Quantum α-fidelity of unitary orbits

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Abstract. The extremum values of quantum α-fidelity under unitary orbits of quantum states are explicitly derived by applying rearrangement inequalities, matrix trace inequalities, and theory of majorization. Furthermore, the α-fidelity is successfully verified to go through the whole closed interval, which works from the minimum value to the maximum value.

Keywords: quantum α-fidelity, unitary orbit, matrix trace inequality, majorization.

1 Introduction

The concept of fidelity was first proposed by A. Uhlmann using the concept of "transition probability" [1]. Moreover, R. Jozsa came up with the definition of fidelity for mixed quantum states in terms of Uhlmann's "transition probability" [2]. It is undeniable that quantum state is always affected by noise, equipment and environment in all transmission process, which inspired people to measure the 'distance' between two quantum states. Fidelity, as a mathematical method to quantify the similarity between two states, is widely used in classical and quantum information theory, such as teleportation, entanglement quantification, quantum phase transitions etc. [3, 4, 5, 6]. In recent years, fidelity has become a critical tool of physical quantities investigations under unitary dynamics with the development of quantum information theory [7]. For example, Zhang et al. investigated two constrained optimization problems in terms of the maximal and minimal fidelity between two quantum states undergoing local unitary dynamics [8, 9].

Besides fidelity, there are several other measures satisfying operational meaning, such as relative entropy, Bures distance, trace distance, and generalized fidelities. α-Fidelity, as a recently proposed [10] measure, is rarely discussed and studied. It is a generalization of quantum fidelity (it reduces to the quantum fidelity when $\alpha = 1/2$) with many properties that can be naturally derived from fidelity. In this paper, we will focus on quantum α-fidelity, which is motivated by the limit formula of quantum fidelity [11].

In [8], the maximums and minimums of the quantum fidelity and the relative entropy between two unitary orbits are explicitly derived. Furthermore, they discussed potential applications in quantum computation and information processing. Motivated by this work, we will address the maximum and minimum values of quantum α-fidelity for unitary orbits. Namely, for two given

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states with fixed spectras, we expect to find that how far or how close the states under unitary evolution with respect to the 'distance' measures.

Based on the matrix trace inequalities (see Proposition 2.2) and the rearrangement inequality (see Proposition 2.5), we first obtain the extremum values of the quantum α-fidelity between unitary orbits of two quantum states (see Theorem 3.1). Moreover, we show that the values of the quantum α-fidelity under unitary orbits fill out a closed interval between the extremum values (see Corollary 3.2).

2 Preliminaries

Quantum α-fidelity and its related properties. Let H be a d-dimensional Hilbert space, $d < \infty$. Denote the set of linear operators acting on H by $L(H)$. Write $P(H) := \{X \in L(H) : X \geq 0\}$ as the set of positive semi-definite operators and $D(H) := \{\rho \in P(H) : \mathrm{Tr}\rho = 1\}$ as the set of density operators (or quantum states) on H. For quantum states $\rho, \sigma \in D(H)$ and $0 < \alpha < \infty$, the quantum α-fidelity between ρ and σ is defined as follows [10]:

$$F_\alpha(\rho,\sigma) := \left(\mathrm{Tr}\left[\left(\sigma^{\frac{-1}{2\alpha'}}\rho\sigma^{\frac{-1}{2\alpha'}}\right)^{\alpha}\right]\right)^{\frac{1}{\alpha}}, \tag{2.1}$$

where $1/\alpha + 1/\alpha' = 1$ with $\alpha' = \alpha/(\alpha - 1)$. It is obvious that the quantum fidelity $F(\rho, \sigma)$ is a special case of α-fidelity, i.e.

$$F(\rho,\sigma) = \mathrm{Tr}^2(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}) = F_{\frac{1}{2}}(\rho,\sigma).$$

If ρ and σ commute, i.e. $\rho = \sum_{i=1}^{d} \tilde{p}_i|i\rangle\langle i|$ and $\sigma = \sum_{i=1}^{d} \tilde{q}_i|i\rangle\langle i|$, the quantum α-fidelity will reduce to the following classical α-fidelity:

$$F_\alpha^C(\tilde{p},\tilde{q}) = \left(\sum_{i=1}^{d} \tilde{p}_i^{\alpha}\tilde{q}_i^{1-\alpha}\right)^{\frac{1}{\alpha}}, \tag{2.2}$$

where $\tilde{p} = \{\tilde{p}_1, \ldots, \tilde{p}_d\}$ and $\tilde{q} = \{\tilde{q}_1, \ldots, \tilde{q}_d\}$ are two probability distributions.

