NONLINEAR STABILITY OF PLANAR VORTEX PATCHES IN AN IDEAL FLUID

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ABSTRACT. In this paper, we prove nonlinear stability of planar vortex patches concentrated near a non-degenerate minimum point of the Robin function in a general bounded domain. These vortex patches are stationary solutions of the two-dimensional incompressible Euler equations. The result is obtained by showing that these concentrated vortex patches are in fact isolated maximizers of the kinetic energy among isovortical patches.

1. INTRODUCTION

In this paper, we consider the incompressible inviscid flow without external force in the plane, the motion of which is governed by the following Euler equations:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P, \ x = (x^1, x^2) \in \mathbb{R}^2, t \in (0, +\infty), \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity field and P is the scalar pressure. Here we assume that the fluid is of unit density.

Let $D \subset \mathbb{R}^2$ be a bounded and simply-connected domain with a smooth boundary, ∂D. When the fluid moves inside D, the impermeability boundary condition is usually imposed:

$$\mathbf{v} \cdot \mathbf{n} = 0, \tag{1.2}$$

where $\mathbf{n}$ is the outward unit normal of ∂D. By introducing the vorticity function $\omega = \partial_{x^1} v_2 - \partial_{x^2} v_1$ and using the identity $\frac{1}{2}\nabla|\mathbf{v}|^2 = (\mathbf{v} \cdot \nabla)\mathbf{v} + J\mathbf{v}\omega$, the first equation of (1.1) becomes

$$\partial_t \mathbf{v} + \nabla(\frac{1}{2}|\mathbf{v}|^2 + P) - J\mathbf{v}\omega = 0, \tag{1.3}$$

where $J(v_1, v_2) = (v_2, -v_1)$ denoting a clockwise rotation through $\pi/2$. Taking the curl on both sides of (1.3) we get

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega = 0. \tag{1.4}$$

Since $\mathbf{v}$ is divergence-free and D is simply-connected, by the Green formula $\mathbf{v}$ can be written as $\mathbf{v} = J\nabla\psi$ for some scalar function ψ. It is obvious that

$$\begin{cases} -\Delta\psi = \omega & \text{in } D, \\ \psi = \text{constant} & \text{on } \partial D. \end{cases} \tag{1.5}$$

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Without loss of generality, we can always assume that ψ vanishes on ∂D by adding a properly chosen constant. Therefore, ψ can be expressed in terms of ω by

$$\psi(x) = G\omega(x) := \int_D G(x,y)\omega(y)dy,$$

where G is the Green function for $-\Delta$ in D with zero Dirichlet boundary condition, which has the following form

$$G(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x-y|} - h(x,y), \;\; x, y \in D.$$

Taking into account the above relation between $\mathbf{v}$, ω and ψ, we are able to deduce the following equation satisfied by ω:

$$\partial_t\omega + J\nabla G\omega \cdot \nabla\omega = 0, \tag{1.6}$$

which is usually called the vorticity equation. To deal with solutions with discontinuity, we need to interpret (1.6) in the weak sense.

Definition 1.1. We call $\omega \in L^\infty((0,+\infty)\times D)$ a weak solution to the vorticity equation (1.6) if

$$\int_D \omega(x,0)\xi(x,0)dx + \int_0^{+\infty}\int_D \omega(\partial_t\xi + \nabla\xi\cdot J\nabla G\omega)dxdt = 0, \;\; \forall\,\xi \in C_c^\infty([0,+\infty)\times D). \tag{1.7}$$

By standard regularity theory for elliptic equations it is easy to check that $\nabla G\omega \in L^\infty((0,+\infty)\times D)$, so the above definition makes sense. By Yudovich [32], for initial vorticity $\omega(\cdot,0) \in L^\infty(D)$, there is a unique weak solution ω to (1.7). Moreover, the distribution function of $\omega(\cdot,t)$ is independent of t, that is,

$$|\{x \in D|\,\omega(x,t) > a\}| = |\{x \in D|\,\omega(x,0) > a\}|, \forall\,a \in \mathbb{R},\,t \geq 0, \tag{1.8}$$

where $|\cdot|$ denotes the two-dimensional Lebesgue measure.

For convenience, we also write $\omega(x,t)$ as $\omega_t(x)$. By (1.8), if the initial vorticity ω_0 has the form $\omega_0 = \lambda I_{A_0}$, where $A_0 \subset D$ is a measurable set, $\lambda \in \mathbb{R}$ represents the vorticity strength and I_{A_0} denotes the characteristic function of A_0, i.e., $I_{A_0}(x) = 1$ for $x \in A_0$ and $I_{A_0} = 0$ elsewhere, then the evolved vorticity ω_t must be of the form $\omega_t = \lambda I_{A_t}$ with $|A_t| = |A_0|$ for all $t \geq 0$. We call such ω_t a vortex patch solution, or vortex patch briefly.

A weak solution of the vorticity equation is said to be steady if it does not depend on the time variable. Thus for any $\omega \in L^\infty(D)$, it is a steady solution if and only if

$$\int_D \omega\nabla\xi \cdot J\nabla\psi dx = 0, \,\forall\,\xi \in C_c^\infty(D). \tag{1.9}$$

The search for dynamically possible steady solutions to the vorticity equation is an important and interesting issue in the study of two-dimensional incompressible Euler equations. It is easy to check that if $\omega \in L^\infty(D)$ satisfies

$$\omega = f(G\omega), \text{ a.e. } x \in D,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, then ω must be a steady solution to the vorticity equation. Burton [12] proved that it is also true for any monotone function

f. As a special case, we immediately deduce that ω is a steady solution of the vorticity equation if it has the form

$$\omega = \lambda I_{\{G\omega > \mu\}}, \tag{1.10}$$

where λ and μ are both positive constants. Here for simplicity we denote $\{x \in D | \; G\omega(x) > \mu\}$ by $\{G\omega > \mu\}$ and similar notations will be used in the sequel.

There are already several papers dealing with the existence of steady vortex patches of the form (1.10). See [16][29] for example. In the present paper, we are mainly concerned with the nonlinear stability of such kind of steady vortex patches. Here by nonlinear stability we mean Lyapunov type. To give the precise definition, let us first define R_ω, the rearrangement class of ω as follows

$$R_\omega := \{w \in L^\infty(D) \; | \; |\{w > a\}| = |\{\omega > a\}|, \; \forall \, a \in \mathbb{R}\}.$$

Definition 1.2. A steady vortex patch ω is called to be stable, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $\omega_0 \in R_\omega$, $\|\omega_0 - \omega\|_{L^1} < \delta$, there holds $\|\omega_t - \omega\|_{L^1} < \varepsilon$ for all $t \geq 0$, where $\omega_t(x) = \omega(x, t)$ is the solution of (1.7) with initial vorticity ω_0.

In the above definition, we use the L^1 norm to measure the "distance" between two solutions at any fixed time, which is very natural for vortex patch solutions. It should be noted that for vortex patch solutions the L^1 norm is equivalent to the L^p norm for any $p \in (1, +\infty)$.

