Gap between the largest and smallest parts of partitions and Berkovich and Uncu's conjectures

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Abstract. We prove three main conjectures of Berkovich and Uncu (Ann. Comb. 23 (2019) 263–284) on the inequalities between the numbers of partitions of n with bounded gap between largest and smallest parts for sufficiently large n. Actually our theorems are stronger than their original conjectures. The analytic version of our results shows that the coefficients of some partition q-series are eventually positive.

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1 Introduction

Let n be a positive integer, a *partition* of n is a nonincreasing finite sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ whose sum is n. Each λ_i is called a *part* of the partition. In [4], Berkovich and Uncu proved various inequalities between the numbers of partitions with the bound on the largest part and some restrictions on occurrences of parts, and also make several conjectures. To be specific, we introduce the following definitions:

1. Let $\mathcal{C}_{L,s,2}(n)$ (resp. $c_{L,s,2}(n)$) be the set (resp. number) of partitions of n with parts in the domain $\{s+1, \ldots, s+L\}$.
2. Let $\mathcal{F}_{L,s,k}(n)$ (resp. $f_{L,s,k}(n)$) denote the set (resp. number) of partitions of n with the smallest part s, the largest part at most $L+s$, and no part equal to k.

In this paper, motivated by the open problems Conjecture 3.2, Conjecture 3.3 and Conjecture 7.1 in [4], we shall prove the following two main theorems.

Theorem 1.1. *For integer $s \geq 1$, $L \geq 3$ and $s+L \geq k \geq \max\{s+1, L\}$, there exists an integer M which only depends on s such that for $n \geq M$,*

$$f_{L,s,k}(n) \geq c_{L,s,2}(n). \tag{1.1}$$

Remark 1.2. *Berkovich and Uncu [4, Theorems 1.1 and 3.1] proved Theorem 1.1 for $s = 1$ (resp. $s = 2$), $k = L$ (resp. $k = L+1$) with $M = 1$ (resp. $M = 10$). They also conjectured the cases $k = s + L - 1$ and $k = L$ of Theorem 1.1 [4, Conjectures 3.2 and 3.3].*

By the elementary theory of partitions [1, Chapters 1–3] it is not difficult to see that the generating functions of $c_{L,s,2}(n)$'s and $f_{L,s,1}(n)$'s read as follows:

$$\sum_{n=0}^{\infty} c_{L,s,2}(n)q^n = \frac{1}{(q^{s+1};q)_L}, \tag{1.2}$$

$$\sum_{n=1}^{\infty} f_{L,s,k}(n)q^n = \frac{q^s(1-q^k)}{(q^s;q)_{L+1}}. \tag{1.3}$$

Here, we use the standard q-notation [1]:

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}).$$

Recall that a series $\sum_{n\geq 0} a_n q^n \in \mathbb{R}[[q]]$ is called *eventually positive* if there exists an integer $M \geq 0$ such that $a_n > 0$ for all $n > M$. For instance, Theorem 1.1 and (1.2) and (1.3) imply that the q-series

$$\frac{q^s(1-q^k)-(1-q^s)}{(q^s;q)_{L+1}}$$

is eventually positive. In general, we derive the following theorem.

Theorem 1.3. *For integers $L \geq 3$, $s \geq 1$, $r \geq 0$ and $k_1 > k_2 \geq 1$, the series*

$$H^*_{L,s,r,k_1,k_2}(q) := \frac{q^r(1-q^{k_1})-(1-q^{k_2})}{(q^s;q)_{L+1}} \tag{1.4}$$

is eventually positive.

Remark 1.4. *Set $r = s$, $k_1 = k$ and $k_2 = s$, we confirm the conjecture raised by Berkovich and Uncu [4, Conjecture 7.1].*

Let $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$ be a set of m positive integers. Denote by $p_{\mathcal{A}}(n)$ the number of nonnegative integer solutions of the diophantine equation $a_1x_1 + \cdots + a_mx_m = n$, i.e.,

$$\sum_{n=0}^{\infty} p_{\mathcal{A}}(n)q^n = \frac{1}{(1-q^{a_1})\dots(1-q^{a_m})}. \tag{1.5}$$

It should be noted that $p_{\mathcal{A}}(n)$ is closely related to the *Frobeinus coin problem*, see [3,5] or https://en.wikipedia.org/wiki/Coin_problem for more details. We shall need the following result, see [2] or [6, Theorem 3.15.2] for an elementary proof.

Theorem 1.5 (Frobeinus-Schur). *If* $\gcd(a_1, \ldots, a_m) = 1$*, then*

$$p_{\mathcal{A}}(n) \sim \frac{n^{m-1}}{(m-1)!a_1a_2\cdots a_m}. \tag{1.6}$$

This paper is organized as follows. In Section 2, we first give two weak forms of Theorem 1.1, we then prove Theorem 1.1 with the aid of these two weak forms. In Section 3, we give a proof of Theorem 1.3.

