# On the polynomiality and asymptotics of moments of sizes for random $(n, dn \pm 1)$-core partitions with distinct parts

Huan XIONG[1] and Wenston J.T. ZANG[2]

[1]Institut de Recherche Mathématique Avancée, UMR 7501
Université de Strasbourg et CNRS, F-67000 Strasbourg, France

[2]Institute of Advanced Study of Mathematics
Harbin Institute of Technology, Heilongjiang 150001, P.R. China

Email: [1]xiong@math.unistra.fr, [2]zang@hit.edu.cn

**Abstract.** Amdeberhan's conjectures on the enumeration, the average size, and the largest size of $(n, n+1)$-core partitions with distinct parts have motivated many research on this topic. Recently, Straub and Nath-Sellers obtained formulas for the numbers of $(n, dn-1)$ and $(n, dn+1)$-core partitions with distinct parts, respectively. Let $X_{s,t}$ be the size of a uniform random $(s,t)$-core partition with distinct parts when $s$ and $t$ are coprime to each other. Some explicit formulas for the $k$-th moments $\mathbb{E}[X^k_{n,n+1}]$ and $\mathbb{E}[X^k_{2n+1,2n+3}]$ were given by Zaleski and Zeilberger when $k$ is small. Zaleski also studied the expectation and higher moments of $X_{n,dn-1}$ and conjectured some polynomiality properties concerning them in *arXiv:1702.05634*.

Motivated by the above works, we derive several polynomiality results and asymptotic formulas for the $k$-th moments of $X_{n,dn+1}$ and $X_{n,dn-1}$ in this paper, by studying the beta sets of core partitions. In particular, we show that these $k$-th moments are asymptotically some polynomials of n with degrees at most $2k$, when $d$ is given and $n$ tends to infinity. Moreover, when $d = 1$, we derive that the $k$-th moment $\mathbb{E}[X^k_{n,n+1}]$ of $X_{n,n+1}$ is asymptotically equal to $\left(n^2/10\right)^k$ when $n$ tends to infinity. The explicit formulas for the expectations $\mathbb{E}[X_{n,dn+1}]$ and $\mathbb{E}[X_{n,dn-1}]$ are also given. The $(n, dn-1)$-core case in our results proves several conjectures of Zaleski on the polynomiality of the expectation and higher moments of $X_{n,dn-1}$.

**Keywords.** partition, hook length, core partition, average size, distinct part

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## 1. Introduction

A partition $\lambda$ is called a *$t$-core partition* if none of its hook lengths is divisible by $t$. Core partitions arise naturally in the study of modular representations of finite groups. For example, they label the blocks of irreducible characters of symmetric groups (see [16]). Furthermore, $\lambda$ is called a $(t_1, t_2, \ldots, t_m)$-core partition if it is simultaneously a $t_1$-core, a $t_2$-core, $\ldots$, a $t_m$-core partition (see [1, 11]). It is well known that, the number of $(t_1, t_2, \ldots, t_m)$-core partitions is finite if and only if the greatest common divisor $\gcd(t_1, t_2, \ldots, t_m) = 1$ (for example, see [11, Theorem 1] or [25, Theorem 1.1]).

In 2002, Anderson [3] proved the following result on the number of $(t_1, t_2)$-core partitions, by studying their connections with certain lattice paths.

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**Theorem 1.1** (Anderson [3]). *Let $t_1$ and $t_2$ be two coprime positive integers. Then the number of $(t_1, t_2)$-core partitions equals*

$$\frac{(t_1 + t_2 - 1)!}{t_1!\, t_2!}.$$

Anderson's work has motivated many research on the enumeration, largest sizes and average sizes of simultaneous core partitions (see [2, 7, 8, 9, 15, 19, 24, 27]). For example, when $t_1$ and $t_2$ are coprime to each other, it was proved by Olsson and Stanton [16] that the largest size of $(t_1, t_2)$-core partitions equals $(t_1^2 - 1)(t_2^2 - 1)/24$, in their study of block inclusions of symmetric groups. Armstrong (see [4]) gave the following conjecture on the average size of such partitions, which was first proved by Johnson [10] and later by Wang [21].

**Theorem 1.2** (Armstrong's Conjecture). *Let $t_1$ and $t_2$ be two coprime positive integers. Then the average size of $(t_1, t_2)$-core partitions equals*

$$\frac{(t_1 - 1)(t_2 - 1)(t_1 + t_2 + 1)}{24}.$$

Recently, the problem on the enumeration of simultaneous core partitions with distinct parts was raised by Amdeberhan [1]. He conjectured explicit formulas for the number, the largest size and the average size of $(n, n + 1)$-core partitions with distinct parts, which were first proved by the first author [23], and later proved independently (and extended) by Straub [20], Nath-Sellers [13], Zaleski [29] and Paramonov [17]. Let $X_{s,t}$ be the size of a uniform random $(s, t)$-core partition with distinct parts when $s$ and $t$ are coprime to each other. Zaleski [29] derived several explicit formulas for the $k$-th moment $\mathbb{E}[X^k_{n,n+1}]$ of $X_{n,n+1}$ when $k \le 16$. The number, the largest size and the average size of $(2n+1, 2n+3)$-core partitions with distinct parts were also well studied (see [5, 17, 20, 26, 28]). Several explicit formulas for the $k$-th (when $k \le 7$) moment $\mathbb{E}[X^k_{2n+1,2n+3}]$ of $X_{2n+1,2n+3}$ were obtained by Zaleski and Zeilberger [28].

In 2016, Straub [20] derived the following generalized Fibonacci recurrence for the number $N_d(n)$ of $(n, dn - 1)$-core partitions with distinct parts.

**Theorem 1.3** (Straub [20]). *Let $N_d(1) = 1$, and $N_d(n)$ be the number of $(n, dn - 1)$-core partitions with distinct parts for two positive integers $d \ge 1$ and $n \ge 2$. Then*

$$\begin{aligned} &N_d(1) = 1,\ N_d(2) = d, \\ &N_d(n) = N_d(n-1) + dN_d(n-2), \quad \text{if } \ n \ge 3. \end{aligned} \tag{1.1}$$

The $(n, dn + 1)$-core analog was obtained later by Nath-Sellers [14].

**Theorem 1.4** (Nath-Sellers [14]). *Let $M_d(-1) = 0$, $M_d(0) = 1$, and $M_d(n)$ be the number of $(n, dn + 1)$-core partitions with distinct parts for two positive integers $d$ and $n$. Then*

$$\begin{aligned} &M_d(-1) = 0,\ M_d(0) = 1, \\ &M_d(n) = M_d(n-1) + dM_d(n-2), \quad \text{if } \ n \ge 1. \end{aligned} \tag{1.2}$$

Table 1 gives the first few values for $N_d(n)$ and $M_d(n)$.

TABLE 1. The number of $(n, dn \pm 1)$-core partitions with distinct parts for $1 \le n \le 6$.

| n | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $N_d(n)$ | 1 | d | $2d$ | $d^2 + 2d$ | $3d^2 + 2d$ | $d^3 + 5d^2 + 2d$ |
| $M_d(n)$ | 1 | $d + 1$ | $2d + 1$ | $d^2 + 3d + 1$ | $3d^2 + 4d + 1$ | $d^3 + 6d^2 + 5d + 1$ |

It is easy to derive that, when $d \neq 2$,

$$M_d(n) = \frac{d(d-1)N_d(n) - N_d(n+1)}{d(d-2)} \tag{1.3}$$

and

$$M_d(n-1) = \frac{(d-1)N_d(n+1) - dN_d(n)}{d(d-2)}. \tag{1.4}$$

Recently, the largest sizes of the above two kinds of partitions were given by the first author [22]. Zaleski conjectured an explicit formula for the average size of $(n, dn-1)$-core partitions with distinct parts in [30]. Furthermore, Zaleski conjectured some polynomiality properties for higher moments of their sizes.

In this paper, we derive results on moments of sizes for random $(n, dn \pm 1)$-core partitions with distinct parts. The $(n, dn-1)$-core case proves several conjectures of Zaleski [30]. Let $\mathcal{C}_{n,dn+1}$ and $\mathcal{C}_{n,dn-1}$ be the sets of $(n, dn+1)$-core and $(n, dn-1)$-core partitions with distinct parts respectively. Our main results are stated next. The $(n, dn-1)$-core case in Theorems 1.5 and 1.6 are equivalent to Zaleski's Conjectures 3.5 and 3.1 in [30], respectively.