In this paper, we confine our attention to the nonlinear stability of steady vortex patch solutions with concentration property, that is, solutions satisfying

$$\omega = \lambda I_{\{G\omega > \mu\}}, \; \lambda|\{G\omega > \mu\}| = 1, \; \{G\omega > \mu\} \subset B_\delta(x_0), \tag{1.11}$$

where $\lambda > 0$ is very large, $\delta > 0$ is very small, and $x_0 \in \overline{D}$ is a fixed point. It can be proved that if for any sufficiently large λ, there exists ω satisfying (1.11), then x_0 is necessarily in the interior of D and must be a critical point of H, the Robin function of D, defined by

$$H(x) := h(x, x).$$

See [14] for a rigorous proof. On the other hand, if x_0 is a non-degenerate critical point of H, then for any sufficiently large λ, there exists ω satisfying (1.11). See [16] for example.

The main result of this paper is as following.

Theorem 1.3. *Assume that ω_λ is a family of steady vortex patch solutions of the vorticity equation satisfying:*

    (i) $\omega_\lambda = \lambda I_{\{G\omega_\lambda > \mu_\lambda\}}$, *where μ_λ is a positive number depending on λ,*
    (ii) $\int_D \omega_\lambda dx = 1$,
    (iii) $\{G\omega_\lambda > \mu_\lambda\} \subset B_{o(1)}(x_0)$, *where $o(1) \to 0$ as $\lambda \to +\infty$, and x_0 is a non-degenerate minimum point of H.*

Then ω_λ is stable in the sense of Definition 1.2 provided that λ is large enough.

The proof of Theorem 1.3 will be given in Section 3.

Remark 1.4. Caffarelli and Friedman [13] proved that if D is a convex domain, then the Robin function H is strictly convex, thus in this case H has a unique minimum point which is non-degenerate.

Remark 1.5. Theorem 1.3 is closely related to the vortex model (see[25], Chapter 4), which describes the motion of the fluid when the vorticity is sufficiently concentrated in N small regions. In the case $N = 1$, the vorticity is simplified as a Dirac measure called a point vortex, the location of which is determined by the following Kirchhoff-Routh equation:

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{1}{2}J\nabla H(x(t)), \ t > 0, \\ x(0) = x_0. \end{cases} \tag{1.12}$$

It is easy to see that

$$\frac{dH(x(t))}{dt} = 0, \ \forall \, t > 0, \tag{1.13}$$

that is, the point vortex moves along the level curve of H. If x_0 is critical point of H, then it is an equilibrium of the Kirchhoff-Routh equation, and the stability of this equilibrium is closely related to $D^2 H(x_0)$, the Hessian of H at x_0. In particular, if $D^2 H(x_0)$ is positive definite (or equivalently, x_0 is a non-degenerate minimum point), then x_0 must be stable. This can be proved by choosing H as the Lyapunov function. See [25], Chapter 3 for example. In such a way, Theorem 1.3 can be interpreted as a desingularized version of the stability for the vortex model.

To prove Theorem 1.3, a stability criterion due to Burton [12] and a local uniqueness result due to Cao, Guo, Peng and Yan [14] are used. In [12], Burton proved that steady vortex patches as isolated maximizers of the kinetic energy among all isovortical patches are stable. In [14], Cao, Guo, Peng and Yan proved that there is only one steady vortex patch supported near x_0 for fixed large λ. For convex domains, Theorem 1.3 is a very straightforward deduction of these two results, since in this case every vortex patch with maximal energy must be supported near the unique non-degenerate minimum point of the Robin function. However, for a general domain it is not obvious at all. One of the novelties of this paper is the reconstruction of steady vortex patches by variational method, from which we can show that vortex patches in Theorem 1.3 are energy maximizers among patches supported near x_0. Another novelty of this paper is that we adapt the method in Wan and Pulvirenti [31] to exclude the possibility that there is a patch with maximal energy but separated support, from which and the local uniqueness result we can further prove that the vortex patches in Theorem 1.3 are in fact energy maximizers among all isovortical patches. This is the key ingredient and allows us to apply Burton's criterion.

The analysis of stability for steady Euler flows in two dimensions is very important in fluid mechanics and has been studied by many authors in history. See for example [2, 3, 12, 21, 22, 24, 28, 31]. Here we recall some of the relevant and significant results associated with the stability of planar vortex patches. The first stability result is due to Kelvin in [21], where he established the linear stability for circular vortex patches in the whole plane. Later Love [24] proved linear stability of a rotating Kirchhoff elliptical vortex patch in the plane. Another excellent work is due to Arnold [2, 3]. Arnold gave several

criteria for nonlinear stability of smooth steady Euler flows in general bounded domains, which can be seen as a nonlinear version of the classical Rayleigh inflection point criterion for linear stability of shear flows in a channel. Moreover, he asserted in [4] in general terms that a steady flow can be seen as a constrained critical point of the kinetic energy; if this critical point is a non-degenerate extreme, then it should be stable. Unfortunately, it seems that his method is not easy to apply to prove the nonlinear stability of vortex patches, since there is strong discontinuity for the vorticity.

In 1985, based on energy conservation, Wan and Pulvirenti in [31] proved nonlinear stability of circular vortex patches in an open disk. In that paper, they established a relative variational principle for the kinetic energy and turned the L^1 perturbation problem into a C^1 perturbation problem. When dealing with the C^1 perturbation case, the key ingredient of their proof is that for a circle, the Green function is explicitly known and rotationally invariant, then the C^1 perturbation case can be handled by spectral analysis of a negative definite operator. In 1987, Tang [28] proved nonlinear stability of both circular vortex patches and rotating elliptic vortex patches in the plane based on the same idea as in [31]. The method developed by Marchioro, Pulvirenti and Tang is still not easy to apply to prove nonlinear stability of steady vortex patches in general bounded domains, since in such cases the Green functions have no explicit expression and good symmetry anymore. However, their idea of turning L^1 perturbation into C^1 perturbation is very clever, and is used in the present paper as a key ingredient to prove Theorem 1.3.

In 2005, Burton in [12] proved that a steady vortex flow as an isolated maximizer of the kinetic energy relative to an "isovortical surface" (rearrangement class of a given function) is stable in some L^p norm. But only in few cases can the isolatedness of the maximizer of the kinetic energy be verified. Burton's result is crucial for the proof of Theorem 1.3 and will be stated precisely in Section 2.

Recently Cao, Guo, Peng and Yan in [14] established a local uniqueness result for steady vortex patches concentrated near a non-degenerate critical point of the Robin function. They considered the following semilinear elliptic equation satisfied by the stream function:

$$\begin{cases} -\Delta\psi_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\}} & \text{in } D, \\ \psi_\lambda = 0 & \text{on } \partial D, \end{cases} \tag{1.14}$$

where $\mu_\lambda > 0$ is an unknown constant and $\lambda|\{\psi_\lambda > \mu_\lambda\}| = 1$. They proved that if the vortex core $\{\psi_\lambda > \mu_\lambda\}$ shrinks to a non-degenerate critical point of the Robin function as $\lambda \to +\infty$, then the solution of (1.14) is unique provided that λ is large enough. The precise statement will be given in Section 2. This local uniqueness result is used in this paper to verify the isolatedness of the maximizer of the kinetic energy.

This paper is organized as follows. In Section 2 we state several known results that will be used later. In Section 3 we give the proof of Theorem 1.3. In Section 4 we give the construction of steady vortex patches concentrated near a given isolated minimum point of the Robin function for completeness.