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. To this end, we first show the following two theorems, namely Theorems 2.1 and 2.2, which can be view as weak forms of Theorem1.1. We then prove Theorem 1.1 with the aid of Theorems 2.1 and 2.2.

Theorem 2.1. *Given integer* s *and* $L \geq 3$*, there exists* $M_{L,s}$ *depending on* L *and* s *such that for any* $\max\{s+1, L\} \leq k \leq s+L$ *and* $n \geq M_{L,s}$*,*

$$f_{L,s,k}(n) \geq c_{L,s,2}(n). \tag{2.1}$$

Proof. By definition, we see that $f_{L,s,k}(n)$ is the number of nonnegative integer solutions of the equation $sx_s + (s+1)x_{s+1} + \cdots + (s+L)x_{s+L} = n$, where $x_s \geq 1$ and $x_k = 0$. Let $A := \{s, s+1, \ldots, s+L\}$, from the definition of $p_A(n)$, we deduce that

$$f_{L,s,k}(n) = p_{A\setminus\{k\}}(n-s). \tag{2.2}$$

Similarly, by the definition of $c_{L,s,2}(n)$,

$$c_{L,s,2}(n) = p_{A\setminus\{s\}}(n). \tag{2.3}$$

As $L \geq 3$ both $A \setminus \{k\}$ and $A \setminus \{s\}$ contain two consecutive integers, thus $\gcd(A \setminus \{k\}) = \gcd(A \setminus \{s\}) = 1$. Hence by Theorem 1.5,

$$f_{L,s,k}(n) \sim \frac{k(n-s)^L}{L!s(s+1)\cdots(s+L)}$$

and

$$c_{L,s,2}(n) \sim \frac{s(n-s)^L}{L!s(s+1)\cdots(s+L)}.$$

Therefore,

$$f_{L,s,k}(n) - c_{L,s,2}(n) \sim \frac{(k-s)(n-s)^L}{L!s(s+1)\cdots(s+L)}. \tag{2.4}$$

From (2.4), we see that there exists $M_{L,s,k}$ such that for $n \geq M_{L,s,k}$, (2.1) holds. When $L \geq s+1$, set

$$M_{L,s} := \max\{M_{L,s,L}, M_{L,s,L+1}, \ldots M_{L,s,s+L}\};$$

and when $L \leq s$, set

$$M_{L,s} := \max\{M_{L,s,s+1}, M_{L,s,s+2}, \dots M_{L,s,s+L}\}.$$

Clearly, for $n \geq M_{L,s}$, (2.1) valid and $M_{L,s}$ only depends on s and L. This completes the proof. ∎

We next give another weak form of Theorem 1.1.

Theorem 2.2. *Let L, s and k be positive integers such that $L \geq 2s^3 + 5s^2 + 1$ and $L \leq k \leq s + L$. Then, for any $n \geq 2s^5 + 8s^4 + s^3 - 14s^2 + 3s + 1$, we have*

$$f_{L,s,k}(n) \geq c_{L,s,2}(n).$$

To prove Theorem 2.2, we shall build an injection $\phi : C_{L,s,2}(n) \to F_{L,s,k}(n)$. More specifically, we shall divide ϕ into five injections $\phi_i : C^i_{L,s,2}(n) \to F^i_{L,s,k}(n)$ for $1 \leq i \leq 5$, where $\{C^1_{L,s,2}(n), \dots, C^5_{L,s,2}(n)\}$ is a set partition of $C_{L,s,2}(n)$, and $(F^1_{L,s,k}(n), \dots, F^5_{L,s,k}(n))$ is a sequence of five disjoint subsets of $F_{L,s,k}(n)$.

We denote each partition $\alpha \in C_{L,s,2}(n)$ by $\alpha = ((s+1)^{f_{s+1}} \dots (s+L)^{f_{s+L}})$, where f_i is the number of occurrences of i in α. The five subsets $C^i_{L,s,2}(n)$ are defined as follows.

(1) $C^1_{L,s,2}(n)$ is the set of partitions in $C_{L,s,2}(n)$ such that $f_k = 0$ and there exists $a \geq 2$ such that $f_{as} \geq 1$.

(2) $C^2_{L,s,2}(n)$ is the set of partitions in $C_{L,s,2}(n)$ such that $f_k = 0$ and $f_{as} = 0$ for all $a \geq 2$. Moreover, there exists an integer j such that $2s^2 + 5s - 1 \geq j \geq s + 1$ and $f_j \geq s$.