**Theorem 1.5** (see Conjecture 3.5 of Zaleski [30])**.** *Let $k$ be a positive integers. The $k$-th power sums*

$$\sum_{\lambda \in \mathcal{C}_{n,dn+1}} |\lambda|^k \qquad \text{and} \qquad \sum_{\lambda \in \mathcal{C}_{n,dn-1}} |\lambda|^k$$

*for sizes of partitions in $\mathcal{C}_{n,dn+1}$ and in $\mathcal{C}_{n,dn-1}$ are of the form*

$$A(n,d)M_d(n) + B(n,d)M_d(n+1), \tag{1.5}$$

*where $A(n,d)$ and $B(n,d)$ are some polynomials of $n$ with degrees at most $2k$, whose coefficients are rational functions in $d$.*

**Remark.** In the above theorem, we use $M_d(n)$ and $M_d(n+1)$ as a basis, while $N_d(n)$ and $N_d(n+1)$ are used in the original statement of Zaleski's conjectures in [30]. As mentioned by Zaleski, some of his conjectures are anomalous for the case $d = 2$. The use of the basis $M_d(n)$ and $M_d(n+1)$ avoids this problem. That is, the form (1.5) always holds for any $d \geq 1$. Also, by (1.3) and (1.4) we know, when $d \neq 2$, $M_d(n)$ and $M_d(n+1)$ in (1.5) can be replaced by $N_d(n)$ and $N_d(n+1)$.

**Theorem 1.6** (see Conjecture 3.1 of Zaleski [30])**.** *Let $n$ and $k$ be two given positive integers. Then the $k$-th power sums*

$$\sum_{\lambda \in \mathcal{C}_{n,dn+1}} |\lambda|^k \qquad \text{and} \qquad \sum_{\lambda \in \mathcal{C}_{n,dn-1}} |\lambda|^k$$

*are polynomials of $d$ with degrees at most $2k + \lfloor n/2 \rfloor$.*

Recall that $X_{n,dn-1}$ and $X_{n,dn+1}$ are sizes of uniform random $(n, dn-1)$-core and $(n, dn+1)$-core partitions with distinct parts, respectively. By Theorems 1.5 and 1.6 we derive the following asymptotic formulas when $d$ is fixed or $n$ is fixed, respectively.

**Theorem 1.7.** *Let $d$ and $k$ be two given positive integers. Then the $k$-th moments of $X_{n,dn+1}$ and $X_{n,dn-1}$ are asymptotically some polynomials of $n$ with degrees at most $2k$, when $n$ tends to infinity. That is, there exist some constants $A_{d,k}$ and $B_{d,k}$ such that*

$$\mathbb{E}[X_{n,dn+1}^k] = A_{d,k}\, n^{2k} + O(n^{2k-1}) \tag{1.6}$$

*and*

$$\mathbb{E}[X_{n,dn-1}^k] = B_{d,k}\, n^{2k} + O(n^{2k-1}). \tag{1.7}$$

**Theorem 1.8.** *Let $n \geq 2$ and $k \geq 1$ be two given integers. Then the $k$-th moments of $X_{n,dn+1}$ and $X_{n,dn-1}$ are asymptotically some polynomials of $d$ with degrees at most $2k$, when $d$ tends to infinity. That is, there exist some constants $C_{n,k}$ and $D_{n,k}$ such that*

$$\mathbb{E}[X_{n,dn+1}^k] = C_{n,k}\, d^{2k} + O(d^{2k-1}) \tag{1.8}$$

*and*

$$\mathbb{E}[X_{n,dn-1}^k] = D_{n,k}\, d^{2k} + O(d^{2k-1}). \tag{1.9}$$

Moreover, when $d = 1$, we derive the leading term in the asymptotic formula of $\mathbb{E}[X_{n,n+1}^k]$.

**Theorem 1.9.** *Let $k$ be a given positive integer. Then the $k$-th moment of $X_{n,n+1}$ satisfies the following asymptotic formula:*

$$\mathbb{E}[X_{n,n+1}^k] = \left(\frac{1}{10}\right)^k n^{2k} + O(n^{2k-1}).$$

We also derive explicit formulas for the expectations of $X_{n,dn+1}$ and $X_{n,dn-1}$.

**Theorem 1.10.** *Let $d$ and $n$ be two given positive integers. The expectation of $X_{n,dn+1}$ equals*

$$\begin{aligned}\mathbb{E}[X_{n,dn+1}] = {} & \frac{d(d+1)(5d+1)(n-1)^2}{24(4d+1)} + \frac{d(d+1)(32d^2+63d+7)(n-1)}{24(4d+1)^2} \\ & + \frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2} - \frac{M_d(n-1)}{M_d(n)} \\ & \cdot \Big(\frac{d(d+1)(d-1)(n-1)^2}{24(4d+1)} + \frac{d(d+1)(14d^2+21d+1)(n-1)}{24(4d+1)^2} + \frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2}\Big).\end{aligned}$$

**Example 1.11.** *Let $d = 2$ and $n = 4$. Then $M_d(n-1) = M_2(3) = 5$ and $M_d(n) = M_2(4) = 11$. By the above theorem the expectation of $X_{n,dn+1}$ should be $54/11$. We can check that this is true since the number of $(4,9)$-core partitions with distinct parts equals $11$, and the sum of their sizes equals $54$:*

$$\mathcal{C}_{4,9} = \{\emptyset, (1), (2), (3), (2,1), (4,1), (5,2), (6,3), (3,2,1), (5,2,1), (4,3,2,1)\}.$$

**Example 1.12.** *Let $d = 3$ and $n = 3$. Then $M_d(n-1) = M_3(2) = 4$ and $M_d(n) = M_3(3) = 7$. By the above theorem the expectation of $X_{n,dn+1}$ should be $34/7$. We can check that this is true since the number of $(3,10)$-core partitions with distinct parts equals $7$, and the sum of their sizes equals $34$:*

$$\mathcal{C}_{3,10} = \{\emptyset, (1), (2), (3,1), (4,2), (5,3,1), (6,4,2)\}.$$

**Theorem 1.13.** *Let $d \geq 1$ and $n \geq 2$ be two given positive integers. The total sum of sizes of partitions in $\mathcal{C}_{n,dn-1}$ is*

$$\begin{aligned}\sum_{\lambda \in \mathcal{C}_{n,dn-1}} |\lambda| = {} & M_d(n) \cdot \Big(\frac{(d^2-1)(5d^2+d-1)n^2}{24d(4\,d+1)} - \frac{(d+1)(8d^4+27d^3+2d^2-1)n}{24d(4\,d+1)^2} + \frac{d^2-1}{12d}\Big) \\ & + M_d(n-1) \cdot \Big(\frac{(d+1)(-d^3+7d^2+d-1)n^2}{24d(4\,d+1)} - \frac{(d+1)(6d^4-19d^3-7d^2+d+1)n}{24d(4\,d+1)^2} \\ & - \frac{(d+1)(d^4+20d^3-6d^2-8d-1)}{12d(4\,d+1)^2}\Big).\end{aligned}$$

**Example 1.14.** *Let $d=1$ and $n=4$. Then $M_d(n-1)=M_1(3)=3$ and $M_d(n)=M_1(4)=5$. By the above theorem the total sum of sizes of $(4,3)$-core partitions with distinct parts should be 3. We can check that this is true since the number of such partitions equals $N_d(n)=N_1(4)=3$, and the sum of their sizes equals 3:*

$$\mathcal{C}_{4,3}=\{\emptyset,(1),(2)\}.$$

**Example 1.15.** *Let $d=2$ and $n=5$. Then $M_d(n-1)=M_2(4)=11$ and $M_d(n)=M_2(5)=21$. By the above theorem the total sum of sizes of $(5,9)$-core partitions with distinct parts should be 92. We can check that this is true since the number of such partitions equals $N_d(n)=N_2(5)=16$, and the sum of their sizes equals 92:*

$$\begin{aligned}\mathcal{C}_{5,9}=\{\emptyset,&(1),(2),(3),(4),(2,1),(3,1),(3,2),(5,1),(6,2),(7,3),\\ &(4,2,1),(6,2,1),(4,3,1),(5,3,2),(5,4,2,1)\}.\end{aligned}$$

By (1.3) and (1.4) we obtain, Theorem 1.13 implies the following conjecture of Zaleski [30] directly.