We have the following well-known properties of quantum α-fidelity which will be used in this paper elsewhere.

Proposition 2.1. [10, 12, 13] Let $\rho \in D(H)$ and $\sigma \in D(H)$, we have

1. $0 \leq F_\alpha(\rho,\sigma) \leq 1$ for $0 < \alpha < 1$ and $F_\alpha(\rho,\sigma) \geq 1$ for $\alpha > 1$.

2. Monotonicity of quantum α-fidelity with respect to the parameter α. $F_{\alpha_1}(\rho,\sigma) \leq F_{\alpha_2}(\rho,\sigma)$ for $1 \leq \alpha_1 < \alpha_2 < \infty$.

3. Jointly concavity. Let $\rho = \sum_x p_x\rho_x$ and $\sigma = \sum_x p_x\sigma_x$, then $F_\alpha(\rho,\sigma) \geq \sum_x p_x F_\alpha(\rho_x,\sigma_x)$ for $0 < \alpha < 1$.

4. Date processing inequality. Let Φ be a completely positive and trace preserving map (or quantum channel) on $D(H)$. Then $F_\alpha(\Phi(\rho),\Phi(\sigma)) \leq F_\alpha(\rho,\sigma)$ for $\alpha > 1$.

5. Unitary invariance. $F_\alpha(\rho,\sigma) = F_\alpha(U\rho U^*, U\sigma U^*)$ for any unitary $U \in L(H)$.

6. Let $\tau \in D(H_2)$. Then $F_\alpha(\rho\otimes\tau, \sigma\otimes\tau) = F_\alpha(\rho,\sigma)$.

7. Let $\rho_1, \sigma_1 \in D(H_1), \rho_2, \sigma_2 \in D(H_2)$. Then

$$F_\alpha(\rho_1\otimes\rho_2, \sigma_1\otimes\sigma_2) = F_\alpha(\rho_1,\sigma_1)\cdot F_\alpha(\rho_2,\sigma_2),$$

$$F_\alpha(\rho_1\oplus\rho_2, \sigma_1\oplus\sigma_2) = F_\alpha(\rho_1,\sigma_1) + F_\alpha(\rho_2,\sigma_2).$$

Proposition 2.2. Matrix trace inequalities.

1. **Golden-Thompson (GT) inequality** [14, 15]. For two Hermitian operators $A, B \in L(H)$, we have
$$\mathrm{Tr}[\exp(A+B)] \le \mathrm{Tr}[\exp(A)\exp(B)].$$
Furthermore, the equality holds if and only if A and B commute.

2. **Araki-Lieb-Thirring (ALT) trace inequality** [16]. For $A, B \in P(H)$ and $q > 0$, we have
$$\mathrm{Tr}\left[\left(B^{\frac{r}{2}}A^rB^{\frac{r}{2}}\right)^{\frac{q}{r}}\right] \le \mathrm{Tr}\left[\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)^{q}\right],\ \text{if } r \in (0,1],$$
and
$$\mathrm{Tr}\left[\left(B^{\frac{r}{2}}A^rB^{\frac{r}{2}}\right)^{\frac{q}{r}}\right] \ge \mathrm{Tr}\left[\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)^{q}\right],\ \text{if } r \ge 1.$$

Proposition 2.3. Submultiplicativity of norm [17]. For any linear operators $A \in L(H_3, H_4)$, $B \in L(H_2, H_3)$, and $C \in L(H_1, H_2)$, and any choice of $p \in [1,\infty]$, we have

$$\|ABC\|_p \le \|A\|_\infty \|B\|_p \|C\|_\infty.$$

It follows that
$$\|AB\|_p \le \|A\|_p\|B\|_p, \tag{2.3}$$
for all choice of $p \in [1,\infty]$ and operators A and B for which the product AB exists. Here $\|\cdot\|_p$ and $\|\cdot\|_\infty$ denote the Schatten p norm and operator norm, respectively.