(3) $C^3_{L,s,2}(n)$ is the set of partitions in $C_{L,s,2}(n)$ such that $f_k = 0$ and $f_{as} = 0$ for all $a \geq 2$. Moreover, for any j such that $2s^2 + 5s - 1 \geq j \geq s + 1$ we have $f_j \leq s - 1$.

(4) $C^4_{L,s,2}(n)$ is the set of partitions in $C_{L,s,2}(n)$ such that $f_k \geq 2$.

(5) $C^5_{L,s,2}(n)$ is the set of partitions in $C_{L,s,2}(n)$ such that $f_k = 1$.

Similarly, we denote each partition $\beta \in F_{L,s,k}(n)$ by $\beta = (s^{g_s}(s+1)^{g_{s+1}} \dots (s+L)^{g_{s+L}})$, where g_i is the number of occurrences of i in β. From the definition of $F_{L,s,k}(n)$, we see that $g_s \geq 1$ and $g_k = 0$. Writing $k = rs + t$ with $0 \leq t \leq s - 1$, we define the five subsets $F^i_{L,s,2}(n)$ as follows.

(1) $F^1_{L,s,k}(n)$ is the set of partitions in $F_{L,s,k}(n)$ such that $r + 1 \geq g_s \geq 2$ and for any $2 \leq i < g_s$, $g_{is} = 0$.

(2) $F^2_{L,s,k}(n)$ is the set of partitions in $F_{L,s,k}(n)$ such that $g_s = 1$ and there exists $i \geq 2$ such that $g_{is} = 1$. Moreover, for any $j \neq 1, i$, we have $g_{js} = 0$.

(3) $F^3_{L,s,k}(n)$ is the set of partitions in $F_{L,s,k}(n)$ such that $g_s = 1$ and $g_{2s} + g_{3s} \geq 2$.

(4) $F^4_{L,s,k}(n)$ is the set of partitions in $F_{L,s,k}(n)$ such that $g_s \geq 2r - 4$.

(5) $F^5_{L,s,k}(n)$ is the set of partitions in $F_{L,s,k}(n)$ such that $g_s = r - 4$ and $g_{2s} \geq 1$.

Since $k \geq L \geq 2s^3 + 5s^2 + 1$, we derive that $r \geq 2s^2 + 5s \geq 7$. Therefore, $2r - 4 > r + 1$, which implies that $F^1_{L,s,k}(n) \cap F^4_{L,s,k}(n) = \emptyset$; also $r - 4 \geq 3$, which implies that $F^1_{L,s,k}(n) \cap F^5_{L,s,k}(n) = \emptyset$, $F^2_{L,s,k}(n) \cap F^5_{L,s,k}(n) = \emptyset$ and $F^3_{L,s,k}(n) \cap F^5_{L,s,k}(n) = \emptyset$.

Now we proceed to construct the five injections explicitly.

Lemma 2.3. *There is an explicit injection $\phi_1 : C^1_{L,s,2}(n) \to F^1_{L,s,k}(n)$.*

Proof. Let $\alpha = ((s+1)^{f_{s+1}} \dots (s+L)^{f_{s+L}}) \in C^1_{L,s,2}(n)$ with $f_k = 0$. Let $a \geq 2$ be the smallest integer such that $f_{as} \geq 1$. We define

$$\phi_1(\alpha) := (s^{g_s} \dots (s+L)^{g_{s+L}}) = (s^a \dots (as)^{f_{as}-1} \dots (s+L)^{f_{s+L}}). \tag{2.5}$$

Clearly, $|\phi_1(\alpha)| = |\alpha| = n$, $g_k = f_k = 0$ and $g_s = a \geq 2$. Moreover from $L \leq k \leq s + L$ and $k = rs + t$, we deduce that $as \leq s + L \leq s + k = (r+1)s + t$. Thus $a \leq r + 1$. Hence $r + 1 \geq a = g_s \geq 2$. From the choice of a, we see that for any $2 \leq i < a$, we have $g_{is} = f_{is} = 0$. From the above analysis, we derive that $\phi_1(\alpha) \in F^1_{L,s,k}(n)$.