**Corollary 1.16** (Conjecture 3.8 of Zaleski [30])**.** *Let $d\geq 1$ and $n\geq 2$ be two given positive integers. When $d\neq 2$, the expectation of $X_{n,dn-1}$ equals*

$$\begin{aligned}\mathbb{E}[X_{n,dn-1}]=&\ \frac{\left(5\,d^3+7\,d^2+d-1\right)n^2}{24(4\,d+1)}-\frac{\left(8\,d^5+21\,d^4+7\,d^3-d^2+3\,d-2\right)n}{24(16\,d^3-24\,d^2-15\,d-2)}\\ &+\frac{17\,d^4+13\,d^3-9\,d^2-7\,d-2}{12(16\,d^3-24\,d^2-15\,d-2)}+\frac{N_d(n+1)}{N_d(n)}\\ &\cdot\left(-\frac{\left(d^2-1\right)n^2}{24(4\,d+1)}-\frac{\left(2\,d^4-9\,d^3-16\,d^2-3\,d+2\right)n}{8(16\,d^3-24\,d^2-15\,d-2)}-\frac{d^4+20\,d^3+9\,d^2-20\,d-10}{12\,(d-2)\,(4\,d+1)^2}\right).\end{aligned}$$

The rest of the paper is arranged as follows. In Section 2 we review some basic results on core partitions. The characterizations for the $\beta$-sets of $(n,dn-1)$ and $(n,dn+1)$-core partitions with distinct parts are given in Section 3. Then in Section 4 we use these characterizations to translate the problems to study two families of functions $G^+_{d,m,a,b}(n)$ and $G^-_{d,m,a,b}(n)$, therefore prove the main results. The explicit formulas for expectations of $X_{n,dn+1}$ and $X_{n,dn-1}$ are derived in Section 5. The asymptotic formulas for moments of $X_{n,n+1}$ are given in Section 6.

## 2. Simultaneous core partitions and their $\beta$-sets

A *partition* is a finite weakly decreasing sequence of positive integers $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$. The numbers $\lambda_i$ $(1\leq i\leq\ell)$ are called the *parts* and $\sum_{1\leq i\leq\ell}\lambda_i$ the *size* of the partition $\lambda$ (see [12, 18]). Each partition $\lambda$ is identified with its *Young diagram*, which is an array of boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$-th row. For the $(i,j)$-box in the $i$-th row and $j$-th column in the Young diagram, its *hook length* $h(i,j)$ is defined to be the number of boxes exactly to the right, and exactly below, and the box itself. Recall that a partition $\lambda$ is called a $(t_1,t_2,\ldots,t_m)$*-core partition* if none of its hook lengths is divisible by $t_1,t_2,\ldots,t_{m-1}$, or $t_m$ (see [1, 11]). For example, Figure 1 gives the Young diagram and hook lengths of the partition $(6,3,3,2)$. Therefore, it is a $(7,10)$-core partition since none of its hook lengths is divisible by 7 or 10.

The *$\beta$-set* of a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ is denoted by

$$\beta(\lambda)=\{\lambda_i+\ell-i:1\leq i\leq\ell\}.$$

In fact, $\beta(\lambda)$ is equal to the set of hook lengths of boxes in the first column of the corresponding Young diagram of $\lambda$ (see [16, 24]). For example, from Figure 1 we know that $\beta((6,3,3,2))=\{9,5,4,2\}$. It is

| | | | | | |
|---|---|---|---|---|---|
| 9 | 8 | 6 | 3 | 2 | 1 |
| 5 | 4 | 2 | | | |
| 4 | 3 | 1 | | | |
| 2 | 1 | | | | |

FIGURE 1. The Young diagram and hook lengths of the partition $(6, 3, 3, 2)$.

easy to see that a partition $\lambda$ is uniquely determined by its $\beta$-set $\beta(\lambda)$. The following results on $\beta$-sets are well known.

**Lemma 2.1** ([16, 22, 23, 24]). *The size of a partition $\lambda$ is determined by its $\beta$-set as the following:*

$$|\lambda| = \sum_{x \in \beta(\lambda)} x - \binom{|\beta(\lambda)|}{2}. \tag{2.1}$$

**Lemma 2.2** ([22, 23]). *The partition $\lambda$ is a partition with distinct parts if and only if there does not exist $x, y \in \beta(\lambda)$ with $x - y = 1$.*

**Lemma 2.3** ([3, 6, 16, 23, 24]). (The abacus condition for t-core partitions.) *A partition $\lambda$ is a $t$-core partition if and only if for any $x \in \beta(\lambda)$ with $x \geq t$, we always have $x - t \in \beta(\lambda)$.*

## 3. The $\beta$-sets of $(n, dn \pm 1)$-core partitions with distinct parts

In this section we focus on $(n, dn - 1)$ and $(n, dn + 1)$-core partitions with distinct parts. The following characterizations for $\beta$-sets are well known. We give a short proof here for completeness.

**Theorem 3.1** ([14, 20, 22, 30]). *Let $n$ and $d$ be two positive integers. Then a finite subset $S$ of $\mathbb{N}$ is a $\beta$-set of some $(n, dn + 1)$-core partition with distinct parts iff the following conditions hold:*

*(i) $S \subseteq \{(i - 1)n + j : 1 \leq i \leq d,\ 1 \leq j \leq n - 1\}$;*

*(ii) If $in + j \in S$ with $1 \leq i \leq d - 1,\ 1 \leq j \leq n - 1$, then $(i - 1)n + j \in S$;*

*(iii) If $j \in S$ with $1 \leq j \leq n - 2$, then $j + 1 \notin S$.*

*Proof.* (1) Suppose that $\lambda$ is an $(n, dn+1)$-core partition with distinct parts and $S = \beta(\lambda)$. By Lemma 2.3 we have $dn+1 \notin S$ and $nx \notin S$ for any $1 \leq x \leq d$ since $0 \notin S$. For $x \geq dn+2$, if $x \in S$, by Lemma 2.3 we know $x - dn$, $x - (dn + 1) \in S$. But by Lemma 2.2 this is impossible since $\lambda$ is a partition with distinct parts. Then the condition (i) holds. Also, (ii) and (iii) hold by Lemmas 2.2 and 2.3.

(2) On the other hand, suppose that the set $S$ satisfies conditions (i), (ii) and (iii). Let $\lambda$ be the partition with $\beta(\lambda) = S$. Since $\beta(\lambda)$ doesn't have elements larger than $dn-1$, $\lambda$ must be a $(dn+1)$-core partition. Also, by (ii) $\lambda$ must be an $n$-core partition. Finally by (i), (ii), (iii) and Lemma 2.2 we know $\lambda$ is a partition with distinct parts. □

Let

$$\mathcal{A}_{d,n} := \{(i, j) : 1 \leq i \leq d,\ 1 \leq j \leq n\}.$$

We say that a subset $I \subseteq \mathcal{A}_{d,n}$ is *nice* if it satisfies the following two conditions:

(1) $(i + 1, j) \in I$ and $i \geq 1$ imply $(i, j) \in I$;

(2) $(1, j) \in I$ and $1 \leq j \leq n - 1$ imply $(1, j + 1) \notin I$.

Let $\mathcal{B}_{d,n}^{+}$ be the set of nice subsets of $\mathcal{A}_{d,n}$. For each $n$-core partition $\lambda$, define

$$\psi_n(\lambda) := \{(i,j):\ 1 \le j \le n-1,\ (i-1)n+j \in \beta(\lambda)\}. \tag{3.1}$$

Then by Theorem 3.1 the map $\psi_n$ gives a bijection between the sets $\mathcal{C}_{n,dn+1}$ and $\mathcal{B}_{d,n-1}^{+}$. Furthermore, by Lemma 2.1 we have

**Lemma 3.2.**

$$\sum_{\lambda \in \mathcal{C}_{n,dn+1}} |\lambda|^k = \sum_{I \in \mathcal{B}_{d,n-1}^{+}} \left( \sum_{(i,j)\in I} ((i-1)n+j) - \frac{|I|^2}{2} + \frac{|I|}{2} \right)^k. \tag{3.2}$$

**Example 3.3.** *Let $d=3$ and $n=3$. By Example 1.12 we know there are 7 of $(3,10)$-core partitions with distinct parts: $\emptyset, (1), (2), (3,1), (4,2), (5,3,1), (6,4,2)$. The corresponding nice subsets of $\mathcal{A}_{3,2}$ are:*

$$\begin{aligned}\mathcal{B}_{3,2}^{+} = \{\ &\emptyset,\ \{(1,1)\},\ \{(1,2)\},\ \{(1,1),(2,1)\},\ \{(1,2),(2,2)\},\\ &\{(1,1),(2,1),(3,1)\},\ \{(1,2),(2,2),(3,2)\}\ \}.\end{aligned}$$

*Let $k=2$. It is easy to check that both sides of (3.2) equals 282.*

Similarly the following are characterizations for $\beta$-sets of $(n, dn-1)$-core partitions with distinct parts. Notice that $dn-1 \notin S$ in the following condition (iv).

