{\times}(\mathcal{A}), \mathcal{W}, \Omega\right) \sim D^b Coh\left(Conf_n(\mathcal{A}_L)_{\mathbf{a}}\right)$$

- Compactification: each partial potential gives rise to a divisor in the dual side.
- (Shen–Zhou–Zaslow) The above two are true for $n = 5$, $G = \mathrm{SL}_2$. Provided 5 divisors + cluster completion =Deligne–Mumford compactification.

Proof of identities for punctured case

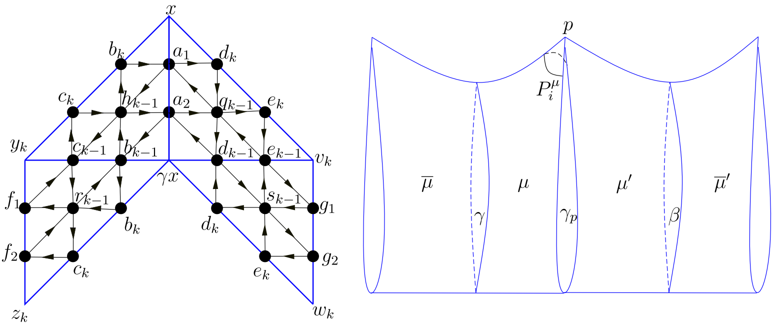

Figure: The half pair of pants μ is glued to $\overline{\mu}$ and μ' is glued to $\overline{\mu}'$.

Proof of identities for punctured case

- Splitting Goncharov–Shen potential by Dehn twists (or cluster transformations).
- Compute gap term for one pair of pants in the beginning

$$\frac{1}{1+\frac{\cosh\frac{d_2}{2}}{\cosh\frac{d_1}{2}}\cdot e^{\frac{1}{2}(\kappa_i(p,\gamma)+\ell_i(\gamma)+\kappa_i(p,\beta)+\ell_i(\beta))}}. \tag{15}$$

  Obtain "$\leq$".
- (Y. Huang–S.) Birman–Series Theorem for convex projective structure. Then obtain "$=$" for $n=3$.

Proof of identities for boundary case

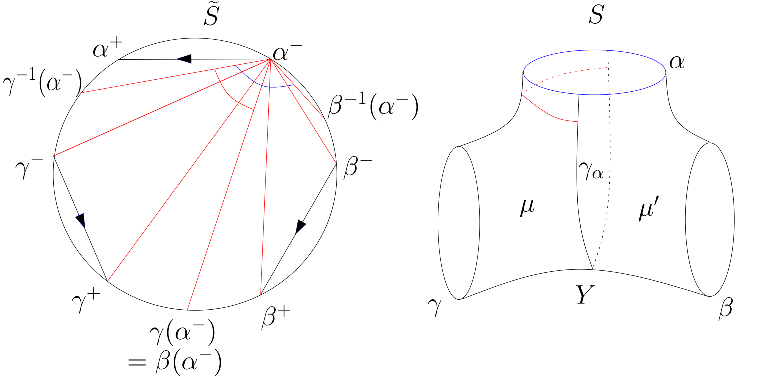

Figure: The pair of pants Y has the boundary components α, β, γ with $\alpha\beta^{-1}\gamma = 1$ and Y is cut into μ, μ' along the simple curve γ_α. Here $\partial\mu$ contains γ_α and γ, and $\partial\mu'$ contains γ_α and β.

Proof of identities for boundary case

- Gap term for Y in the beginning is $B_i(\alpha^-;\alpha^+,\gamma^+,\beta^+) = \frac{P_i(\alpha^-;\alpha^+,\beta^+)}{P_i(\alpha^-;\alpha^+,\gamma^+)}$.
- Compute explicitly, play i-th character as play with hyperbolic geometry.
- $$d_1 := \log \frac{P_i(\alpha^-;\gamma^+,\gamma(\alpha^-))}{P_i(\alpha^-;\beta(\alpha^-),\beta^+)}, \quad d_2 := \log \frac{P_i(\alpha^-;\gamma^{-1}(\beta^+),\gamma^{-1}(\alpha^-))}{P_i(\alpha^-;\gamma^{-1}(\alpha^-),\gamma^+)}. \tag{16}$$
- $$K_i(p,\delta) = \frac{1+\sum_{c=1}^{i-1}\prod_{j=1}^{c} T_{n-i,j,i-j}(\delta p,\delta^+,p)}{1+\sum_{c=1}^{i-1}\prod_{j=1}^{c} T_{n-i,j,i-j}(p,\delta p,\delta^+)} \cdot \frac{\prod_{j=1}^{n-i-1} T_{n-i-j,j,i}(p,\delta p,\delta^+)}{\prod_{j=1}^{i-1} T_{j,n-i,i-j}(p,\delta p,\delta^+)}. \tag{17}$$
- Limit of the path from loxodromic to parabolic boundary monodromy, obtain the gap term for the punctured case.