It remains to show that ϕ_1 is an injection. Let

$$I^1_{L,s,k}(n) = \{\phi_1(\alpha) \colon \alpha \in C^1_{L,s,2}(n)\}$$

be the image set of ϕ_1, which has been shown to be a subset of $F^1_{L,s,k}(n)$. We wish to construct a map $\psi_1 \colon I^1_{L,s,k}(n) \to C^1_{L,s,2}(n)$ such that for any $\alpha \in C^1_{L,s,2}(n)$,

$$\psi_1(\phi_1(\alpha)) = \alpha.$$

Let $\beta = (s^{g_s} \dots (s+L)^{g_{s+L}}) \in I^1_{L,s,k}(n)$, that is, there exists $\alpha \in C^1_{L,s,2}(n)$ such that $\phi_1(\alpha) = \beta$. From the construction (2.5), we see that $sg_s \leq s + L$ and $sg_s \neq k$. Define

$$\psi_1(\beta) = ((s+1)^{g_{s+1}} \dots (sg_s)^{g_{sg_s}+1} \dots (s+L)^{g_{s+L}}).$$

It is easy to check that $\psi_1(\beta) \in C^1_{L,s,2}(n)$ and $\psi_1(\phi_1(\alpha)) = \alpha$. This completes the proof. ∎

Example 2.4. *For $s = 3$, $L = 110$, $k = 112$ and $n = 1205$, let*

$$\alpha := (9^7, 15^3, 16^2, 20^9, 30^8, 40^2, 80, 97^5).$$

It is trivial to check that $\alpha \in C^1_{110,3,2}(1205)$. Applying ϕ_1 to α, we see that $a = 3$. Hence

$$\phi_1(\alpha) = (3^3, 9^6, 15^3, 16^2, 20^9, 30^8, 40^2, 80, 97^5),$$

which is in $F^1_{110,3,112}(1205)$. Moreover, applying ψ_1 to $\phi_1(\alpha)$, we recover α.

Lemma 2.5. *There is an explicit injection $\phi_2 : C^2_{L,s,2}(n) \to F^2_{L,s,k}(n)$.*

Proof. For $\alpha = ((s+1)^{f_{s+1}} \dots (s+L)^{f_{s+L}}) \in C^2_{L,s,2}(n)$, by definition we see $f_k = 0$, for any $a \geq 2$ we have $f_{as} = 0$. Moreover, there exists $s+1 \leq j \leq 2s^2+5s-1$ such that $f_j \geq s$. We choose such j to be minimum, that is, $j = \min\{i\colon f_i \geq s\}$. Define

$$\phi_2(\alpha) = (s^{g_s} \dots (s+L)^{g_{s+L}}) = (s^1 \dots (j)^{f_j - s} \dots ((j-1)s)^1 \dots (s+L)^{f_{s+L}}). \tag{2.6}$$

From $k \geq L \geq 2s^3+5s^2+1$ and $j \leq 2s^2+5s-1$, we deduce that $k > s(2s^2+5s) > (j-1)s$. Thus $f_k = g_k = 0$. Moreover, it is clear to see that $g_s = g_{(j-1)s} = 1$ and for any $i \neq 1, j-1$, $g_{is} = f_{is} = 0$. Furthermore, $|\phi_2(\alpha)| = |\alpha| - js + s + (j-1)s = n$. Hence we derive that $\phi_2(\alpha) \in F^2_{L,s,k}(n)$.

It remains to show that ϕ_2 is an injection. Let

$$I^2_{L,s,k}(n) = \{\phi_2(\alpha)\colon \alpha \in C^2_{L,s,2}(n)\}$$

be the image set of ϕ_2, which has been shown to be a subset of $F^2_{L,s,k}(n)$. We wish to construct a map $\psi_2\colon I^2_{L,s,k}(n) \to C^2_{L,s,2}(n)$ such that for any $\alpha \in C^2_{L,s,2}(n)$,

$$\psi_2(\phi_2(\alpha)) = \alpha.$$

Let $\beta = (s^{g_s} \dots (s+L)^{g_{s+L}}) \in I^2_{L,s,k}(n)$, that is, there exists $\alpha \in C^2_{L,s,2}(n)$ such that $\phi_2(\alpha) = \beta$. From the definition of $F^2_{L,s,k}(n)$, we see that there exists a unique $i \geq 2$ such that $g_{is} = 1$. By (2.6), we have $k \geq L > 2s^2+5s-1 \geq i+1 \geq s+1$. Moreover, since j is not a multiple of s, we see that $i+1$ is not a multiple of s. Hence we may define

$$\psi_2(\beta) = ((s+1)^{g_{s+1}} \dots (i+1)^{g_{i+1}+s} \dots (is)^0 \dots (s+L)^{g_{s+L}}).$$

It is easy to check that $\psi_2(\beta) \in C^2_{L,s,2}$ and $\psi_2(\phi_2(\alpha)) = \alpha$. This completes the proof. ∎

Example 2.6. *Let $s = 3$, $L = 103$, $k = 103$ and $n = 1286$. Let*

$$\alpha = (10^1, 11^3, 20^7, 28^2, 31^7, 46^9, 52^3, 65^4)$$

which is in $C^2_{103,3,2}(1286)$. It is clear that $j = 11$. Applying ϕ_2 to α, we see that

$$\phi_2(\alpha) = (3^1, 10^1, 20^7, 28^2, 30^1, 31^7, 46^9, 52^3, 65^4)$$

which is in $F^2_{103,3,103}(1286)$. Applying ψ_2 to $\phi_2(\alpha)$, we have $i = 10$ and $\psi_2(\phi_2(\alpha)) = \alpha$.