**Theorem 3.4** ([14, 20, 22, 30])**.** *Let $n \ge 2$ and $d \ge 1$ be two positive integers. Then a finite subset $S$ of $\mathbb{N}$ is a $\beta$-set of some $(n, dn-1)$-core partition with distinct parts iff the following conditions hold:*

*(iv) $S \subseteq \{(i-1)n+j : 1 \le i \le d,\ 1 \le j \le n-2\} \cup \{in-1 : 1 \le i \le d-1\}$;*

*(v) If $in+j \in S$ with $i \ge 1$ and $1 \le j \le n-1$, then $(i-1)n+j \in S$;*

*(vi) If $j \in S$ with $1 \le j \le n-2$, then $j+1 \notin S$.*

Let $\mathcal{B}_{d,n}^{-}$ be the set of nice subsets $I$ of $\mathcal{A}_{d,n}$ with $(d,n) \notin I$. Then by Theorem 3.4 the map $\psi_n$ defined in (3.1) gives a bijection between the sets $\mathcal{C}_{n,dn-1}$ and $\mathcal{B}_{d,n-1}^{-}$. Furthermore, by Lemma 2.1 we obtain

**Lemma 3.5.**

$$\sum_{\lambda \in \mathcal{C}_{n,dn-1}} |\lambda|^k = \sum_{I \in \mathcal{B}_{d,n-1}^{-}} \left( \sum_{(i,j)\in I} ((i-1)n+j) - \frac{|I|^2}{2} + \frac{|I|}{2} \right)^k. \tag{3.3}$$

**Example 3.6.** *Let $d=3$ and $n=3$. Then there are 6 of $(3,8)$-core partitions with distinct parts: $\emptyset, (1), (2), (3,1), (4,2), (5,3,1)$. The corresponding nice subsets of $\mathcal{A}_{3,2}$ are:*

$$\mathcal{B}_{3,2}^{-} = \{\ \emptyset,\ \{(1,1)\},\ \{(1,2)\},\ \{(1,1),(2,1)\},\ \{(1,2),(2,2)\}, \{(1,1),(2,1),(3,1)\}\ \}.$$

*Let $k=2$. Then both sides of (3.3) equals 138.*

## 4. Polynomiality of moments of sizes for core partitions

In this section, we will prove the main results.

For each nice subset $I$ of $\mathcal{A}_{d,n}$, let $|I|$ be the cardinality of $I$. Define

$$\sigma_m(I) := \sum_{(i,j)\in I} ((i-1)m + j)$$

and

$$G^+_{d,m,a,b}(n) := \sum_{I\in\mathcal{B}^+_{d,n}} \sigma_m(I)^a\, |I|^b$$

for $d, m, a, b, n \geq 0$.

To compute the $k$-th power sum of sizes of partitions in $\mathcal{C}_{n,dn+1}$, by Lemma 3.2 we just need to compute the functions $G^+_{d,n,a,b}(n-1)$ with four variables $d, n, a, b$. The basic idea is induction on $n$. To do this, we need one more parameter $m$ here. That is, we study a more general family of functions $G^+_{d,m,a,b}(n)$ with five variables $d, m, a, b, n$. First we derive formulas for generating functions of $G^+_{d,m,a,b}(n)$.

**Theorem 4.1.** *Assume that $a$ and $b$ are two nonnegative integers. For each $1 \leq i \leq 2a+b+1$, there exists some polynomial $P_{a,b,i}(d,m,q)$ of $d$, $m$ and $q$ with $\deg_m(P_{a,b,i}) \leq 2a+b+1-i$, such that the generating function of $G^+_{d,m,a,b}(n)$ equals:*

$$\Psi_{d,m,a,b} := \sum_{n\geq 0} G^+_{d,m,a,b}(n)\, q^n = \sum_{i=1}^{2a+b+1} \frac{P_{a,b,i}(d,m,q)}{(1-q-dq^2)^i}. \tag{4.1}$$

*Proof.* We will prove this result by induction on $a+b$. When $a+b=0$, we have $a=b=0$. For $n \geq 2$,

$$\begin{aligned} G^+_{d,m,0,0}(n) &= \sum_{I\in\mathcal{B}^+_{d,n}} 1 = |\mathcal{B}^+_{d,n}| \\ &= \sum_{I\in\mathcal{B}^+_{d,n-1}} 1 + \sum_{I\in\mathcal{B}^+_{d,n}\setminus\mathcal{B}^+_{d,n-1}} 1 \\ &= \sum_{I\in\mathcal{B}^+_{d,n-1}} 1 + \sum_{(1,n)\in I\in\mathcal{B}^+_{d,n}} 1. \end{aligned}$$

When $(1,n) \in I \in \mathcal{B}^+_{d,n}$, we know $(1,n-1) \notin I$ and therefore $I \cap \mathcal{A}_{d,n-1} \in \mathcal{B}^+_{d,n-2}$. Thus for each $1 \leq i \leq d$,

$$|\{I \in \mathcal{B}^+_{d,n} :\ (i,n)\in I,\ (i+1,n)\notin I\}| = |\mathcal{B}^+_{d,n-2}|.$$

Therefore

$$G^+_{d,m,0,0}(n) = |\mathcal{B}^+_{d,n-1}| + d\,|\mathcal{B}^+_{d,n-2}| = G^+_{d,m,0,0}(n-1) + d\,G^+_{d,m,0,0}(n-2) \tag{4.2}$$

for $n \geq 2$. By definition it is easy to derive:

$$G^+_{d,m,0,0}(0) = 1, \quad G^+_{d,m,0,0}(1) = d+1. \tag{4.3}$$

Therefore

$$\Psi_{d,m,0,0} - (d+1)q - 1 = q\,(\Psi_{d,m,0,0} - 1) + dq^2\Psi_{d,m,0,0}$$

and thus

$$\Psi_{d,m,0,0} = \frac{dq+1}{1-q-dq^2}. \tag{4.4}$$

Then the theorem is true for $a+b=0$. Next assume that $a+b>0$ and (4.1) holds for all pairs $(a',b')$ with $a'+b'<a+b$. For $n\geq 2$, considering the largest integer $i$ such that $(i,n)\in I$ (or $(1,n)\notin I$), we obtain

$$
\begin{aligned}
G^{+}_{d,m,a,b}(n) &= \sum_{I\in\mathcal{B}^{+}_{d,n}} \sigma_m(I)^a|I|^b = \sum_{(1,n)\notin I\in\mathcal{B}^{+}_{d,n}} \sigma_m(I)^a|I|^b + \sum_{(1,n)\in I\in\mathcal{B}^{+}_{d,n}} \sigma_m(I)^a|I|^b \\
&= \sum_{I\in\mathcal{B}^{+}_{d,n-1}} \sigma_m(I)^a|I|^b + \sum_{i=1}^{d} \sum_{\substack{(i,n)\in I\in\mathcal{B}^{+}_{d,n}\\ (i+1,n)\notin I}} \sigma_m(I)^a|I|^b \\
&= \sum_{I\in\mathcal{B}^{+}_{d,n-1}} \sigma_m(I)^a|I|^b + \sum_{I\in\mathcal{B}^{+}_{d,n-2}} \sum_{i=1}^{d} \left(\sigma_m(I)+\binom{i}{2}m+in\right)^a (|I|+i)^b \\
&= G^{+}_{d,m,a,b}(n-1) + d\,G^{+}_{d,m,a,b}(n-2) + \sum_{\substack{a'+b'<a+b\\ 0\leq a'\leq a\\ 0\leq b'\leq b}} A^{a,b}_{a',b'}(d,m,n)\,G^{+}_{d,m,a',b'}(n-2) \qquad (4.5)
\end{aligned}
$$