Majorization and Schur convexity/concavity. Our main reference is [20]. Let $\mathbb{R}^d$ denotes the set of all real d-dimensional vectors. We say that $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d)$ is majorized by $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_d)$ if for all $k = 1, \ldots, d$,

$$\sum_{i=1}^k \tilde{p}_i^\downarrow \le \sum_{i=1}^k \tilde{q}_i^\downarrow,\ \sum_{i=1}^d \tilde{p}_i^\downarrow = \sum_{i=1}^d \tilde{q}_i^\downarrow,$$

where $\tilde{p}^\downarrow$ is a rearrangement of $\tilde{p}$ in decreasing order, i.e. $\tilde{p}^\downarrow = \{\tilde{p}_1^\downarrow \ldots \tilde{p}_d^\downarrow : \tilde{p}_1^\downarrow \ge \ldots \ge \tilde{p}_d^\downarrow\}$. We denote $\tilde{p} \prec \tilde{q}$ if $\tilde{p}$ is majorized by $\tilde{q}$.

A $d \times d$ matrix $D = (D_{ij})_{i,j=1}^d$ is called doubly stochastic if

$$D_{ij} \geq 0 \text{ and } \sum_{i=1}^d D_{ij} = \sum_{j=1}^d D_{ij} = 1.$$

It is obviously that $D\tilde{p} \prec \tilde{p}, \tilde{p} \in \mathbb{R}^d$ if D is a d-dimensional doubly stochastic matrix [20].

A function ϕ is called Schur convex (resp. Schur concave) if it preserves the majorization order, i.e. $\tilde{p} \prec \tilde{q}$ implies $\phi(\tilde{p}) \leq \phi(\tilde{q})$ (resp. if the majorization order is reversed). We need the following useful criterion:

Proposition 2.4. [20] Let $\mathscr{D} = \{\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d) \in \mathbb{R}^d : \tilde{p}_1 \geq \tilde{p}_2 \ldots \geq \tilde{p}_d\}$ be a subset of $\mathbb{R}^d$. A function $\phi : \mathscr{D} \to \mathbb{R}$ of the form $\phi(\tilde{q}) = \sum_{i=1}^d \nu_i g(\tilde{q}_i)$ is Schur convex (resp. concave), if g is increasing (resp. decreasing) convex. Where $\nu = (\nu_1, \ldots, \nu_d) \in \mathscr{D}$ and $\nu_i \geq 0, i = 1, \ldots, d$.

For a given state $\rho \in D(H)$, let $\lambda(\rho) = (\lambda_1, \ldots, \lambda_d)$ be the eigenvalues of ρ. Let $\lambda^{\downarrow}(\rho)$ (resp. $\lambda^{\uparrow}(\rho)$) be a rearrangement of $\lambda(\rho)$ in decreasing order (resp. increasing order). Moreover, for two given states ρ, σ with eigenvalues $\lambda(\rho) = (\lambda_1, \ldots, \lambda_d)$ and $\lambda(\sigma) = (\mu_1, \ldots, \mu_d)$, we define

$$\langle \lambda(\rho), \lambda(\sigma) \rangle := \sum_{i=1}^d \lambda_i \mu_i.$$

Further, we have the following result:

Proposition 2.5. [18, 19] For two given states $\rho, \sigma \in D(H)$, we have

$$\langle \lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\sigma) \rangle \leq \mathrm{Tr}[\rho\sigma] \leq \langle \lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\sigma) \rangle.$$

3 Quantum α-fidelity between unitary orbits

In this section, we will discuss the quantum α-fidelity between unitary orbits from two given states ρ and σ.