Lemma 2.7. *There is an explicit injection $\phi_3 : C^3_{L,s,2}(n) \to F^3_{L,s,k}(n)$.*

Proof. Given $\alpha = ((s+1)^{f_{s+1}} \dots (s+L)^{f_{s+L}}) \in C^3_{L,s,2}(n)$, by definition we see $f_k = 0$, and for any $a \geq 2$ we have $f_{as} = 0$. Moreover, $f_j \leq s-1$ for any $2s^2+5s-1 \geq j \geq s+1$. We claim that there exists $i \geq 2s^2+5s+1$ such that $f_i \geq 1$. Otherwise, we see that

$n=|\alpha|\le (s-1)(s+1)+(s-1)(s+2)+\cdots+(s-1)(2s^2+5s-1)=2s^5+8s^4+s^3-14s^2+3s$, which is contradict to $n\ge 2s^5+8s^4+s^3-14s^2+3s+1$. Hence our claim has been verified.

From the above claim, we may set $j=\min\{i\colon i\ge 2s^2+5s+1\}$ and $j=cs+d$, where $1\le d\le s-1$. From $j\ge 2s^2+5s+1$ we see that $c\ge 2s+5$. Moreover, it is trivial to check that

$$j=cs+d=s+(s+1)(s+d)+s(c-s-d-2). \tag{2.7}$$

Notice that $c-s-d-2\ge 2s+5-s-(s-1)-2=4$. It is well known that $c-s-d-2$ can be uniquely written as $2x+3y$, where $0\le y\le 1$. Now we may define $\phi_3(\alpha)$ as follows.

$$\begin{aligned}\phi_3(\alpha) &= (s^{g_s}\dots(s+L)^{g_{s+L}})\\ &= (s^1\dots(s+d)^{f_{s+d}+s+1}\dots(2s)^x\dots(3s)^y\dots j^{f_j-1}\dots(s+L)^{f_{s+L}}). \end{aligned}\tag{2.8}$$

From $c-s-d-2\ge 4$ we see that $g_{2s}+g_{3s}=x+y\ge 2$. Moreover, $k\ge L>3s$ yields that $g_k=f_k=0$. Furthermore, we may calculate $|\phi_3(\alpha)|$ as follows.

$$\begin{aligned}|\phi_3(\alpha)| &= |\alpha|+s+(s+1)(s+d)+x\cdot 2s+y\cdot 3s-j\\ &= n+s+(s+1)(s+d)+s(2x+3y)-j\\ &= n+s+(s+1)(s+d)+s(c-s-d-2)-j\\ &= n. \end{aligned}\tag{2.9}$$

The last equation follows from (2.7). Hence $\phi_3(\alpha)\in F^3_{L,s,k}(n)$.

It remains to show that ϕ_3 is an injection. Let

$$I^3_{L,s,k}(n)=\{\phi_3(\alpha)\colon \alpha\in C^3_{L,s,2}(n)\}$$

be the image set of ϕ_3, which has been shown to be a subset of $F^3_{L,s,k}(n)$. We wish to construct a map $\psi_3\colon I^3_{L,s,k}(n)\to C^3_{L,s,2}(n)$ such that for any $\alpha\in C^3_{L,s,2}(n)$,

$$\psi_3(\phi_3(\alpha))=\alpha.$$

Let $\beta=(s^{g_s}\dots(s+L)^{g_{s+L}})\in I^3_{L,s,k}(n)$, that is, there exists $\alpha\in C^3_{L,s,2}(n)$ such that $\phi_3(\alpha)=\beta$. From the definition of $F^3_{L,s,k}(n)$, we see that $g_{2s}+g_{3s}\ge 2$. By (2.8) and the fact $f_i\le s-1$ for all $s+1\le i\le 2s^2+5s-1$, we see that there exists a unique $s+1\le i\le 2s-1$ such that $g_i\ge s+1$. Moreover, by (2.9), we see that

$$s+2sx+3sy+(s+d)(s+1)=j.$$

Thus

$$s+2sg_{2s}+3sg_{3s}+i(s+1)=j\ne k.$$

Hence we may define

$$\psi_3(\beta)=((s+1)^{g_{s+1}}\dots i^{g_i-s-1}\dots 2s^0\dots 3s^0\dots w^{g_w+1}\dots(s+L)^{g_{s+L}}),$$

where $w=s+2sg_{2s}+3sg_{3s}+i(s+1)$. It is easy to check that $\psi_3(\beta)\in C^3_{L,s,2}(n)$ and $\psi_3(\phi_3(\alpha))=\alpha$. This completes the proof. ∎