2. Preliminaries

In this section, we mainly recall several known results on the existence, local uniqueness and stability of planar vortex patches in general bounded domains.

Let us set some notations first. For any function $f : D \to \mathbb{R}$, we denote $supp(f) := \overline{\{f \neq 0\}}$. For any planar set A, we use $diam(A)$ to denote the diameter of A, i.e.,

$$diam(A) = \sup_{x,y \in A} |x - y|.$$

Define $E(\omega)$ as the kinetic energy of the fluid with vorticity ω by setting

$$E(\omega) := \frac{1}{2} \int_D \int_D G(x,y)\omega(x)\omega(y)dxdy, \tag{2.1}$$

By integration by parts, it is easy to check that

$$E(\omega) = \frac{1}{2} \int_D \psi(x)\omega(x)dx = \frac{1}{2} \int_D |\nabla G\omega(x)|^2 dx. \tag{2.2}$$

2.1. **Existence of Steady Vortex Patches.** To our knowledge, there are mainly two methods dealing with the existence of steady vortex flows in two dimensions. The first one is called the stream function method, whose starting point is to solve (1.14) for a give nonlinearity f. See [7, 16, 15, 26, 27] for example. The other one is to solve a variational problem for the kinetic energy subject to some appropriate constraints for the vorticity. See [6, 8, 9, 10, 11, 29] for example.

In this subsection, for our purpose, we recall the existence of steady vortex patches of the form (1.10) via the vorticity method. The result and idea are mostly based on Turkington [29].

To this end, we define

$$K_\lambda \triangleq \{\omega \in L^\infty(D)| \ 0 \le \omega \le \lambda, \ \int_D \omega(x)dx = 1\},$$

where λ is positive and large enough so that K_λ is not empty. Let ε be the positive number determined by $\lambda\pi\varepsilon^2 = 1$.

Turkington [29] proved the following result.

Theorem 2.1. *E attains it maximum on K_λ, and each maximizer ω_λ has the form $\omega_\lambda = \lambda I_{\{G\omega_\lambda > \mu_\lambda\}}$ (thus must be a steady solution to the vorticity equation), where μ_λ is a real number depending on λ. Moreover, $supp(\omega_\lambda)$ shrinks to a global minimum point of H as $\lambda \to +\infty$; more precisely, $diam(supp(\omega_\lambda)) \le C\varepsilon$ and $\int_D x\omega(x)dx \to x_0$ as $\lambda \to +\infty$, where C is a positive number not depending on λ and x_0 is a global minimum point of H.*

Remark 2.2. It should be noted that $\lim_{x \to \partial D} H(x) = +\infty$, so H attains its global minimum in D. But there may be more than one global minimum point of H, and we do not know which one $supp(\omega_\lambda)$ shrinks to as $\lambda \to +\infty$.

By performing a similar procedure to the one in [29], in the following theorem we prove existence of a family of steady vortex patches concentrated near an isolated minimum point of H. Similar results have been obtained by Elcrat and Miller [17, 18].

To state our result, we need some notations slightly different. Here and in the sequel let $x_1 \in D$ be a strict local minimum point of H. We choose a sufficiently small positive number r such that x_1 is the unique minimum point of H on $\overline{B_r(x_1)}$ and $B_r(x_1) \subset\subset D$. Define

$$N_\lambda \triangleq \{\omega \in L^\infty(D)|\ 0 \le \omega \le \lambda,\ \int_D \omega(x)dx = 1,\ supp(\omega) \subset B_r(x_1)\}.$$

Theorem 2.3. *E attains it maximum on N_λ, and each maximizer ω_λ has the form $\omega_\lambda = \lambda I_{\{G\omega_\lambda > \mu_\lambda\} \cap B_r(x_1)}$, where μ_λ is a real number depending on λ. Moreover, $supp(\omega_\lambda)$ shrinks to x_1 as $\lambda \to +\infty$; more precisely, $diam(supp(\omega_\lambda)) \le C\varepsilon$ for some C independent of λ and $\int_D x\omega_\lambda(x)dx \to x_1$ as $\lambda \to +\infty$. If λ is large enough, then ω_λ is a steady solution to the vorticity equation, that is, ω_λ satisfies (1.9).*

For readers' convenience, we will give a complete proof of Theorem 2.3 in Section 4.

Remark 2.4. In fact, we will prove in Section 4 that $\mu_\lambda \to +\infty$ as $\lambda \to +\infty$. Taking into account the fact that $supp(\omega_\lambda)$ shrinks to x_1, by using maximum principle we immediately deduce that ω_λ has the form $\omega_\lambda = \lambda I_{\{G\omega_\lambda > \mu_\lambda\}}$ if λ is sufficiently large.

2.2. **Burton's Stability Criterion.** In [12], Burton proved that steady vortex flow as isolated maximizer of the kinetic energy on an isovortical surface is stable in some L^p norm. In the case of vortex patches, the corresponding result can be stated as follows:

Theorem 2.5. *Let ω be a vortex patch in D (that is, $\omega = \lambda I_A$ for some $\lambda > 0$ and $A \subset D$). Suppose that ω is an isolated maximizer of the kinetic energy E on R_ω, that is, there exists $\delta_0 > 0$, such that for any $\bar\omega \in R_\omega$, $0 < \|\bar\omega - \omega\|_{L^1} < \delta_0$, there holds $E(\bar\omega) < E(\omega)$. Then ω is stable in the sense of Definition 1.2.*

Remark 2.6. Burton's result is a very general stability criterion. According to [12], Theorem 5, any steady vortex flow as isolated maximizer or minimizer relative to the rearrangement class of a given L^p function with $p \ge 3/2$ is stable in the L^p norm. But in the case of vortex patches, Theorem 2.5 is enough for our use.

The key assumption in Theorem 2.5 is the "isolatedness" of the energy maximizer, which in general terms is an non-degeneracy condition. In [12], only one example of strict global maximizer is given, i.e., D is an open disk and ω is a non-negative radially symmetric decreasing function. For general maximizers, especially local maximizers, the isolatedness assumption is not easy to be verified, since "isolatedness" is equivalent to uniqueness in some sense, which is usually a more difficult problem than existence.

2.3. **Local Uniqueness of Steady Vortex Patches.** Fortunately, Cao et al. [14] proved the following local uniqueness result of steady vortex patches near a non-degenerate critical of the Robin function based on fine estimates for the corresponding stream function, which may be used to prove isolatedness of the energy maximizer on isovortical patches.

Theorem 2.7. *Let x^* be a non-degenerate critical point of H. Then there exists $\lambda_0 > 0$, such that for each $\lambda > \lambda_0$, the ω_λ satisfying the following properties is unique:*

   (i) $\omega_\lambda = \lambda I_{\{G\omega_\lambda > \mu_\lambda\}}$, where μ_λ is an unknown positive number depending on λ;

(ii) $\int_D \omega_\lambda dx = 1$;

(iii) *the support of ω_λ shrinks to x^*, or equivalently, $supp(\omega_\lambda) \subset B_{o(1)}(x^*)$.*

Remark 2.8. When λ is not large, there is no uniqueness result on steady vortex patches in general bounded domains. However, we conjecture that this is true when D is a convex domain.

3. Proof of the Theorem 1.3

In this section, we give the proof of Theorem 1.3.