where

$$
A^{a,b}_{a',b'}(d,m,n) = \binom{a}{a'}\binom{b}{b'} \sum_{i=1}^{d} \left(\binom{i}{2}m+in\right)^{a-a'} i^{b-b'}
$$

are polynomials of $d,m,n$ such that

$$
\deg_m A^{a,b}_{a',b'} + \deg_n A^{a,b}_{a',b'} \leq a-a'.
$$

It is obvious that, when $a+b>0$,

$$
G^{+}_{d,m,a,b}(0)=0, \quad G^{+}_{d,m,a,b}(1) = \sum_{i=1}^{d} \left(\binom{i}{2}m+i\right)^a i^b = \sum_{k=0}^{a} B^{a,b}_k(d)\ m^k, \qquad (4.6)
$$

where

$$
B^{a,b}_k(d) = \binom{a}{k} \sum_{i=1}^{d} \binom{i}{2}^k i^{a-k+b}.
$$

Considering the generating function, by (4.5) we have

$$
\begin{aligned}
&\Psi_{d,m,a,b} - q\,G^{+}_{d,m,a,b}(1) \\
&\quad = q\Psi_{d,m,a,b} + dq^2\Psi_{d,m,a,b} + \sum_{\substack{a'+b'<a+b\\ 0\leq a'\leq a\\ 0\leq b'\leq b}} \sum_{n\geq 2} A^{a,b}_{a',b'}(d,m,n) G^{+}_{d,m,a',b'}(n-2)\ q^n \\
&\quad = q\Psi_{d,m,a,b} + dq^2\Psi_{d,m,a,b} + q^2 \sum_{\substack{a'+b'<a+b\\ 0\leq a'\leq a\\ 0\leq b'\leq b}} \sum_{n\geq 0} A^{a,b}_{a',b'}(d,m,n+2) G^{+}_{d,m,a',b'}(n)\ q^n. \qquad (4.7)
\end{aligned}
$$

When $a'+b'<a+b$, by induction hypothesis and

$$
\sum_{n\geq 0} na_n\, q^n = q\,(\sum_{n\geq 0} a_n q^n)'
$$

we obtain

$$\sum_{n\geq 0} n^j G^+_{d,m,a',b'}(n)\ q^n = \sum_{i=1}^{2a'+b'+1+j} \frac{C_{j,a',b',i}(d,m,q)}{(1-q-dq^2)^i}$$

for each $j \geq 0$, where $C_{j,a',b',i}(d,m,q)$ are some polynomials of $d$, $m$ and $q$ with

$$\deg_m(C_{j,a',b',i}(d,m,q)) \leq 2a'+b'+1+j-i.$$

Therefore for $a'+b' < a+b$, we have

$$\sum_{n\geq 0} A^{a,b}_{a',b'}(d,m,n+2)\, G^+_{d,m,a',b'}(n)\ q^n = \sum_{i=1}^{2a'+b'+1+a-a'} \frac{D_{a',b',i}(d,m,q)}{(1-q-dq^2)^i}, \tag{4.8}$$

where $D_{a',b',i}(d,m,q)$ are some polynomials of $d$, $m$ and $q$ with

$$\deg_m(D_{a',b',i}(d,m,q)) \leq 2a'+b'+1+a-a'-i = a+a'+b'+1-i \leq 2a+b-i.$$

Then by (4.6), (4.7) and (4.8) we obtain

$$\Psi_{d,m,a,b} = \sum_{n\geq 0} G^+_{d,m,a,b}(n) q^n = \sum_{i=1}^{2a+b+1} \frac{P_{a,b,i}(d,m,q)}{(1-q-dq^2)^i},$$

where $P_{a,b,i}(d,m,q)$ are some polynomials of $d$, $m$ and $q$ with

$$\deg_m(P_{a,b,i}(d,m,q)) \leq 2a+b+1-i. \qquad \square$$

By the above theorem, to derive the explicit expression for $G^+_{d,m,a,b}(n)$, we need to study the expansion of $1/(1-q-dq^2)^k$. Let $x_d = (1+\sqrt{1+4d})/2$ and $y_d = (1-\sqrt{1+4d})/2$ be two roots of $x^2-x-d$. By the partial fraction decomposition, we obtain the following results.

**Lemma 4.2.** *Let $d$ and $k$ be given positive integers. Then*

$$\frac{1}{(1-q-dq^2)^k} = \sum_{i=1}^{k} \frac{\binom{2k-1-i}{k-1} d^{k-i}}{(1+4d)^{\frac{2k-i}{2}}} \sum_{n\geq 0} \binom{n+i-1}{i-1} \left(x_d^{n+i} + (-1)^i y_d^{n+i}\right) q^n. \tag{4.9}$$

*Proof.* For $a, b \geq 0$, let

$$F_{a,b} = \frac{1}{(1-x_d\, q)^a (1-y_d\, q)^b}.$$

It is easy to see that

$$F_{a+1,b+1} = \frac{x_d}{x_d - y_d} F_{a+1,b} + \frac{y_d}{y_d - x_d} F_{a,b+1}$$

for all $a, b \geq 0$. Therefore by induction we derive

$$F_{a,b} = \sum_{i=1}^{a} \frac{(-1)^{a-i}\binom{a+b-1-i}{b-1} x_d^b\, y_d^{a-i}}{(x_d-y_d)^{a+b-i}(1-x_d\, q)^i} + \sum_{j=1}^{b} \frac{(-1)^a \binom{a+b-1-j}{a-1} x_d^{b-j}\, y_d^a}{(x_d-y_d)^{a+b-j}(1-y_d\, q)^i}$$

for all $a, b \geq 1$. Let $a = b = k$. Then by $x_d\, y_d = -d$, $x_d - y_d = \sqrt{1+4d}$, and

$$\frac{1}{(1-zq)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} z^n q^n,$$

we derive (4.9). $\square$

**Lemma 4.3.** *Let $k$ be a positive integer. Then*

$$\frac{1}{(1-q-dq^2)^k} = \sum_{n\geq 0} c_n\, q^n$$

*where $c_n$ is of the form $A(n,d)M_d(n)+B(n,d)M_d(n+1)$, such that $A(n,d)$ and $B(n,d)$ are polynomials of $n$ with degrees at most $k-1$, whose coefficients are rational functions in $d$. In particular, we have (notice that $M_d(n+2) = M_d(n+1) + dM_d(n)$)*

$$\frac{1}{1-q-dq^2} = \sum_{n\geq 0} M_d(n)\, q^n; \tag{4.10}$$

$$\frac{1}{(1-q-dq^2)^2} = \sum_{n\geq 0} \frac{1}{4d+1}\big((n+1)\, M_d(n+2) + (n+3)\, d\, M_d(n)\big)\, q^n; \tag{4.11}$$

*and*

$$\frac{1}{(1-q-dq^2)^3} = \sum_{n\geq 0}\left(\Big(\frac{3d(n+1)}{(4d+1)^2} + \frac{1}{4d+1}\binom{n+2}{2}\Big)\cdot M_d(n+2) + \frac{3d^2(n+3)}{(4d+1)^2}\, M_d(n)\right) q^n. \tag{4.12}$$

*Proof.* By the recurrence relation (1.2), it is easy to see that

$$M_d(n) = \frac{1}{\sqrt{1+4d}}\,(x_d^{n+1} - y_d^{n+1}) \tag{4.13}$$

and

$$2M_d(n+1) - M_d(n) = x_d^{n+1} + y_d^{n+1}. \tag{4.14}$$

Therefore Lemma 4.2 implies

$$\frac{1}{(1-q-dq^2)^k} = \sum_{n\geq 0} c_n q^n$$

where $c_n$ is of the form $\sum_{i=0}^{k} A_i(n,d)M_d(n+i)$, such that each $A_i(n,d)$ is a polynomial of $n$ with degree at most $k-1$, whose coefficients are rational functions in $d$. But by (1.2), each $M_d(n+i)$ can be written as some linear combination of $M_d(n)$ and $M_d(n+1)$, whose coefficients are rational functions in $d$. Therefore we prove the main result of the lemma. In particular, let $k = 1, 2, 3$ in Lemma 4.2, we derive (4.10), (4.11) and (4.12). □

Notice that for each $i \in \mathbb{Z}$, $M_d(n+i)$ can be written as some linear combination of $M_d(n)$ and $M_d(n+1)$, whose coefficients are rational functions in $d$. The next result follows from Theorem 4.1 and Lemma 4.3 directly.