Proof of identities for boundary case

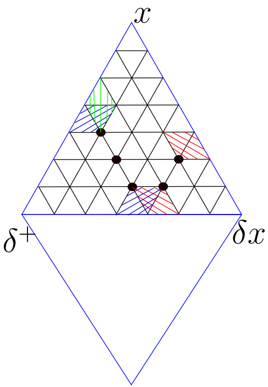

Figure: $\kappa_i(p,\delta) = \log K_i(p,\delta)$

Fuchsian rigidity

- (Y. Huang–S.) Theorem: For $(\rho,\xi) \in \mathcal{X}_{\mathrm{PGL}_n,S_{g,m}}$ is n-Fuchsian iff all the triple ratios on any ideal triangle equal to 1.
- (Y. Huang–S.) Theorem: For $(\rho,\xi) \in \mathcal{X}_{\mathrm{PGL}_3,S_{g,m}}$ is 3-Fuchsian iff all the edge functions on any ideal edge are equal.
- Conjecture: The above edge function Fuchsian rigidity is true for any n.

Boundedness of triple ratio

- We find when surface S is closed, for $H_n(S)$, any triple ratio coordinate is bounded under the mapping class group orbit. Proved by private communication with Labourie and Zhang.
- Reason: Ordered triple of points up to $\pi_1(S)$ is isomorphic to a compact set $T^1(S)$.
- (Y. Huang, S.) Theorem: For $(\rho, \xi) \in p(\mathcal{A}_{\mathrm{SL}_n, S_{g,m}}(\mathbb{R}_{>0}))$, any triple ratio coordinate is bounded under the mapping class group orbit.

Boundedness of triple ratio

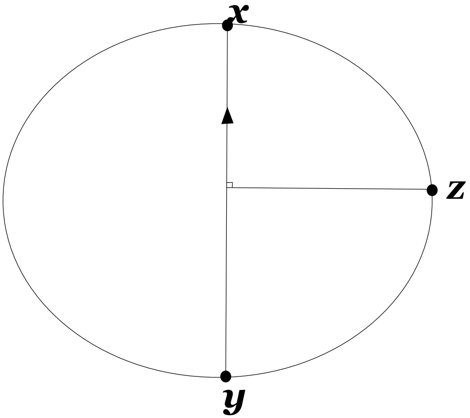

Figure: $\{(x,y,z)|x<y<z<x\in S^1\}/\pi_1(S)\cong\mathbb{T}^1(S)$

Discrete spectral

- (Y. Huang–S.) Corollary: Given $(\rho, \xi) \in p(\mathcal{A}_{\mathrm{SL}_n, S_{g,m}}(\mathbb{R}_{>0}))$, the simple closed $\{\log \frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}\}_\gamma$ spectral is discrete in $\mathbb{R}_{>0}$.
- Proof: Boundedness of triple ratio+our generalized Mcshane identity.
- Using identity to define a Thurston–type metric.
  Non-domination property of simple root length spectral.

Collar lemma for $n = 3$

- (Y. Huang–S.) Theorem: Given $(\rho, \xi) \in p(\mathcal{A}_{\mathrm{SL}_3, S_{g,m}}(\mathbb{R}_{>0}))$, for two intersecting oriented simple closed geodesics α, β. Let $u_1 = T(x, \alpha x, \alpha^+) \cdot \frac{\lambda_1(\rho(\alpha))}{\lambda_2(\rho(\alpha))}$, $u_2 = T(x, \alpha^{-1} x, \alpha^-) \cdot \frac{\lambda_1(\rho(\alpha^{-1}))}{\lambda_2(\rho(\alpha^{-1}))}$, $u_3 = T(x, \beta x, \beta^+) \cdot \frac{\lambda_1(\rho(\beta))}{\lambda_2(\rho(\beta))}$, $u_4 = T(x, \beta^{-1} x, \beta^-) \cdot \frac{\lambda_1(\rho(\beta^{-1}))}{\lambda_2(\rho(\beta^{-1}))}$. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then
- $$\left((u_i u_j)^{\frac{1}{2}} - 1\right) \cdot \left((u_k u_l)^{\frac{1}{2}} - 1\right) > 4. \tag{18}$$
- Especially, when $\{i, j\} = \{1, 2\}$

$$\left(\left(\frac{\lambda_1(\rho(\alpha))}{\lambda_3(\rho(\alpha))}\right)^{\frac{1}{2}} - 1\right) \cdot \left(\left(\frac{\lambda_1(\rho(\beta))}{\lambda_3(\rho(\beta))}\right)^{\frac{1}{2}} - 1\right) > 4. \tag{19}$$

Details in arXiv:1901.02032.

Thank you!