When D is a convex domain, there is no gap between local uniqueness and isolatedness of the energy maximizer, so in this case the proof of Theorem 1.3 is much simpler. For clarity, we give the short proof here.

Proof of Theorem 1.3 (the case D is convex). By Theorem 2.5, it suffices to show that ω_λ is an isolated maximizer of E over R_{ω_λ}.

When D is convex, there is only one critical point of H, which is exactly the unique global minimum point. So the x_0 in Theorem 1.3 must be the unique minimum point of H. Now combining Theorem 2.1 and Theorem 2.7, we immediately deduce that the ω_λ in Theorem 1.3 must be the uniqueness maximizer of E over K_λ if λ is large enough. Taking into the fact that $R_{\omega_\lambda} \subset K_\lambda$, we reach the conclusion that ω is in fact the unique maximizer of E over R_{ω_λ}, which is the desired result.

$\square$

When D is a general bounded domain, there is a gap between local uniqueness and isolatedness of energy maximizer. To eliminate the gap, we follow the idea of Wan and Pulvirenti [31] to turn L^1 perturbation into C^1 perturbation.

To make the idea more adaptable, we give a more general stability criterion for steady vortex patches, that is, Lemma 3.1 below. For convenience, we introduce the following notation. Let ω be a vortex patch enclosed by a C^1 closed curve denoted by γ_ω (thus γ_ω is a planar set). A δ neighbourhood of γ_ω is defined by:

$$\gamma_{\omega,\delta} \triangleq \{x \in \mathbb{R}^2 \mid dist(x, \gamma_\omega) < \delta\}. \tag{3.1}$$

Lemma 3.1. *Suppose that ω_0 is a steady vortex patch in D satisfying the following conditions:*

(C1) ω_0 has the form $\omega_0 = \lambda I_{\{\psi_0 > \mu\}}$ for some $\mu > 0$, where $\psi_0 = \int_D G(x,y)\omega_0(y)dy$;

(C2) $\partial\{\psi_0 > \mu\}$ is a C^1 closed curve and $\frac{\partial\psi_0}{\partial\vec{n}} < 0$ on this curve, where $\vec{n}$ is the outward unit normal of $\partial\{\psi_0 > \mu\}$;

(C3) there exists $\delta > 0$, such that if $\omega_1 \in R_{\omega_0}$ is another vortex patch(not necessarily steady) enclosed by a C^1 simple curve γ_{ω_1} and $\gamma_{\omega_1} \subset \gamma_{\omega_0,\delta}$, then $E(\omega_1) \leq E(\omega_0)$, the equality holds if and only if $\omega_0 \equiv \omega_1$.

Then ω_0 is stable.

Proof. By Theorem 2.5, it suffices to show that ω_0 is an isolated maximizer of E over R_{ω_0}. We show this by contradiction in the following.

Suppose that ω_0 is not an isolated maximizer of E over R_{ω_0}, then we can choose a sequence $\{\omega_n\} \subset R_{\omega_0}$ such that $0 < \|\omega_n - \omega_0\|_{L^1} < \frac{1}{n}$, and

$$E(\omega_n) \geq E(\omega_0). \tag{3.2}$$

For such a sequence, we have the following claim:

Claim: Denote $\psi_n = G\omega_n$ and $\psi_0 = G\omega_0$, then, if n is large, there exists a unique $\nu_n > 0$ such that

(i) $\partial\{\psi_n > \nu_n\}$ is a C^1 closed curve,
(ii) $|\{\psi_n > \nu_n\}| = |\{\psi_0 > \mu\}|$,
(iii)$\partial\{\psi_n > \nu_n\} \subset \gamma_{\omega_0,\delta}$, where δ is the one in $(C3)$.

Proof of the claim: First, notice that ψ_n and ψ_0 satisfy the following equations:

$$\begin{cases} -\Delta\psi_n = \omega_n, \\ -\Delta\psi_0 = \omega_0. \end{cases} \tag{3.3}$$

For vortex patches, $\|\omega_n - \omega_0\|_{L^1} \to 0$ implies $\|\omega_n - \omega_0\|_{L^p} \to 0$ for any $1 \leq p < +\infty$. Then by L^p estimates we have $\|\psi_n - \psi_0\|_{W^{2,p}} \to 0$ for any $1 < p < +\infty$. Choosing p large enough, by the Sobolev embedding $W^{2,p}(D) \hookrightarrow C^{1,\alpha}(\overline{D})$ for some $\alpha \in (0,1)$, we obtain

$$\|\psi_n - \psi_0\|_{C^1} \to 0. \tag{3.4}$$

By $(C2)$ we can take $\delta_0 > 0$ small, such that the set $\{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}$ is an annulus-like domain and $\frac{\partial\psi_0}{\partial\vec{n}} < 0$ on each closed curve $\{\psi_0 = a\}, \mu - \delta_0 \leq a \leq \mu + \delta_0$. This is true by the continuity of ψ_0 and $\nabla\psi_0$. Since $\|\psi_n - \psi_0\|_{C^1} \to 0$, we have $\frac{\partial\psi_n}{\partial\vec{n}} < 0$ on each curve $\{\psi_0 = a\}, \mu - \delta_0 \leq a < \mu + \delta_0$ if n is large enough. Thus $\nabla\psi_n \neq 0$ in the annulus-like domain $\{\mu - \delta_0 \leq \psi_0 \leq \mu + \delta_0\}$.

Now choose $\varepsilon < \delta_0$ small and define $M := \max_{\{\psi_0 = \mu - \varepsilon\}} \psi_n, m := \min_{\{\psi_0 = \mu + \varepsilon\}} \psi_n$. By the implicit function theorem, $\{\psi_n = M\}$ and $\{\psi_n = m\}$ are both C^1 curves locally. By the properties that ψ_n strictly increases along the direction $\nabla\psi_0$ in the annulus-domain $\{\mu - \delta_0 \leq \psi_0 \leq \mu + \delta_0\}$, the curve $\{\psi_n = M\}$ can not go outside $\{\psi_0 \geq \mu - \varepsilon\}$.

In fact, suppose that there exists $x_0 \in \{\psi_0 < \mu - \varepsilon\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}$ and $\psi_n(x_0) = M$, we can find $x_1 \in \{\psi_0 = \mu - \varepsilon\}$ by solving the following ODE:

$$\begin{cases} \frac{dx(t)}{dt} = -\nabla\psi_0(x), \\ x(0) = x_0. \end{cases} \tag{3.5}$$

Since $\frac{\partial\psi_n}{\partial\vec{n}} < 0$, we have $\psi_n(x_1) > \psi_n(x_0)$. But by the definition of M, $\psi_n(x_1) \leq M = \psi_n(x_0)$, which is a contradiction.

On the other hand, by taking n large enough the curve $\{\psi_n = M\}$ can not enter $\{\psi_0 > \mu\}$. In fact, suppose that there exists x_0 such that $x_0 \in \{\psi_0 = \mu\}$ and $x_0 \in \{\psi_n = M\}$, then we have $\psi_n(x_0) = M$. But by (3.4) $\psi_n(x_0) \to \mu$ and $M \to \mu - \varepsilon$ as $n \to +\infty$, from which we deduce that $\psi_n(x_0) > M$ if n is large enough, which is a contradiction.