Example 2.8. *For example, let $s = 3$, $L = 105$, $k = 105$ and $n = 1057$. Let*

$$\alpha = (4^2, 7^2, 11^2, 13^2, 16^2, 19^2, 32^2, 55, 58^3, 61^4, 76^5)$$

be a partition in $C^3_{105,3,2}(1057)$. It is easy to see that $j = 55$. Hence $c = 18$ and $d = 1$ and $c - s - d - 2 = 12 = 2 * 6$. So $x = 6$ and $y = 0$. Applying ϕ_3 on α,

$$\phi_3(\alpha) = (3, 4^6, 6^6, 7^2, 11^2, 13^2, 16^2, 19^2, 32^2, 58^3, 61^4, 76^5).$$

It is trivial to check that $\phi_3(\alpha) \in F^3_{105,3,105}(1057)$. Applying ψ_3 to $\phi_3(\alpha)$ we recover α.

Lemma 2.9. *There is an explicit injection $\phi_4 : C^4_{L,s,2}(n) \to F^4_{L,s,2}(n)$.*

Proof. Given $\alpha = ((s+1)^{f_{s+1}} \dots (s+L)^{f_{s+L}}) \in C^4_{L,s,2}(n)$, by definition we see $f_k \geq 2$. Recall that $k = rs + t$, where $0 \leq t \leq s - 1$ and $r \geq 2s^2 + 5s \geq 7$. We may define $\phi_4(\alpha)$ as follows.

$$\phi_4(\alpha) = (s^{g_s} \dots (s+L)^{g_{s+L}}) = (s^{f_k(r-2)} \dots (2s+t)^{f_{2s+t}+f_k} \dots k^0 \dots (s+L)^{f_{s+L}}). \quad (2.10)$$

It is clear that $g_s = f_k(r-2) \geq 2(r-2)$. Moreover, $2s + t < 7s + t \leq rs + t = k$ and

$$|\phi_4(\alpha)| = |\alpha| + sf_k(r-2) + (2s+t)f_k - kf_k = n.$$

Hence $\phi_4(\alpha) \in F^4_{L,s,k}(n)$.

It remains to show that ϕ_4 is an injection. Let

$$I^4_{L,s,k}(n) = \{\phi_4(\alpha) \colon \alpha \in C^4_{L,s,2}(n)\}$$

be the image set of ϕ_4, which has been shown to be a subset of $F^4_{L,s,k}(n)$. We wish to construct a map $\psi_4 \colon I^4_{L,s,k}(n) \to C^4_{L,s,2}(n)$ such that for any $\alpha \in C^4_{L,s,2}(n)$,

$$\psi_4(\phi_4(\alpha)) = \alpha.$$

Let $\beta = (s^{g_s} \dots (s+L)^{g_{s+L}}) \in I^4_{L,s,k}(n)$, that is, there exists $\alpha \in C^4_{L,s,2}(n)$ such that $\phi_4(\alpha) = \beta$. From (2.10), the construction of ϕ_4, we see that g_s is a multiple of $(r-2)$. Moreover, $g_{2s+t} \geq g_s/(r-2)$. We may define ψ_4 as follows.

$$\psi_4(\beta) = ((s+1)^{g_{s+1}} \dots (2s+t)^{g_{2s+t} - g_s/(r-2)} \dots k^{g_s/(r-2)} \dots (s+L)^{g_{s+L}}).$$

It is easy to check that $\psi_4(\beta) \in C^4_{L,s,2}(n)$ and $\psi_4(\phi_4(\alpha)) = \alpha$. This completes the proof. ∎

Example 2.10. *For $s = 3$, $L = 108$, $k = 109$ and $n = 1138$, set*

$$\alpha = (4^6, 7^5, 12^4, 18^3, 25^3, 42^5, 73^5, 109^3).$$

Then $k = 36 * 3 + 1$, so $r = 36$ and $t = 1$. Applying ϕ_4 to α, we derive that

$$\phi_4(\alpha) = (3^{102}, 4^6, 7^8, 12^4, 18^3, 25^3, 42^5, 73^5).$$

It is trivial to check that $\phi_4(\alpha) \in F^4_{108,3,109}(1138)$. Applying ψ_4 to $\phi_4(\alpha)$ we recover α.

Lemma 2.11. *There is an explicit injection* $\phi_5 : C^5_{L,s,2}(n) \to F^5_{L,s,k}(n)$.