Definition. [8, 21] Let $U(H)$ be a set of $d \times d$ unitary matrices on H. For $\rho \in D(H)$, its unitary orbit is defined as

$$U_\rho = \{U\rho U^* : U \in U(H)\}. \tag{3.4}$$

Our purpose is to study the maximum and minimum values of quantum α-fidelity between the two unitary orbits U_ρ and U_σ. For any two unitary operators V and W, since we have

$$\max F_\alpha(V\rho V^*, W\sigma W^*) = \max F_\alpha(\rho, U\sigma U^*) \text{ and } \min F_\alpha(V\rho V^*, W\sigma W^*) = \min F_\alpha(\rho, U\sigma U^*),$$

where unitary operator U satisfies $U = V^*W$, it suffices to analyze the following optimums:

$$\max_{U\in U(H)} F_\alpha(\rho, U_\sigma) \text{ and } \min_{U\in U(H)} F_\alpha(\rho, U_\sigma).$$

Theorem 3.1. *Let $\rho, \sigma \in D(H)$, the quantum α-fidelity between the unitary orbits U_ρ and U_σ satisfies the following relations,*

$$\max_{U\in U(H)} F_\alpha(\rho, U_\sigma) = \begin{cases} F_\alpha^C(\lambda^\downarrow(\rho), \lambda^\downarrow(\sigma)), \ \ for\ each\ \alpha \in (0,1), \\ F_\alpha^C(\lambda^\downarrow(\rho), \lambda^\uparrow(\sigma)), \ \ for\ each\ \alpha \in (1,\infty), \end{cases} \tag{3.5}$$

and

$$\min_{U\in U(H)} F_\alpha(\rho, U_\sigma) = \begin{cases} F_\alpha^C(\lambda^\downarrow(\rho), \lambda^\uparrow(\sigma)), \ \ for\ each\ \alpha \in (0,1), \\ F_\alpha^C(\lambda^\downarrow(\rho), \lambda^\downarrow(\sigma)), \ \ for\ each\ \alpha \in (1,\infty). \end{cases} \tag{3.6}$$

Proof. Without loss of generality, we assume operators $\rho > 0$ and $\sigma > 0$ (if $\sigma \geq 0$, we consider $\sigma + \varepsilon\mathbb{I} > 0$ and $\lim_{\varepsilon\to 0} \sigma + \varepsilon\mathbb{I} = \sigma$).

Since the eigenvectors of two positive matrices on a Hilbert space H_d can be related by a unitary matrix, it suffices to suppose that ρ and σ have the following spectral decompositions:

$$\rho = \sum_{i=1}^{d} \lambda_i^\downarrow(\rho)|i\rangle\langle i| \ \text{ and } \ \sigma = \sum_{i=1}^{d} \lambda_i^\downarrow(\sigma) W_0|i\rangle\langle i|W_0^*,$$

where $\lambda_i(\rho)>0$, $\lambda_i(\sigma)>0$, $i = 1, \ldots, d$ and W_0 is a unitary operator.

(I). For Equation (3.5), let $A = \frac{1-\alpha}{\alpha} U \ln \sigma U^*$ and $B = \ln \rho$, then there are two unitary matrices U_1 and U_2 such that [22],

$$\exp\left(\frac{A}{2}\right)\exp(B)\exp\left(\frac{A}{2}\right) = \exp(U_1 A U_1^* + U_2 B U_2^*),$$

which induces

$$U\sigma^{\frac{1-\alpha}{2\alpha}} U^* \rho U \sigma^{\frac{1-\alpha}{2\alpha}} U^* = \exp\left(U_2 \ln \rho U_2^* + \frac{1-\alpha}{\alpha} U_1 U \ln \sigma U^* U_1^*\right).$$

Hence, for $0 < \alpha < 1$, we have

$$\begin{aligned} F_\alpha(\rho, U\sigma U^*) &= \left(\mathrm{Tr}\left[\left(U\sigma^{\frac{1-\alpha}{2\alpha}} U^* \rho U \sigma^{\frac{1-\alpha}{2\alpha}} U^*\right)^\alpha\right]\right)^{\frac{1}{\alpha}} \\ &= (\mathrm{Tr}\,[\exp(\alpha U_2 \ln \rho U_2^* + (1-\alpha) U_1 U \ln \sigma U^* U_1^*])^{\frac{1}{\alpha}} \\ &\text{denote } \tilde{U} = U_2^* U_1 U \\ &= \left(\mathrm{Tr}\left[\exp(\alpha \ln \rho + (1-\alpha)\tilde{U} \ln \sigma \tilde{U}^*\right]\right)^{\frac{1}{\alpha}} \\ &\text{by GT inequality} \\ &\leq \left(\mathrm{Tr}\left[\rho^\alpha \tilde{U} \sigma^{(1-\alpha)} \tilde{U}^*\right]\right)^{\frac{1}{\alpha}} \\ &\text{by ALT inequality, by letting } r = q = \alpha, A = \rho, B = \tilde{U}\sigma^{\frac{(1-\alpha)}{\alpha}}\tilde{U}^* \\ &\leq F_\alpha(\rho, \tilde{U}\sigma\tilde{U}^*). \end{aligned}$$