**Theorem 4.4.** *Let $a, b \geq 0$ be some given integers. Then $G^+_{d,m,a,b}(n)$ is of the form*

$$A(m,n,d)\, M_d(n) + B(m,n,d)\, M_d(n+1),$$

*where $A(m,n,d)$ and $B(m,n,d)$ are polynomials of $m$ and $n$ with degrees at most $2a+b$ (that is, $\deg_m + \deg_n \leq 2a+b$), whose coefficients are rational functions in $d$.*

Next we give some examples of explicit expressions for $G^+_{d,m,a,b}(n)$ when $a$ and $b$ are small.

**Example 4.5.** *Let $a = 1,\ b = 0$. We have*

$$G^+_{d,m,1,0}(n) = \sum_{I\in\mathcal{B}^+_{d,n}} \sigma_m(I) = \sum_{I\in\mathcal{B}^+_{d,n-1}} \sigma_m(I) + \sum_{I\in\mathcal{B}^+_{d,n-2}} \sum_{i=1}^{d} \left(\sigma_m(I) + \binom{i}{2}m + in\right)$$

$$= G^+_{d,m,1,0}(n-1) + d\,G^+_{d,m,1,0}(n-2) + \left(\binom{d+1}{3}m + \binom{d+1}{2}n\right) G^+_{d,m,0,0}(n-2).$$

*Also*

$$G^+_{d,m,1,0}(0) = 0,\ G^+_{d,m,1,0}(1) = \sum_{j=1}^{d}\sum_{i=1}^{j}(1+(i-1)m) = \binom{d+1}{3}m + \binom{d+1}{2}.$$

*Then by* (4.7) *we have*

$$\begin{aligned}\Psi_{d,m,1,0} &= \frac{\left(\binom{d+1}{3}m + \binom{d+1}{2}\right) q}{1-q-dq^2} + \frac{\left(\binom{d+1}{3}m + 2\binom{d+1}{2}\right) q^2(dq+1)}{(1-q-dq^2)^2} + \frac{\binom{d+1}{2}q^3(d^2q^2+2dq+d+1)}{(1-q-dq^2)^3}\\ &= \frac{\binom{d+1}{3}mq - \binom{d+1}{2}q}{(1-q-dq^2)^2} + \frac{\binom{d+1}{2}(2q-q^2)}{(1-q-dq^2)^3}.\end{aligned}$$

*Therefore by* (1.2) *and Lemma 4.3,*

$$\begin{aligned}G^+_{d,m,1,0}(n) &= \frac{1}{4d+1}\left(\binom{d+1}{3}m + \binom{d+1}{2}\cdot\frac{n+1}{2}\right) n\,M_d(n)\\ &+ \frac{d}{4d+1}\binom{d+1}{2}\left(\frac{2(d-1)m}{3} + n + 1\right)(n+1)\,M_d(n-1).\end{aligned} \tag{4.15}$$

**Example 4.6.** *Let* $a=0,\ b=1$. *We have*

$$\begin{aligned}G^+_{d,m,0,1}(n) :&= \sum_{I\in\mathcal{B}^+_{d,n}} |I| = \sum_{I\in\mathcal{B}^+_{d,n-1}} |I| + \sum_{I\in\mathcal{B}^+_{d,n-2}}\sum_{i=1}^{d}(|I|+i)\\ &= G^+_{d,m,0,1}(n-1) + d\,G^+_{d,m,0,1}(n-2) + \binom{d+1}{2}G^+_{d,m,0,0}(n-2).\end{aligned}$$

*Also*

$$G^+_{d,m,0,1}(0) = 0,\ G^+_{d,m,0,1}(1) = \sum_{i=1}^{d} i = \binom{d+1}{2}.$$

*Then the generating function satisfies*

$$\Psi_{d,m,0,1} - qG^+_{d,m,0,1}(1) = q\Psi_{d,m,0,1} + dq^2\Psi_{d,m,0,1} + \binom{d+1}{2}q^2\Psi_{d,m,0,0}.$$

*Therefore,*

$$\Psi_{d,m,0,1} = \frac{\binom{d+1}{2}\,q}{1-q-dq^2} + \frac{\binom{d+1}{2}\,q^2(dq+1)}{(1-q-dq^2)^2}.$$

*Finally,*

$$G^+_{d,m,0,1}(n) = \frac{1}{4d+1}\binom{d+1}{2}\Big(n\,M_d(n) + d(2n+2)\,M_d(n-1)\Big). \tag{4.16}$$

**Example 4.7.** *Let $a = 0,\ b = 2$. We have*

$$G^+_{d,m,0,2}(n) = \sum_{I\in\mathcal{B}^+_{d,n}} |I|^2 = \sum_{I\in\mathcal{B}^+_{d,n-1}} |I|^2 + \sum_{I\in\mathcal{B}^+_{d,n-2}} \sum_{i=1}^{d} (|I|+i)^2$$
$$= G^+_{d,m,0,2}(n-1) + d\,G^+_{d,m,0,2}(n-2) + 2\binom{d+1}{2} G^+_{d,m,0,1}(n-2) + \frac{1}{4}\binom{2d+2}{3} G^+_{d,m,0,0}(n-2).$$

*Also*

$$G^+_{d,m,0,2}(0) = 0,\ G^+_{d,m,0,2}(1) = \sum_{i=1}^{d} i^2 = \frac{1}{4}\binom{2d+2}{3}.$$

*Then the generating function satisfies*

$$\Psi_{d,m,0,2} - qG^+_{d,m,0,2}(1) = q\Psi_{d,m,0,2} + dq^2\Psi_{d,m,0,2} + 2\binom{d+1}{2}q^2\Psi_{d,m,0,1} + \frac{1}{4}\binom{2d+2}{3}q^2\Psi_{d,m,0,0}.$$

*Therefore,*

$$\Psi_{d,m,0,2} = \frac{\frac{1}{4}\binom{2d+2}{3}q}{1-q-dq^2} + \frac{2\binom{d+1}{2}^2 q^3 + \frac{1}{4}\binom{2d+2}{3}q^2(dq+1)}{(1-q-dq^2)^2} + \frac{2\binom{d+1}{2}^2 q^4(dq+1)}{(1-q-dq^2)^3}.$$

Finally,

$$G^+_{d,m,0,2}(n) = \left(\frac{1}{4}\binom{2d+2}{3}\frac{1}{4d+1} - \binom{d+1}{2}^2 \frac{6}{(4d+1)^2}\right)\cdot n\,M_d(n)$$
$$+ \frac{1}{4}\binom{2d+2}{3}\frac{2d}{4d+1}\cdot(n+1)\,M_d(n-1)$$
$$+ \binom{d+1}{2}^2 \frac{1}{(4d+1)^2}\,\left(n^2(4d+1)+3n-4d+2\right)\cdot M_d(n-1). \tag{4.17}$$

Next we show that, $G^+_{d,m,a,b}(n)$ is a polynomial of $d$ when other variables are fixed.

**Theorem 4.8.** *Let $m, n \geq 1$ and $a, b \geq 0$ be some given integers. Then $G^+_{d,m,a,b}(n)$ is a polynomial of $d$ with degree $2a+b+\lfloor\frac{n+1}{2}\rfloor$.*

*Proof.* The $a = b = 0$ case is guaranteed by (4.2) and (4.3). Therefore we can assume that $a+b \geq 1$. We will prove this result by induction on $n$. It is easy to see that

$$G^+_{d,m,a,b}(1) = \sum_{i=1}^{d}\left(\binom{i}{2}m+i\right)^a i^b,$$
$$G^+_{d,m,a,b}(2) = \sum_{i=1}^{d}\left(\binom{i}{2}m+i\right)^a i^b + \sum_{i=1}^{d}\left(\binom{i}{2}m+2i\right)^a i^b$$

are polynomials of $d$ with degrees $2a+b+1$, which shows that the theorem is true for $n = 1$ and 2.