Proof. Given $\alpha = ((s+1)^{f_{s+1}} \dots (s+L)^{f_{s+L}}) \in C^5_{L,s,2}(n)$, by definition we see $f_k = 1$. Recall that $k = rs + t$, where $0 \le t \le s-1$ and $r \ge 2s^2 + 5s \ge 7$. When $t \ne 0$, define $\phi_5(\alpha)$ as follows.

$$\phi_5(\alpha) = (s^{g_s} \dots (s+L)^{g_{s+L}}) = (s^{r-4} \dots (2s)^{f_{2s}+1} \dots (2s+t)^{f_{2s+t}+1} \dots k^0 \dots (s+L)^{f_{s+L}}). \tag{2.11}$$

And when $t = 0$, we set $\phi_5(\alpha)$ as given below.

$$\phi_5(\alpha) = (s^{g_s} \dots (s+L)^{g_{s+L}}) = (s^{r-4} \dots (2s)^{f_{2s}+2} \dots k^0 \dots (s+L)^{f_{s+L}}). \tag{2.12}$$

In either case, we see that $g_s = r-4$ and $g_{2s} \ge 1$. Moreover, it is trivial to check that $|\phi_5(\alpha)| = n$. This yields $\phi_5(\alpha) \in F^5_{L,s,k}(n)$.

It remains to show that ϕ_5 is an injection. Let

$$I^5_{L,s,k}(n) = \{\phi_5(\alpha) \colon \alpha \in C^5_{L,s,2}(n)\}$$

be the image set of ϕ_5, which has been shown to be a subset of $F^5_{L,s,k}(n)$. We wish to construct a map $\psi_5 \colon I^5_{L,s,k}(n) \to C^5_{L,s,2}(n)$ such that for any $\alpha \in C^5_{L,s,2}(n)$,

$$\psi_5(\phi_5(\alpha)) = \alpha.$$

Let $\beta = (s^{g_s} \dots (s+L)^{g_{s+L}}) \in I^5_{L,s,k}(n)$, that is, there exists $\alpha \in C^5_{L,s,2}(n)$ such that $\phi_5(\alpha) = \beta$. When $t \ne 0$, from (2.11) we see $g_s = r-4$, $g_{2s} \ge 1$ and $g_{2s+t} \ge 1$. We may define ψ_5 as follows.

$$\psi_5(\beta) = ((s+1)^{g_{s+1}} \dots (2s)^{g_{2s}-1} \dots (2s+t)^{g_{2s+t}-1} \dots k^1 \dots (s+L)^{g_{s+L}}).$$

When $t = 0$, from (2.12), we have $g_s = r-4$ and $g_{2s} \ge 2$. Hence the map ψ_5 is defined as follows.

$$\psi_5(\beta) = ((s+1)^{g_{s+1}} \dots (2s)^{g_{2s}-2} \dots k^1 \dots (s+L)^{g_{s+L}}).$$

It is easy to check that in either case $\psi_5(\beta) \in C^5_{L,s,2}(n)$ and $\psi_5(\phi_5(\alpha)) = \alpha$. This completes the proof. ∎

Example 2.12. *For* $s = 3$, $L = 103$, $k = 105$ *and* $n = 1217$, *set*

$$\alpha = (6^2, 9^5, 12^8, 17^4, 35^6, 42^5, 73^5, 105^1, 106^1).$$

Then $k = 35 * 3$, *so* $r = 35$ *and* $t = 0$. *Applying* ϕ_5 *to* α, *we derive that*

$$\phi_5(\alpha) = (3^{31}, 6^4, 9^5, 12^8, 17^4, 35^6, 42^5, 73^5, 106^1).$$

It is trivial to check that $\phi_5(\alpha) \in F^5_{103,3,105}(1217)$. *Applying* ψ_5 *to* $\phi_5(\alpha)$ *we recover* α.

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Given integer $s \geq 1$, $L \geq 2s^3 + 5s^2 + 1$, $s + L \geq k \geq L$ and $n \geq 2s^5 + 8s^4 + s^3 - 14s^2 + 3s + 1$, for any $\alpha \in C_{L,s,2}(n)$, we define

$$\phi(\alpha) = \phi_i(\alpha) \text{ if } \alpha \in C^i_{L,s,2}(n) \text{ for } i = 1, \ldots, 5. \tag{2.13}$$

From Lemmas 2.3-2.11, we deduce that $\phi(\alpha) \in F_{L,s,k}(n)$ and ϕ is an injection. This completes the proof. ∎

We show that Theorem 1.1 is a consequence of Theorems 2.1 and 2.2.