Therefore, $\{\psi_n = M\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}$ must be a C^1 closed curve and

$$\{\psi_0 > \mu\} \subset \{\psi_n > M\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} \subset \{\psi_0 > \mu - \varepsilon\}. \tag{3.6}$$

Similarly $\{\psi_n = m\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}$ must be a C^1 closed curve and

$$\{\psi_0 > \mu + \varepsilon\} \subset \{\psi_n > m\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} \subset \{\psi_0 > \mu\}. \tag{3.7}$$

Hence

$$|\{\psi_n > m\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}| \leq |\{\psi_0 > \mu\}|$$
$$\leq |\{\psi_n > M\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}|. \tag{3.8}$$

By the continuity of ψ_n we can choose $\nu_n \in [m, M]$ such that

$$|\{\psi_n > \nu_n\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}| = |\{\psi_0 > \mu\}|,$$

and

$$\{\psi_n = \nu_n\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} \subset \{\mu - \varepsilon < \psi_0 < \mu + \varepsilon\}.$$

Note that such ν_n must be unique because $\frac{\partial \psi_n(x)}{\partial \vec{n}} < 0$ on the curve $\{\psi_n = \nu_n\}$. By (3.4) $\sup_{\{\psi_0 < \mu - \delta_0\}} \psi_n < \nu_n$ if n is large enough and by strong maximum principle $\inf_{\{\psi_0 > \mu + \delta_0\}} \psi_n > \nu_n$, so $\{\psi_n > \nu_n\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} = \{\psi_n > \nu_n\}$ if n is large enough.

That is, for any $\varepsilon > 0$, if n is large, we can choose a unique ν_n, such that $\partial\{\psi_n > \nu_n\} = \{\psi_n = \nu_n\}$ is a C^1 closed curve, moreover, $|\{\psi_n > \nu_n\}| = |\{\psi_0 > \mu\}|$ and $\{\psi_n = \nu_n\} \subset \{\mu - \varepsilon < \psi_0 < \mu + \varepsilon\}$. Hence the claim is proved.

Now we continue our proof of Lemma 3.1. Define $\bar{\omega}_n = \lambda I_{\{\psi_n > \nu_n\}}$, where ν_n is the one chosen in the above Claim. It is obvious that $\bar{\omega}_n \in R_{\omega_0}$. Now we compare $E(\bar{\omega}_n)$ and $E(\omega_n)$ as follows:

$$E(\bar{\omega}_n) - E(\omega_n) = \frac{1}{2}\int_D \bar{\omega}_n \bar{\psi}_n dx - \frac{1}{2}\int_D \omega_n \psi_n dx$$
$$= \int_D \psi_n(\bar{\omega}_n - \omega_n)dx + \frac{1}{2}\int_D (\bar{\psi}_n - \psi_n)(\bar{\omega}_n - \omega_n)dx, \tag{3.9}$$

where we used $\int_D \psi_n \bar{\omega}_n dx = \int_D \bar{\psi}_n \omega_n dx$ by the symmetry of the Green function. By integration by parts we have

$$\frac{1}{2}\int_D (\bar{\psi}_n - \psi_n)(\bar{\omega}_n - \omega_n)dx = \frac{1}{2}\int_D |\nabla(\bar{\psi}_n - \psi_n)|^2 dx,$$

therefore we get

$$E(\bar{\omega}_n) - E(\omega_n) \geq \int_D \psi_n(\bar{\omega}_n - \omega_n)dx + \frac{1}{2}\int_D |\nabla(\bar{\psi}_n - \psi_n)|^2 dx.$$

Since $|\{\psi_n > \nu_n\}| = |\{\psi_0 > \mu\}|$ and $\bar{\omega}_n = \lambda I_{\{\psi_n > \nu_n\}}$, the integral $\int_D \psi_n \omega_n dx$ attains its maximum if and only if $\omega_n = \bar{\omega}_n$, thus we obtain

$$\int_D \psi_n(\bar{\omega}_n - \omega_n)dx \geq 0. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$E(\bar{\omega}_n) \geq E(\omega_n), \tag{3.11}$$

and the equality holds if and only if $\omega_n \equiv \bar{\omega}_n$. By (3.2) and (3.11) we have

$$E(\bar{\omega}_n) \geq E(\omega_0).$$

On the other hand, by (iii) in the above Claim we can take n large enough such that $\gamma_{\bar{\omega}_n} \subset \gamma_{\omega_0,\delta}$, then by $(C3)$ we have

$$\bar{\omega}_n \equiv \omega_0,$$

which implies that

$$E(\bar{\omega}_n) = E(\omega_0) = E(\omega_n),$$

hence $\omega_n \equiv \bar{\omega}_n \equiv \omega_0$. This leads to a contradiction since $|\omega_n - \omega_0|_{L^1} > 0$ for each n. Therefore Lemma 3.1 is proved.

$\square$

Lemma 3.1 is a general stability criterion which does not require the vortex patch to be concentrated. However, $(C1) - (C3)$ are not easy to be verified in general. To continue, we need the following result from [14].

Lemma 3.2 ([14]). *Let ω_λ be the steady vortex patch in Theorem 1.3. Then $\{G\omega_\lambda > \mu_\lambda\}$ is a simply-connected domain with a C^1 boundary, and $\frac{\partial G\omega_\lambda}{\partial \vec{n}} < 0$ on this boundary, provided that λ is large enough.*

Having made the above preparations, now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 (general case). By Lemma 3.1, it suffices to show that ω_λ satisfies $(C1)$-$(C3)$ in Lemma 3.1. By Lemma 3.2, $(C1)$ and $(C2)$ holds true. We need only to verify $(C3)$.

By Theorem 2.7, the steady vortex patch in Theorem 1.3 is the same as the one in Theorem 2.3 when λ is large enough, so we need just to verify $(C3)$ for the ω_λ in Theorem 2.3.

Suppose that there exists another vortex patch $\bar{\omega}_\lambda$ enclosed by a C^1 closed curve $\gamma_{\bar{\omega}_\lambda}$, $\gamma_{\bar{\omega}_\lambda} \subset \gamma_{\omega_\lambda,\delta}$ and $E(\omega_\lambda) = E(\bar{\omega}_\lambda)$. To finish the proof, it suffices to show that $\omega_\lambda \equiv \bar{\omega}_\lambda$. Without loss of generality, we assume that δ is sufficiently small such that $\bar{\omega}_\lambda \in N_\lambda$, then it is obvious that $\bar{\omega}_\lambda$ is a maximizer of E on N_λ. By Theorem 2.3, $\bar{\omega}_\lambda$ must satisfy $\bar{\omega}_\lambda = \lambda I_{\{\bar{\psi}_\lambda > \bar{\mu}_\lambda\}}$ for some $\bar{\mu}_\lambda \in \mathbb{R}$ and the support of $\bar{\omega}_\lambda$ shrinks to x_1 as $\lambda \to +\infty$. Taking into account Theorem 2.7, we deduce that $\omega_\lambda \equiv \bar{\omega}_\lambda$ if λ is large enough, which is the desired result. $\square$

4. Proof of Theorem 2.3

In this section, we give the proof of Theorem 2.3 for completeness. The idea is basically from [29].