Since the compactness of unitary group $U(H)$ and the continuity of the map $U \to F_\alpha(\rho, U\sigma U)$ on the unitary group equipped with the operator norm (see the proof of Theorem 3.2. We have shown that the map is continuous respect to the L_1 norm. However, all norms are equivalent in finite dimensional vector spaces), there is a unitary U_0 such that the maximum can be obtained. Then by the above inequality, we have

$$\max_{U\in U(H)} F_\alpha(\rho, U\sigma U^*) = F_\alpha(\rho, U_0\sigma U_0^*) = F_\alpha(\rho, \tilde{U}_0\sigma\tilde{U}_0^*).$$

Thus,

$$\mathrm{Tr}\left[\exp(\alpha\ln\rho + (1-\alpha)\tilde{U}_0\ln\sigma\tilde{U}_0^*\right] = \mathrm{Tr}\left[\rho^\alpha\tilde{U}_0\sigma^{(1-\alpha)}\tilde{U}_0^*\right].$$

By the condition of equality of the GT inequality, we have $[\rho^\alpha, \tilde{U}_0\sigma^{(1-\alpha)}\tilde{U}_0^*] = 0$. It is clear that $\tilde{U}_0$ must be equal to W_0^* since $[\rho^\alpha, W_0^*\sigma^{1-\alpha}W_0] = 0$. Therefore, we have shown that if $[\rho, W_0^*\sigma W_0] = 0$ and $0 < \alpha < 1$, then

$$\max_{U\in U(H)} F_\alpha(\rho, U\sigma U^*) = F_\alpha(\rho, W_0^*\sigma W_0).$$

Therefore, for $0 < \alpha < 1$, we have

$$\begin{aligned}\max_{U\in U(H)} F_\alpha(\rho, U\sigma U^*) &= F_\alpha(\rho, W_0^*\sigma W_0) \\ &\quad \text{by } [\rho^\alpha, W_0^*\sigma^{1-\alpha}W_0] = 0 \\ &= F_\alpha^C(\lambda^\downarrow(\rho), \lambda^\downarrow(\sigma)).\end{aligned}$$

Now, for $1 < \alpha < \infty$, note that $\{(\lambda_1^\uparrow(\sigma))^{1-\alpha}, \ldots, (\lambda_d^\uparrow(\sigma))^{1-\alpha}\}$ is a decreasing sequence. Using the ALT inequality (by letting $r = q = \alpha, A = \rho, B = U\sigma^{\frac{(1-\alpha)}{\alpha}}U^*$), we have

$$\begin{aligned}F_\alpha(\rho, U\sigma U^*) &\le \left(\mathrm{Tr}\left[\rho^\alpha U\sigma^{1-\alpha}U^*\right]\right)^{\frac{1}{\alpha}} \\ &\quad \text{by Proposition 2.5} \\ &\le \left(\sum_{i=1}^d \lambda_i^\downarrow(\rho^\alpha)\lambda_i^\downarrow(U\sigma^{1-\alpha}U^*)\right)^{\frac{1}{\alpha}} \\ &= \left(\sum_{i=1}^d (\lambda_i^\downarrow(\rho))^\alpha(\lambda_i^\uparrow(\sigma))^{1-\alpha}\right)^{\frac{1}{\alpha}}.\end{aligned}$$

Hence for $1 < \alpha < \infty$,

$$\max_{U\in U(H)} F_\alpha(\rho, U_\sigma) \le F_\alpha^C(\lambda^\downarrow(\rho), \lambda^\uparrow(\sigma)).$$

Obviously, the above upper bound will be saturated when we take $UW_0|i\rangle = |d-i+1\rangle$.