Proof of Theorem 1.1. From Theorem 2.1, for any positive integer s and L, there exists an integer $M_{L,s}$ such that for $n \geq M_{L,s}$,

$$f_{L,s,k}(n) \geq c_{L,s,2}(n). \tag{2.14}$$

Moreover, by Theorem 2.2, for $L \geq 2s^3 + 5s^2 + 1$ and $n \geq 2s^5 + 8s^4 + s^3 - 14s^2 + 3s + 1$, (2.14) also holds. Hence, if we set

$$M = \max\{M_{3,s}, M_{4,s}, \ldots, M_{2s^3+5s^2,s}, 2s^5 + 8s^4 + s^3 - 14s^2 + 3s + 1\},$$

then M only depends on s, and (2.14) holds for all $n \geq M$. ∎

3 Proof of Theorem 1.3

Define the sequences (a_n), (b_n) and (c_n) by

$$\sum_{n=0}^{\infty} a_n q^n := \frac{1-q}{(q^s;q)_{L+1}}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} b_n q^n := \frac{1}{(1-q^s)(q^{s+2};q)_{L-1}}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} c_n q^n := \frac{1}{(q^{s+1};q)_L}. \tag{3.3}$$

We have the following result, which will play a curial role in the proof of Theorem 1.3.

Lemma 3.1. *For any $s \geq 1$ and $L \geq 3$,*

$$a_n \sim \frac{n^{L-1}}{(L-1)!s(s+1)\cdots(s+L)}. \tag{3.4}$$

Proof. Writing $1 - q = (1 - q^{s+1}) - (q - q^{s+1})$ we see that

$$\frac{1-q}{(q^s;q)_{L+1}} = \frac{1}{(1-q^s)(q^{s+2};q)_{L-1}} - \frac{q}{(q^{s+1};q)_L}. \tag{3.5}$$

It follows from (3.1)-(3.3) that

$$a_n = b_n - c_{n-1} \quad \text{for} \quad n \geq 1. \tag{3.6}$$

For $L \geq 3$, as $\gcd(s, s+2, \ldots, s+L) = 1$ and $\gcd(s+1, \ldots, s+L) = 1$, by Theorem 1.5, we have

$$b_n \sim \frac{(s+1)n^{L-1}}{(L-1)!s(s+1)\cdots(s+L)}, \tag{3.7}$$

and

$$c_n \sim \frac{sn^{L-1}}{(L-1)!s(s+1)\cdots(s+L)}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we deduce that

$$a_n = b_n - c_{n-1} \sim \frac{n^{L-1}}{(L-1)!s(s+1)\cdots(s+L)}. \tag{3.9}$$

This completes the proof. ∎

We proceed to show Theorem 1.3 with the aid of Lemma 3.1.

Proof of Theorem 1.3. For fixed integer k, define

$$\sum_{n=0}^{\infty} d_k(n)q^n = \frac{1-q^k}{(q^s;q)_{L+1}}. \tag{3.10}$$

Comparing with (3.1) we have

$$\sum_{n=0}^{\infty} d_k(n)q^n = (1+q+\cdots+q^{k-1})\sum_{n=0}^{\infty} a_n q^n. \tag{3.11}$$

Thus, for $n \geq k$, Lemma 3.1 implies that

$$\begin{aligned} d_k(n) &= a_n + a_{n-1} + \cdots + a_{n-k+1} \\ &\sim \frac{n^{L-1} + \cdots + (n-k+1)^{L-1}}{(L-1)!s(s+1)\cdots(s+L)} \\ &\sim \frac{kn^{L-1}}{(L-1)!s(s+1)\cdots(s+L)}. \end{aligned} \tag{3.12}$$

For fixed integer $r \geq 0$, set

$$\sum_{n=0}^{\infty} e_{k,r}(n)q^n := \frac{q^r(1-q^k)}{(q^s;q)_{L+1}}. \tag{3.13}$$

Then, for $n \geq r$,

$$e_{k,r}(n) = d_k(n-r). \tag{3.14}$$

By (3.12), we deduce that

$$e_{k,r}(n) \sim \frac{kn^{L-1}}{(L-1)!s(s+1)\cdots(s+L)}. \tag{3.15}$$

Therefore, it follows from (1.4), (3.10) and (3.13) that

$$H^*_{L,s,r,k_1,k_2}(q) = \sum_{n=0}^{\infty} \left(e_{k_1,r}(n) - d_{k_2}(n)\right) q^n. \tag{3.16}$$

From (3.12) and (3.15), we derive that

$$e_{k_1,r}(n) - d_{k_2}(n) \sim \frac{(k_1-k_2)n^{L-1}}{(L-1)!s(s+1)\cdots(s+L)}.$$

Since $k_1 > k_2$, there exists M such that $e_{k_1,r}(n) - d_{k_2}(n) > 0$ for $n > M$. Thus $H^*_{L,s,r,k_1,k_2}(q)$ is eventually positive. This completes the proof. ∎

Remark 3.2. *For some special cases it would be interesting to determine the smallest M in Theorem 1.1, see also Conjecture 5.3 in [4].*

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