Lemma 4.1. *There exists $\omega_\lambda \in N_\lambda$ such that $E(\omega_\lambda) = \sup_{\omega \in N_\lambda} E(\omega)$. Moreover, $\omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\} \cap B_r(x_1)}$ for some $\mu_\lambda > 0$ depending on ω_λ, where $\psi_\lambda = G\omega_\lambda$.*

Proof. First we show the existence of a maximizer, that is, there exists $\omega_\lambda \in N_\lambda$ such that $E(\omega_\lambda) = \sup_{\omega \in N_\lambda} E(\omega)$. Since $G(x, y) \in L^1(D \times D)$, we have for any $\omega \in N_\lambda$

$$E(\omega) = \frac{1}{2} \int_D \int_D G(x, y)\omega(x)\omega(y)dxdy \le \frac{1}{2}\lambda^2 \int_D \int_D |G(x, y)|dxdy \le C\lambda^2,$$

which implies that E is bounded from above over N_λ. Let $\{\omega_n\} \subset N_\lambda$ be a sequence satisfying

$$\lim_{n \to +\infty} E(\omega_n) = \sup_{\omega \in N_\lambda} E(\omega).$$

Since N_λ is a bounded set in $L^\infty(D)$, thus is sequentially compact in the weak star topology in $L^\infty(D)$. Without loss of generality, we assume that $\omega_n \to \omega_\lambda$ weakly star in $L^\infty(D)$ for some $\omega_\lambda \in L^\infty(D)$ as $n \to +\infty$. We claim that $\omega_\lambda \in N_\lambda$. In fact, $\omega_n \to \omega_\lambda$ weakly star in $L^\infty(D)$ means

$$\lim_{n \to +\infty} \int_D \omega_n \phi dx = \int_D \omega_\lambda \phi dx, \ \forall \phi \in L^1(D). \tag{4.1}$$

For any $\phi \in C_0^\infty(D \setminus B_r(x_1))$, by the definition of N_λ we have

$$\lim_{n \to +\infty} \int_D \omega_n \phi dx = \int_D \omega_\lambda \phi dx = 0,$$

which implies $supp(\omega_\lambda) \subset B_r(x_1)$. By choosing $\phi \equiv 1$ in (4.1), we have

$$\int_D \omega_\lambda dx = \lim_{n \to +\infty} \int_D \omega_n dx = 1.$$

Now we prove that $0 \le \omega_\lambda \le \lambda$ a.e. in D. We prove $\omega_\lambda \le \lambda$ first. Suppose that $|\{\omega_\lambda > \lambda\}| > 0$, then there exist $\varepsilon_0, \varepsilon_1 > 0$ such that $|\{\omega_\lambda > \lambda + \varepsilon_0\}| > \varepsilon_1$. Denote $B^* = \{\omega_\lambda > \lambda + \varepsilon_0\} \subset D$, then for $\phi = I_{B^*}$ by weak star convergence we have

$$\lim_{n \to +\infty} \int_D (\omega_\lambda - \omega_n)\phi dx = \lim_{n \to +\infty} \int_{B^*} (\omega_\lambda - \omega_n)dx = 0.$$

On the other hand,

$$\int_{B^*} (\omega_\lambda - \omega_n)dx \ge \varepsilon_0 \varepsilon_1,$$

which is a contradiction. So we have $\omega_\lambda \le \lambda$ a.e. in D. Repeating this procedure, we obtain $\omega_\lambda \ge 0$ a.e. in D. Therefore we have proved $\omega_\lambda \in N_\lambda$.

Finally, by the property of weak star convergence, we have

$$\lim_{n \to +\infty} \frac{1}{2} \int_D \int_D G(x, y)\omega_n(x)\omega_n(y)dxdy = \lim_{n \to +\infty} \frac{1}{2} \int_D \int_D G(x, y)\omega_\lambda(x)\omega_\lambda(y)dxdy,$$

which gives $E(\omega_\lambda) = \sup_{\omega \in N_\lambda} E(\omega)$. So E attains its maximum over N_λ.

Now we show that for any maximizer ω_λ, there exists $\mu_\lambda > 0$ such that $\omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\} \cap B_r(x_1)}$. To this end, we define a family of test functions $\omega^s(x) = \omega_\lambda + s(z_0(x) - z_1(x))$, $s > 0$,

where z_0, z_1 satisfies

$$\begin{cases} z_0, z_1 \in L^\infty(D), \\ \int_D z_0 dx = \int_D z_1 dx, \\ z_0, z_1 \geq 0, \\ supp(z_0), supp(z_1) \subset B_r(x_1), \\ z_0 = 0 \quad in \ D\backslash\{\omega_\lambda \leq \lambda - \delta\}, \\ z_1 = 0 \quad in \ D\backslash\{\omega_\lambda \geq \delta\}. \end{cases} \quad (4.2)$$

Here δ is a positive parameter. Note that for fixed z_0, z_1 and δ, if s is sufficiently small, then $\omega^s \in N_\lambda$. So we have

$$0 \geq \frac{dE(\omega^s)}{ds}\big|_{s=0^+} = \int_D z_0 \psi_\lambda dx - \int_D z_1 \psi_\lambda dx,$$

where $\psi_\lambda = G\omega_\lambda$. This gives

$$\sup_{\{\omega_\lambda < \lambda\} \cap B_r(x_1)} \psi_\lambda \leq \inf_{\{\omega_\lambda > 0\} \cap B_r(x_1)} \psi_\lambda.$$

Since $\overline{B_r(x_1)}$ is connected and $\overline{\{\omega_\lambda < \lambda\} \cap B_r(x_1)} \cup \overline{\{\omega_\lambda > 0\} \cap B_r(x_1)} = \overline{B_r(x_1)}$, we have $\overline{\{\omega_\lambda < \lambda\} \cap B_r(x_1)} \cap \overline{\{\omega_\lambda > 0\} \cap B_r(x_1)} \neq \varnothing$, then by the continuity of ψ_λ we deduce that

$$\sup_{\{\omega_\lambda < \lambda\} \cap B_r(x_1)} \psi_\lambda = \inf_{\{\omega_\lambda > 0\} \cap B_r(x_1)} \psi_\lambda.$$

Define

$$\mu_\lambda := \sup_{\{\omega_\lambda < \lambda\} \cap B_r(x_1)} \psi_\lambda = \inf_{\{\omega_\lambda > 0\} \cap B_r(x_1)} \psi_\lambda,$$

then it is easy to check that

$$\begin{cases} \omega_\lambda = 0 \quad a.e. \ in \ \{\psi_\lambda < \mu_\lambda\} \cap B_r(x_1), \\ \omega_\lambda = \lambda \quad a.e. \ in \ \{\psi_\lambda > \mu_\lambda\} \cap B_r(x_1). \end{cases} \quad (4.3)$$

On the level set $\{\psi_\lambda = \mu_\lambda\} \cap B_r(x_1)$, we have $\nabla\psi_\lambda = 0$ a.e. by the property of Sobolev functions, thus $\omega_\lambda = -\Delta\psi_\lambda = 0$ a.e.. To summarize, we have obtained

$$\begin{cases} \omega_\lambda = 0 \quad a.e. \ \{\psi_\lambda \leq \mu_\lambda\} \cap B_r(x_1), \\ \omega_\lambda = \lambda \quad a.e. \ \{\psi_\lambda > \mu_\lambda\} \cap B_r(x_1), \end{cases} \quad (4.4)$$

or equivalently, $\omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\}} \cap B_r(x_1)$.