When $n \geq 3$, we assume that this result is true for $n-1$ and $n-2$. Therefore $G^+_{d,m,a,b}(n-1)$ and $d\,G^+_{d,m,a,b}(n-2)$ are polynomials of $d$ with degrees $2a+b+\lfloor\frac{n}{2}\rfloor$ and $2a+b+\lfloor\frac{n-1}{2}\rfloor+1 = 2a+b+\lfloor\frac{n+1}{2}\rfloor$ respectively. Also, for $a' \leq a$, $b' \leq b$ with $a'+b' < a+b$, we have

$$\binom{a}{a'}\binom{b}{b'}\sum_{i=1}^{d}\left(\binom{i}{2}m+in\right)^{a-a'} i^{b-b'}\,G^+_{d,m,a',b'}(n-2)$$

is a polynomial of $d$ with degree

$$2(a-a')+(b-b')+1+2a'+b'+\lfloor\frac{n-2+1}{2}\rfloor = 2a+b+\lfloor\frac{n+1}{2}\rfloor.$$

Therefore by (4.5) we prove the theorem. □

For $a,b,m,n\geq 0$ and $d\geq 1$, let

$$G^-_{d,m,a,b}(n) := \sum_{I\in\mathcal{B}^-_{d,n}} \sigma_m(I)^a|I|^b.$$

Then

$$G^-_{d,m,0,0}(n) = N_d(n+1) = M_d(n)+(d-1)M_d(n-1).$$

When $a+b>0$, it is obvious that

$$G^-_{d,m,a,b}(0)=0,\quad G^-_{d,m,a,b}(1)=G^+_{d-1,m,a,b}(1). \tag{4.18}$$

For $n\geq 2$, we have

$$\begin{aligned} G^-_{d,m,a,b}(n) &= \sum_{I\in\mathcal{B}^+_{d,n-1}} \sigma_m(I)^a|I|^b + \sum_{I\in\mathcal{B}^+_{d,n-2}}\sum_{i=1}^{d-1}\left(\sigma_m(I)+\binom{i}{2}m+in\right)^a(|I|+i)^b \\ &= G^+_{d,m,a,b}(n-1)+(d-1)G^+_{d,m,a,b}(n-2) \\ &\quad + \sum_{a'+b'<a+b} B^{a,b}_{a',b'}(d,m,n)G^+_{d,m,a',b'}(n-2), \end{aligned} \tag{4.19}$$

where

$$B^{a,b}_{a',b'}(d,m,n) = \binom{a}{a'}\binom{b}{b'}\sum_{i=1}^{d-1}\left(\binom{i}{2}m+in\right)^{a-a'} i^{b-b'}.$$

Similarly as the $G^+_{d,m,a,b}(n)$ case, we obtain the following results for $G^-_{d,m,a,b}(n)$.

**Theorem 4.9.** *Let $a,b\geq 0$ be some given integers. Then $G^-_{d,m,a,b}(n)$ is of the form*

$$A(m,n,d)M_d(n)+B(m,n,d)M_d(n+1),$$

*where $A(m,n,d)$ and $B(m,n,d)$ are polynomials of $m$ and $n$ with degrees at most $2a+b$, whose coefficients are rational functions in $d$.*

**Theorem 4.10.** *Let $m,n\geq 1$ and $a,b\geq 0$ be some given integers. Then $G^-_{d,m,a,b}(n)$ is a polynomial of $d$ with degree $2a+b+\lfloor\frac{n+1}{2}\rfloor$.*

Now we are ready to prove the main theorems.

*Proofs of Theorems 1.5 and 1.6.* By Lemmas 3.2 and 3.5 we know

$$\sum_{\lambda\in\mathcal{C}_{n,dn+1}}|\lambda|^k \qquad\text{and}\qquad \sum_{\lambda\in\mathcal{C}_{n,dn-1}}|\lambda|^k$$

can be written as some linear combinations of $G^+_{d,n,a',b'}(n-1)$ and $G^-_{d,n,a',b'}(n-1)$ respectively, where $2a'+b'\leq 2k$. Notice that for each $i\in\mathbb{Z}$, $M_d(n+i)$ can be written as some linear combination of $M_d(n)$ and $M_d(n+1)$, whose coefficients are rational functions in $d$. Replace $n$ by $n-1$, and $m$ by $n$ in Theorems 4.4 and 4.9, we obtain that $G^+_{d,n,a',b'}(n-1)$ and $G^-_{d,n,a',b'}(n-1)$ are of the form

$$A(n,d)M_d(n)+B(n,d)M_d(n+1),$$

where $A(n,d)$ and $B(n,d)$ are polynomials of $n$ with degrees $2a'+b'\le 2k$, whose coefficients are rational functions in $d$. Therefore Theorem 1.5 is true. Also, Theorem 1.6 follows from Theorems 4.8 and 4.10. □

## 5. Explicit formulas for expectations of $X_{n,dn+1}$ and $X_{n,dn-1}$

In this section we give proofs of Theorems 1.10 and 1.13.

*Proof of Theorem 1.10.* Let $k=1$ in Lemma 3.2. We have

$$\sum_{\lambda\in\mathcal{C}_{n,dn+1}}|\lambda| = \sum_{I\in\mathcal{B}^{+}_{d,n-1}}\left(\sum_{(i,j)\in I}((i-1)n+j)-\frac{|I|^2}{2}+\frac{|I|}{2}\right)$$
$$= G^{+}_{d,n,1,0}(n-1)-\frac{1}{2}G^{+}_{d,n,0,2}(n-1)+\frac{1}{2}G^{+}_{d,n,0,1}(n-1).$$

Then by (4.15), (4.16) and (4.17) we derive

$$\begin{aligned}\sum_{\lambda\in\mathcal{C}_{n,dn+1}}|\lambda| = {}& M_d(n-1)\cdot\Big(\frac{-d(d+1)(d-1)(n-1)^2}{24(4d+1)}\\ &-\frac{d(d+1)(14d^2+21d+1)(n-1)}{24(4d+1)^2}-\frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2}\Big)\\ &+M_d(n)\cdot\Big(\frac{d(d+1)(5d+1)(n-1)^2}{24(4d+1)}\\ &+\frac{d(d+1)(32d^2+63d+7)(n-1)}{24(4d+1)^2}+\frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2}\Big),\end{aligned}$$

which implies Theorem 1.10. □

*Proof of Theorem 1.13.* Let $k=1$ in Lemma 3.5. We have

$$\sum_{\lambda\in\mathcal{C}_{n,dn-1}}|\lambda| = \sum_{I\in\mathcal{B}^{-}_{d,n-1}}\left(\sum_{(i,j)\in I}((i-1)n+j)-\frac{|I|^2}{2}+\frac{|I|}{2}\right)$$
$$= G^{-}_{d,n,1,0}(n-1)-\frac{1}{2}G^{-}_{d,n,0,2}(n-1)+\frac{1}{2}G^{-}_{d,n,0,1}(n-1).$$

But by the definitions of $G^{+}_{d,m,a,b}$ and $G^{-}_{d,m,a,b}$ we obtain

$$G^{-}_{d,n,1,0}(n-1)=G^{+}_{d,n,1,0}(n-1)-G^{+}_{d,n,1,0}(n-3)-M_d(n-2)\left(\binom{d}{2}n+d(n-1)\right),$$

$$\begin{aligned}G^{-}_{d,n,0,2}(n-1)&=G^{+}_{d,n,0,2}(n-1)-\sum_{I\in\mathcal{B}^{+}_{d,n-3}}(|I|+d)^2\\ &=G^{+}_{d,n,0,2}(n-1)-G^{+}_{d,n,0,2}(n-3)-2d\,G^{+}_{d,n,0,1}(n-3)-d^2M_d(n-2),\end{aligned}$$

and

$$\begin{aligned}G^{-}_{d,n,0,1}(n-1)&=G^{+}_{d,n,0,1}(n-1)-\sum_{I\in\mathcal{B}^{+}_{d,n-3}}(|I|+d)\\ &=G^{+}_{d,n,0,1}(n-1)-G^{+}_{d,n,0,1}(n-3)-dM_d(n-2).\end{aligned}$$

Then by (4.15), (4.16) and (4.17) we derive Theorem 1.13. □

## 6. Asymptotic formulas for moments of $X_{n,dn+1}$ and $X_{n,dn-1}$

In this section we study asymptotic behavior for moments of $X_{n,dn+1}$ and $X_{n,dn-1}$. First we give proofs of Theorems 1.7 and 1.8.