(II). Now we move forward to study Equation (3.6). For $0 < \alpha < 1$, with Proposition 2.5, we have

$$\langle\lambda^\downarrow(\rho^\alpha), \lambda^\uparrow(U\sigma^{1-\alpha}U^*)\rangle \le \mathrm{Tr}[\rho^\alpha U\sigma^{1-\alpha}U^*].$$

Again by the ALT inequality, we attain

$$F_\alpha(\rho, U\sigma U^*) \ge \left(\mathrm{Tr}[\rho^\alpha U\sigma^{1-\alpha}U^*]\right)^{\frac{1}{\alpha}}.$$

Thus, we obtain

$$\min_{U\in U(H)} F_\alpha(\rho, U\sigma U^*) \geq \min_{U\in U(H)} \langle \lambda^{\downarrow}(\rho^\alpha), \lambda^{\uparrow}(U\sigma^{1-\alpha}U^*)\rangle$$
$$= \left(\sum_{i=1}^d (\lambda_i^{\downarrow}(\rho))^\alpha (\lambda_i^{\uparrow}(\sigma))^{1-\alpha}\right)^{\frac{1}{\alpha}}.$$

The above lower bound will be reached by letting $UW_0|i\rangle = |d-i+1\rangle$.

For $1 < \alpha < \infty$, firstly, we assume that $\bar{U}_0$ is an unitary matrix such that

$$F_\alpha(\rho, \bar{U}_0\sigma\bar{U}_0^*) = \min_U F_\alpha(\rho, U\sigma U^*),$$

then

$$F_\alpha(\rho, \bar{U}_0\sigma\bar{U}_0^*) \leq F_\alpha(\rho, W_0^*\sigma W_0).$$

We recall that $W_0^*\sigma W_0 = \sum_{i=1}^d \lambda_i^{\downarrow}(\sigma)|i\rangle\langle i|$. Hence it is clear that

$$\begin{aligned} F_\alpha(\rho, U_0\sigma U_0^*) &\leq F_\alpha(\rho, W_0^*\sigma W_0) \\ &= \left(\sum_{i=1}^d (\lambda_i^{\downarrow}(\rho))^\alpha (\lambda_i^{\downarrow}(\sigma))^{1-\alpha}\right)^{\frac{1}{\alpha}} \\ &= F_\alpha^C(\lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\sigma)). \end{aligned} \tag{3.7}$$

On the other hand, by letting $\nu_i = \tilde{p}_i^\alpha$ and $g(\tilde{q}_i) = \tilde{q}_i^{1-\alpha}$ in Proposition 2.4, it is easy to see that the function $\phi : \mathscr{D} \to \mathbb{R}$ defined by

$$\phi(\tilde{q}) = \sum_{i=1}^d \tilde{p}_i^\alpha \tilde{q}_i^{1-\alpha}$$

is Schur concave ($\alpha > 1$). Thus for probability distributions $\tilde{p}, \tilde{q}, \tilde{q}' \in \mathscr{D}$, we have

$$F_\alpha^C(\tilde{p}, \tilde{q}) \geq F_\alpha^C(\tilde{p}, \tilde{q}'), \text{ if } \tilde{q} \prec \tilde{q}'. \tag{3.8}$$

Moreover, we define a completely positive and trace preserving map (quantum channel) on $D(H)$ as follows:

$$\Phi(K) := \sum_{i=1}^d \langle i|K|i\rangle |i\rangle\langle i|, \ K \in D(H).$$

Note that $\Phi(\rho) = \rho$ and $[\rho, \Phi(K)] = 0$. Thus, the data processing inequality yields

$$\begin{aligned} F_\alpha(\rho, \bar{U}_0\sigma\bar{U}_0^*) &\geq F_\alpha(\rho, \Phi(\bar{U}_0\sigma\bar{U}_0^*)) \\ &\text{by } \ [\rho, \Phi(\bar{U}_0\sigma\bar{U}_0^*)] = 0 \\ &= F_\alpha^C(\lambda^{\downarrow}(\rho), D_{\bar{u}_0}\lambda^{\downarrow}(\sigma)) \\ &\text{by Proposition 2.5} \\ &\geq F_\alpha^C(\lambda^{\downarrow}(\rho), (D_{\bar{u}_0}\lambda^{\downarrow}(\sigma))^{\downarrow}) \end{aligned}$$