Finally, by taking r small and λ large we have $\mu_\lambda > 0$. $\qquad\square$

Now we estimate the size and location of $supp(\omega_\lambda)$ as $\lambda \to +\infty$. This is somewhat different from [29]. Define $\zeta_\lambda := \psi_\lambda - \mu_\lambda$, $\Omega := \{\psi_\lambda > \mu_\lambda\} \cap B_r(x_1)$ which is called the vortex core, and

$$T(\omega_\lambda) := \frac{1}{2}\int_D \zeta_\lambda \omega_\lambda dx,$$

which represents the kinetic energy of ω_λ on $\{\psi_\lambda > \mu_\lambda\}$. Note that $\{\zeta_\lambda > 0\} \subset\subset D$ since $\mu_\lambda > 0$.

Obviously we have the identity $E(\omega_\lambda) = T(\omega_\lambda) + \frac{1}{2}\mu_\lambda$. Moreover, integration by parts gives

$$
\begin{aligned}
T(\omega_\lambda) =& \frac{1}{2}\int_{\{\zeta_\lambda>0\}} \zeta_\lambda\omega_\lambda dx \\
=& \frac{1}{2}\int_{\{\zeta_\lambda>0\}} \zeta_\lambda(-\Delta\zeta_\lambda)dx \\
=& \frac{1}{2}\int_{\{\zeta_\lambda>0\}} |\nabla\zeta_\lambda|^2 dx \\
=& \frac{1}{2}\int_{D} |\nabla\zeta_\lambda^+|^2 dx.
\end{aligned}
$$

Here we used $\{\zeta_\lambda > 0\} \subset\subset D$.

Lemma 4.2. $T(\omega_\lambda) \le C$, where C is a positive number not depending on λ.

Proof. First we apply Hölder inequality to obtain

$$
T(\omega_\lambda) = \frac{1}{2}\int_D \zeta_\lambda\omega_\lambda dx = \frac{1}{2}\lambda\int_\Omega \zeta_\lambda dx \le \frac{1}{2}\lambda|\Omega|^{\frac{1}{2}}\left(\int_\Omega |\zeta_\lambda|^2 dx\right)^{\frac{1}{2}} = \frac{1}{2}\lambda|\Omega|^{\frac{1}{2}}\left(\int_{B_r(x_1)} |\zeta_\lambda^+|^2 dx\right)^{\frac{1}{2}}.
$$

On the other hand, by Sobolev embedding $W^{1,1}(B_r(x_1)) \hookrightarrow L^2(B_r(x_1))$, we have

$$
\begin{aligned}
\left(\int_{B_r(x_1)} |\zeta_\lambda^+|^2 dx\right)^{\frac{1}{2}} \le& C\left(\int_{B_r(x_1)} \zeta_\lambda^+ dx + \int_{B_r(x_1)} |\nabla\zeta_\lambda^+|dx\right) \\
=& C\left(\int_\Omega \zeta_\lambda^+ dx + \int_\Omega |\nabla\zeta_\lambda^+|dx\right).
\end{aligned}
$$

Here and in the sequel we use C to denote various positive numbers independent of λ. Therefore

$$
\begin{aligned}
T(\omega_\lambda) \le& \frac{1}{2}C\lambda|\Omega|^{\frac{1}{2}}\int_\Omega \zeta_\lambda dx + \frac{1}{2}C\lambda|\Omega|^{\frac{1}{2}}\int_\Omega |\nabla\zeta_\lambda|dx \\
\le& CT(\omega_\lambda)|\Omega|^{\frac{1}{2}} + \frac{1}{2}C\lambda|\Omega|\left(\int_\Omega |\nabla\zeta_\lambda|^2 dx\right)^{\frac{1}{2}} \\
\le& CT(\omega_\lambda)\lambda^{-\frac{1}{2}} + \frac{1}{2}C\left(T(\omega_\lambda)\right)^{\frac{1}{2}},
\end{aligned}
$$

where we used $\lambda|\Omega| = \int_D \omega_\lambda = 1$. By choosing λ large enough such that $C\lambda^{-1/2} < \frac{1}{2}$, we deduce that $T(\omega_\lambda) \le C$, which is the desired result.

$\square$

Lemma 4.3. $E(\omega_\lambda) \ge -\frac{1}{4\pi}\ln\varepsilon - C$, where $\varepsilon = 1/\sqrt{\lambda\pi}$.

Proof. Define $\bar\omega_\lambda = \lambda I_{B_\varepsilon(x_1)}$. It is easy to see that $\bar\omega_\lambda \in N_\lambda$, so we have $E(\omega_\lambda) \ge E(\bar\omega_\lambda)$. Now we calculate $E(\bar\omega_\lambda)$:

$$
\begin{aligned}
E(\bar{\omega}_\lambda) =& \frac{1}{2} \int_D \int_D G(x,y)\bar{\omega}_\lambda(x)\bar{\omega}_\lambda(y)dxdy \\
=& -\frac{1}{4\pi} \int_D \int_D \ln|x-y|\bar{\omega}_\lambda(x)\bar{\omega}_\lambda(y)dxdy - \frac{1}{2} \int_D \int_D h(x,y)\bar{\omega}_\lambda(x)\bar{\omega}_\lambda(y)dxdy \quad (4.5) \\
=& -\frac{\lambda^2}{4\pi} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} \ln|x-y|dxdy - \frac{1}{2} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} h(x,y)dxdy.
\end{aligned}
$$

Since $|x-y| \le 2\varepsilon$ for $x,y \in B_\varepsilon(x_1)$, we have

$$
\begin{aligned}
-\frac{\lambda^2}{4\pi} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} \ln|x-y|dxdy \ge& -\frac{\lambda^2}{4\pi} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} \ln|2\varepsilon|dxdy \\
=& -\frac{1}{4\pi}\ln\varepsilon - \frac{1}{4\pi}\ln 2.
\end{aligned} \quad (4.6)
$$

On the other hand, the integral $\int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} h(x,y)dxdy$ converges to $h(x_1,x_1)$ as $\lambda \to +\infty$, therefore is bounded, or equivalently

$$
\left| \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} h(x,y)dxdy \right| \le C. \quad (4.7)
$$

Taking into account (4.5) (4.6) and (4.7) we get

$$
E(\omega_\lambda) \ge -\frac{1}{4\pi}\ln\varepsilon - C.
$$

$\square$

From Lemma 4.2, Lemma 4.3 and the identity $E(\omega_\lambda) = T(\omega_\lambda) + \mu_\lambda/2$, we immediately obtain

Lemma 4.4. $\mu_\lambda \ge -\frac{1}{2\pi}\ln\varepsilon - C.$

Now we show that the size of $supp(\omega_\lambda)$ is of order ε.