*Proof of Theorem 1.7.* By the recurrence relations (1.1) and (1.2) it is easy to derive

$$M_d(n) = \frac{1}{\sqrt{1+4d}}\left(\left(\frac{1+\sqrt{1+4d}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+4d}}{2}\right)^{n+1}\right) \tag{6.1}$$

and

$$N_d(n) = M_d(n) - M_d(n-2). \tag{6.2}$$

Then by Theorem 1.5 we have

$$\mathbb{E}[X_{n,dn+1}^k] = A(n,d) + B(n,d)\cdot \frac{\left(\frac{1+\sqrt{1+4d}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{1+4d}}{2}\right)^{n+2}}{\left(\frac{1+\sqrt{1+4d}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+4d}}{2}\right)^{n+1}}$$

where $A(n,d)$ and $B(n,d)$ are some polynomials of $n$ with degrees at most $2k$. Therefore (1.6) holds. Similarly (1.7) follows from Theorem 1.5, (6.1) and (6.2). □

*Proof of Theorem 1.8.* By the recurrence relations (1.1) and (1.2) it is easy to see that $M_d(n)$ and $N_d(n)$ are polynomials of of $d$ with degrees $\lfloor n/2 \rfloor$ when $n$ is given. Then Theorem 1.8 follows from Theorem 1.6. □

Next we consider the asymptotic formula for $G^+_{1,0,a,b}(n)$.

**Theorem 6.1.** *Suppose that $a$ and $b$ are two given nonnegative integers. Let $\alpha := (1+\sqrt{5})/2$. Then*

$$G^+_{1,0,a,b}(n) = 2^{-a}\ 5^{-(a+b+1)/2}\ n^{2a+b}\ \alpha^{n+2-a-b} + O(n^{2a+b-1}\alpha^n). \tag{6.3}$$

*Proof.* We will prove (6.3) by induction on $a+b$. When $a+b=0$, we have $a=b=0$. Let $d=1$ and $m=0$ in (4.4) we derive

$$G^+_{1,0,0,0}(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} = 5^{-1/2}\alpha^{n+2} + O(n^{-1}\alpha^n). \tag{6.4}$$

Next assume that $a+b>0$, and (6.3) holds for all pairs $(a',b')$ with $a'+b' < a+b$.

By (4.13) and Theorem 4.4, for any $a',b' \geq 0$, there exist some constants $C_{a',b'}$ and $D_{a',b'}$ such that

$$G^+_{1,0,a',b'}(n) = C_{a',b'}\ n^{2a'+b'}\ \alpha^n + D_{a',b'} n^{2a'+b'-1}\ \alpha^n + O(n^{2a'+b'-2}\alpha^n). \tag{6.5}$$

Let $d=1$ and $m=0$ in (4.5) we derive

$$\begin{aligned} G^+_{1,0,a,b}(n) &= G^+_{1,0,a,b}(n-1) + G^+_{1,0,a,b}(n-2) + a\,n\,G^+_{1,0,a-1,b}(n-2) + b\,G^+_{1,0,a,b-1}(n-2) \\ &\quad + \sum_{a'+b'\leq a+b-2} \binom{a}{a'}\binom{b}{b'} n^{a-a'} G^+_{1,0,a',b'}(n-2), \end{aligned} \tag{6.6}$$

where $G^+_{d,m,a',b'}(n) := 0$ if $a'<0$ or $b'<0$. But by (6.5), when $a'+b' \leq a+b-2$,

$$n^{a-a'} G^+_{1,0,a',b'}(n-2) = O(n^{2a+b-2}\alpha^n).$$

Notice that $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$. Also by (6.5), we have

$$G^{+}_{1,0,a,b}(n) - G^{+}_{1,0,a,b}(n-1) - G^{+}_{1,0,a,b}(n-2) = (\alpha+2)\,(2a+b)\,C_{a,b}\,n^{2a+b-1}\alpha^{n-2} + O(n^{2a+b-2}\alpha^n)$$

and

$$a\,n\,G^{+}_{1,0,a-1,b}(n-2) + b\,G^{+}_{1,0,a,b-1}(n-2) = (a\,C_{a-1,b} + b\,C_{a,b-1})\,n^{2a+b-1}\alpha^{n-2} + O(n^{2a+b-2}\alpha^n),$$

where $C_{a',b'} := 0$ if $a' < 0$ or $b' < 0$.

Therefore by (6.6), we have

$$\Big((\alpha+2)\,(2a+b)C_{a,b} - (aC_{a-1,b} + bC_{a,b-1})\Big)\,n^{2a+b-1}\,\alpha^{n-2} = O(n^{2a+b-2}\,\alpha^n),$$

which means that

$$(\alpha+2)\,(2a+b)C_{a,b} - (aC_{a-1,b} + bC_{a,b-1}) = 0. \tag{6.7}$$

By induction hypothesis we have

$$C_{a-1,\,b} = 2^{-a+1}\;5^{-(a+b)/2}\;\alpha^{3-a-b} \qquad \text{if } a \geq 1;$$

and

$$C_{a,\,b-1} = 2^{-a}\;5^{-(a+b)/2}\;\alpha^{3-a-b} \qquad \text{if } b \geq 1.$$

Notice that $\sqrt{5}\alpha = \alpha + 2$. Then by (6.7) we obtain

$$C_{a,b} = 2^{-a}\;5^{-(a+b+1)/2}\;\alpha^{2-a-b}.$$

Therefore (6.3) holds. □

Now we are ready to prove Theorem 1.9.

*Proof of Theorem 1.9.* By Lemma 3.2 and Theorem 6.1 we have

$$\begin{aligned}
\sum_{\lambda\in\mathcal{C}_{n,n+1}} |\lambda|^k &= \sum_{a=0}^{k} \binom{k}{a} \left(-\frac{1}{2}\right)^{k-a} G^{+}_{1,0,a,2(k-a)}(n-1) + O(n^{2k-1}\alpha^n) \\
&= \left(\sum_{a=0}^{k} \binom{k}{a} (-1)^{k-a}\,2^{-k}\;5^{-(2k-a+1)/2}\;\alpha^{2-2k+a}\right)(n-1)^{2k}\,\alpha^{n-1} + O(n^{2k-1}\alpha^n) \\
&= 2^{-k}\;5^{-(2k+1)/2}\,\alpha^{2-2k}\cdot(\sqrt{5}\alpha-1)^k\cdot n^{2k}\,\alpha^{n-1} + O(n^{2k-1}\alpha^n).
\end{aligned}$$

Notice that $\sqrt{5}\alpha - 1 = \alpha^2$. Then the above formula becomes

$$\sum_{\lambda\in\mathcal{C}_{n,n+1}} |\lambda|^k = 2^{-k}\;5^{-(2k+1)/2}\,\alpha\cdot n^{2k}\,\alpha^n + O(n^{2k-1}\alpha^n).$$

Therefore

$$\begin{aligned}
\mathbb{E}(X^k_{n,n+1}) &= \frac{1}{M_1(n)} \sum_{\lambda\in\mathcal{C}_{n,n+1}} |\lambda|^k \\
&= \frac{\sqrt{5}}{\alpha^{n+1} - (1-\alpha)^{n+1}} \cdot \Big(2^{-k}\;5^{-(2k+1)/2}\,\alpha\cdot n^{2k}\,\alpha^n + O(n^{2k-1}\alpha^n)\Big) \\
&= \left(\frac{1}{10}\right)^k n^{2k} + O(n^{2k-1}).
\end{aligned}$$

□

## 7. Further Directions

We derive several polynomiality results and asymptotic formulas for moments of sizes of random $(n, dn\pm1)$-core partitions with distinct parts, which prove several conjectures of Zaleski [30]. In the past few years, the numbers, the largest sizes and the average sizes of $(n, n+1)$, $(2n+1, 2n+3)$-core partitions with distinct parts were also well studied by many mathematicians (see [5, 13, 17, 20, 23, 26, 28, 29]). But for general $(s, t)$-core partitions with distinct parts, even for the $(n, n+3)$-core case, we know very little. We hope that the methods used and results obtained in this paper provide some clues for studying the general $(s, t)$-core case.

Also, Zaleski [30, Conjecture 3.4] conjectured that the distribution of $(n, dn-1)$-core partitions with distinct parts is asymptotically normal as $n$ tends to infinity when $d$ is given. At this moment, we are unable to prove this asymptotic distribution conjecture. By the idea from Zeilberger [31], to try to prove this conjecture, we need to have a better understanding of the leading terms in the asymptotic formulas of $\mathbb{E}[X^k_{n,dn+1}]$ and $\mathbb{E}[X^k_{n,dn-1}]$, which means that we should study the coefficients of the generating functions in (4.1). It would be interesting to find a proof of this distribution conjecture and furthermore study the distribution of general $(s, t)$-core partitions with distinct parts.

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