where $D_{\bar{u}_0}$ is a $d \times d$ doubly stochastic matrix with $(D_{\bar{u}_0})_{ij} = |\langle i|\bar{U}_0 W_0|j\rangle|^2$. It is obviously that $D_{\bar{u}_0}\lambda^{\downarrow}(\sigma) \prec \lambda^{\downarrow}(\sigma)$. Thus applying Equation 3.8, we can get

$$F_\alpha^C(\lambda^{\downarrow}(\rho), (D_{\bar{u}_0}\lambda^{\downarrow}(\sigma))^{\downarrow}) \geq F_\alpha^C(\lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\sigma)). \tag{3.9}$$

Combing Equations 3.7 and 3.9, we give rise to

$$\min_{U \in U(H)} F_\alpha(\rho, U\sigma U^*) = F_\alpha^C(\lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\sigma)).$$

Thus $\min_{U \in U(H)} F_\alpha(\rho, U\sigma U^*)$ is achieved precisely when $U = W_0^*$. □

Remark. Theorem 3.1 gives an explicit formula to attain the maximum and minimum values of the quantum α-fidelity of the unitary orbits of quantum states by the eigenvalues of quantum states.

A quantum system usually undergoes unitary evolution with $\{U_t = \exp(\mathrm{i}tH) : t \in \mathbb{R}\}$ by some Hamiltonian H. Actually, a unitary matrix U_t is path-connected with $\mathbb{I}_d$ via a path $U_t = \exp(tL)$ with skew-Hermitian matrix L [23]. Clearly, the unitary matrix U_t is continuous in t, thus, it is not difficult to state that $F_\alpha(\rho, U_t\sigma U_t^*)$ is also continuous in t (see proof of Theorem 3.2). Indeed, since the extremum values of $F_\alpha(\rho, U_t\sigma U_t^*)$ do exist (see Theorem 3.1), it is natural to get the following closed intervals.

Theorem 3.2. *For $\rho, \sigma \in D(H)$, the set $\{F_\alpha(\rho, U\sigma U^*), U \in U(H)\}$ is identical to the interval*

$$[F_\alpha^C(\lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\sigma)), F_\alpha^C(\lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\sigma))], \ \textit{for each } \alpha \in (0,1), \tag{3.10}$$

or

$$[F_\alpha^C(\lambda^{\downarrow}(\rho), \lambda^{\downarrow}(\sigma)), F_\alpha^C(\lambda^{\downarrow}(\rho), \lambda^{\uparrow}(\sigma))], \ \textit{for each } \alpha \in (1,\infty). \tag{3.11}$$

Proof. Due to Theorem 3.1, the extremum values of $F_\alpha(\rho, U\sigma U^*), U \in U(H)$ exist. By Stone's theorem on one-parameter unitary groups [24], one can find some skew operators L_0 and L_1 such that the minimum is reached for the unitary $U = e^{L_0}$ and the maximum is reached for the unitary $U = e^{L_1}$. Define the following function of t:

$$t \to F_\alpha(\rho, e^{(1-t)L_0 + tL_1}\sigma e^{-(1-t)L_0 - tL_1}).$$

If this function is continuous in t in the interval $[0,1]$, then by using the intermediate value theorem we can prove our theorem.

Now let $A_t = \sqrt{\rho}U_t\sigma^{\frac{1-\alpha}{\alpha}}U_t^*\sqrt{\rho}$, where $t \to U_t$ is a path in the unitary matrix group. Thus, we have

$$\mathrm{Tr}(A_t^\alpha) = F_\alpha^\alpha(\rho, U_t\sigma U_t^*).$$

Due to the above argument, it is sufficient to prove that the function $t \to \mathrm{Tr}(A_t^\alpha)$ is continuous in t for any $\alpha \in (0,\infty)$. Without loss of generality, we assume that all the operators are taken on the support of operators.

Firstly, by Stone's theorem, $U_t = e^{tL}$ is continuous in t with respect to the Schatten 1 norm, i.e, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|U_{t+\delta} - U_t\|_1 \leq \varepsilon/2$. (In fact, due to Stone's theorem, U_t is continuous with respect to the strong operator norm. However, we only consider the finite dimensional space and all norms are equivalent in finite dimensional vector spaces.)