Lemma 4.5. *There exists some $R_0 > 0$ such that $diam(supp(\omega_\lambda)) < R_0\varepsilon$ when λ is large enough.*

Proof. For any $x \in supp(\omega_\lambda)$, we have by definition $\psi_\lambda(x) \ge \mu_\lambda$, that is,

$$
\int_D G(x,y)w_\lambda(y)dy \ge -\frac{1}{2\pi}\ln\varepsilon - C. \quad (4.8)
$$

On the other hand, since $h(x,y)$ is bounded from below in $D \times D$, we have

$$
\int_D G(x,y)w_\lambda(y)dy \le -\frac{1}{2\pi}\int_D \ln|x-y|\omega_\lambda(y)dy + C. \quad (4.9)
$$

Combining (4.8) and (4.9) and by simple calculation, we can easily get

$$
\int_D \ln\frac{\varepsilon}{|x-y|}\omega_\lambda(y)dy \ge C. \quad (4.10)
$$

Let $R > 1$ be a number to be determined. we divide the integral on the left side of (4.10) into two parts

$$\int_{B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y) dy + \int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y) dy \geq C. \qquad (4.11)$$

The first integral in (4.11) can be estimated by the rearrangement inequality as follows

$$\int_{B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y) dy \leq \lambda \int_{B_\varepsilon(x)} \ln \frac{\varepsilon}{|x-y|} dy = \frac{1}{2}.$$

So we obtain

$$\int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y) dy \geq C. \qquad (4.12)$$

Notice that $|x-y| \geq R\varepsilon$ for any $y \in B_{R\varepsilon}(x)$, so we obtain from (4.12)

$$\int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y) dy \leq \int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{1}{R} \omega_\lambda(y) dy,$$

or equivalently,

$$\int_{D \setminus B_{R\varepsilon}(x)} \omega_\lambda(y) dy \leq \frac{C}{\ln R}.$$

Taking into account the fact $\int_D \omega_\lambda dx = 1$, we get

$$\int_{B_{R\varepsilon}(x)} \omega_\lambda(y) dy \geq 1 - \frac{C}{\ln R}.$$

Choosing R large such that $1 - \frac{C}{\ln R} > \frac{1}{2}$, we obtain

$$\int_{B_{R\varepsilon}(x)} \omega_\lambda(y) dy > \frac{1}{2}. \qquad (4.13)$$

Since $\int_D \omega_\lambda = 1 dx$ and (4.13) holds true for arbitrary $x \in supp(\omega_\lambda)$, we deduce that

$$diam(supp\omega_\lambda) < 2R\varepsilon.$$

Thus the lemma is proved by choosing $R_0 = 2R$.

$\square$

Now we estimate the location of $supp(\omega_\lambda)$.

Lemma 4.6. $\lim_{\lambda \to +\infty} \int_D x\omega_\lambda(x) dx = x_1$.

Proof. Denote $x_\lambda := \int_D x\omega_\lambda(x) dx$, then obviously $x_\lambda \in \overline{B_r(x_1)}$. For any sequence $\{x_{\lambda_j}\}, \lambda_j \to +\infty$, there exists a subsequence $\{x_{\lambda_{j_k}}\}$ such that $x_{\lambda_{j_k}} \to z_1 \in \overline{B_r(x_1)}$. For simplicity, we still denote the subsequence by $\{x_{\lambda_k}\}$. It suffices to show that $z_1 = x_1$.

Define $\bar{\omega}_\lambda := \lambda I_{B_\varepsilon(x_1)}$. Since $E(\bar{\omega}_\lambda) \le E(\omega_\lambda)$, we obtain

$$
\begin{aligned}
&\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\bar{\omega}_{\lambda_k}(x)\bar{\omega}_{\lambda_k}(y)dxdy - \int_D\int_D h(x,y)\bar{\omega}_{\lambda_k}(x)\bar{\omega}_{\lambda_k}(y)dxdy \\
&\le \int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_{\lambda_k}(x)\omega_{\lambda_k}(y)dxdy - \int_D\int_D h(x,y)\omega_{\lambda_k}(x)\omega_{\lambda_k}(y)dxdy.
\end{aligned}
\tag{4.14}
$$

By Riesz's rearrangement inequality (see [23], 3.7),

$$
\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\bar{\omega}_{\lambda_k}(x)\bar{\omega}_{\lambda_k}(y)dxdy \ge \int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_{\lambda_k}(x)\omega_{\lambda_k}(y)dxdy. \tag{4.15}
$$

Combining (4.14) and (4.15) we get

$$
\int_D\int_D h(x,y)\bar{\omega}_{\lambda_k}(x)\bar{\omega}_{\lambda_k}(y)dxdy \ge \int_D\int_D h(x,y)\omega_{\lambda_k}(x)\omega_{\lambda_k}(y)dxdy. \tag{4.16}
$$

Passing $k \to +\infty$ in (4.16), we have $h(z_1, z_1) \le h(x_1, x_1)$, that is, $H(z_1) \le H(x_1)$. Since x_1 is the unique minimum point of $H(x)$ in $\overline{B_r(x_1)}$, we obtain $z_1 = x_1$, which is the desired result. $\qquad\square$

Lemma 4.7. ω_λ *is a steady solution of the vorticity equation, that is, satisfies* (1.9), *provided that λ is large enough.*

Proof. For any $\xi \in C_c^\infty(D)$, we define a family of C^1 transformations $\Phi_t(x), t \in (-\infty, +\infty)$, from D to D by solving the following ODE,

$$
\begin{cases}
\frac{d\Phi_t(x)}{dt} = J\nabla\xi(\Phi_t(x)), & t \in \mathbb{R}, \\
\Phi_0(x) = x,
\end{cases}
\tag{4.17}
$$

where J denotes clockwise rotation through $\pi/2$ as before. Note that (4.17) is solvable for all t since $J\nabla\xi$ is a smooth vector field with compact support in D. It is easy to see that $J\nabla\xi$ is divergence-free, so by Liouville theorem(see [25], Appendix 1.1), $\Phi_t(x)$ is an area-preserving transformation for any fixed t, that is,

$$
\Phi_t(A) = A, \ \forall A \subset D.
$$

Now we define a family of test functions

$$
\omega_t(x) := \omega_\lambda(\Phi_t(x)). \tag{4.18}
$$

It is easy to check that $\omega_t \in R_{\omega_\lambda}$. Since $supp(\omega_\lambda)$ shrinks to x_1 as $\lambda \to +\infty$, so $dist(supp(\omega_\lambda, \partial B_r(x_1))) > 0$ if λ is large enough. Therefore we have $\omega_t \in N_\lambda$ if $|t|$ is small, from which we obtain

$$
\left.\frac{dE(\omega_t)}{dt}\right|_{t=0} = 0. \tag{4.19}
$$

For $|t| << 1$, we expand $E(\omega_t)$ at $t = 0$ as follows

$$
\begin{aligned}
E(\omega_t) =& \frac{1}{2} \int_D \int_D G(x, y) \omega_\lambda(\Phi_t(x)) \omega_\lambda(\Phi_t(y)) dx dy \\
=& \frac{1}{2} \int_D \int_D G(\Phi_{-t}(x), \Phi_{-t}(y)) \omega_\lambda(x) \omega_\lambda(y) dx dy \\
=& E(\omega_\lambda) + t \int_D \omega_\lambda \nabla G \omega_\lambda \cdot J \nabla \xi dx + o(t).
\end{aligned}
\tag{4.20}
$$

Combining (4.19) and (4.20) together we immediately get

$$
\int_D \omega_\lambda \nabla G \omega_\lambda \cdot J \nabla \xi dx = 0,
$$

which completes the proof Lemma 4.6. □

Proof of Theorem 2.3. It follows from Lemma 4.1, Lemma 4.4, Lemma 4.5 and Lemma 4.6. □

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