Applying Proposition 2.3 with $p = 1$, we can get

$$\begin{aligned}\|A_{t+\delta} - A_t\|_1 &= \|\sqrt{\rho}[(U_{t+\delta} - U_t)\sigma^{\frac{1-\alpha}{\alpha}} U^*_{t+\delta} + U_t\sigma^{\frac{1-\alpha}{\alpha}}(U^*_{t+\delta} - U^*_t)]\sqrt{\rho}\|_1 \\ &\leq \|\sqrt{\rho}\|^2_\infty(\|U_{t+\delta} - U_t\|_1 \cdot \|\sigma^{\frac{1-\alpha}{\alpha}} U^*_{t+\delta}\|_\infty + \|U_t\sigma^{\frac{1-\alpha}{\alpha}}\|_\infty \cdot \|U^*_{t+\delta} - U^*_t\|_1) \\ &\leq \|\sqrt{\rho}\|^2_\infty \cdot \|\sigma^{\frac{1-\alpha}{\alpha}}\|_\infty(\|U_{t+\delta} - U_t\|_1 + \|U^*_{t+\delta} - U^*_t\|_1) \\ &\leq \varepsilon.\end{aligned}$$

Furthermore, for a non-negative operator A, when $\alpha \in (0,1)$, we have

$$A^\alpha = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty x^\alpha(\frac{1}{x} - \frac{1}{x+A})\mathrm{d}x.$$

Combining the operator identity $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$, we derive

$$\begin{aligned}\|A^\alpha_{t+\delta} - A^\alpha_t\|_1 &= \left\|\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty x^\alpha(\frac{1}{x+A_t} - \frac{1}{x+A_{t+\delta}})\mathrm{d}x\right\|_1 \\ &= \left\|\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty x^\alpha\frac{1}{x+A_t}(A_{t+\delta} - A_t)\frac{1}{x+A_{t+\delta}}\mathrm{d}x\right\|_1 \\ &\leq \frac{1}{\pi}\int_0^\infty x^\alpha\left\|\frac{1}{x+A_t}\right\|_\infty \cdot \|A_{t+\delta} - A_t\|_1 \cdot \left\|\frac{1}{x+A_{t+\delta}}\right\|_\infty \mathrm{d}x.\end{aligned}$$

For the positive operator $A_t \in B(H)$, let $\lambda^\downarrow_d(A_t)$ denotes A_t's minimal eigenvalue, then we have

$$\left\|\frac{1}{x+A_t}\right\|_\infty = \frac{1}{x+\lambda^\downarrow_d(A_t)}, \text{ for } x \geq 0.$$

Therefore

$$\begin{aligned}\|A^\alpha_{t+\delta} - A^\alpha_t\|_1 &\leq \frac{1}{\pi}\|A_{t+\delta} - A_t\|_1\int_0^\infty x^\alpha\left(\frac{1}{x+\min\{\lambda^\downarrow_d(A_t), \lambda^\downarrow_d(A_{t+\delta})\}}\right)^2\mathrm{d}x \\ &\leq \frac{1}{\pi}\varepsilon\int_0^\infty x^\alpha\left(\frac{1}{x+\min\{\lambda^\downarrow_d(A_t), \lambda^\downarrow_d(A_{t+\delta})\}}\right)^2\mathrm{d}x \\ &\text{denote } a = \min\{\lambda^\downarrow_d(A_t), \lambda^\downarrow_d(A_{t+\delta})\} \text{ and } y = x+a \\ &\leq \frac{1}{\pi}\varepsilon\int_a^\infty \frac{1}{y^{2-\alpha}}\mathrm{d}y := b\varepsilon,\end{aligned}$$

where $b = 1/\pi\int_a^\infty 1/y^{2-\alpha}\mathrm{d}y$. We note that for $\alpha \in (0,1)$, the integral $\int_a^\infty 1/y^{2-\alpha}\mathrm{d}y < \infty$.

Finally, we have

$$|\mathrm{Tr}(A^\alpha_{t+\delta}) - \mathrm{Tr}(A^\alpha_t)| \leq \|A^\alpha_{t+\delta} - A^\alpha_t\|_1 \leq b\varepsilon.$$

Thus the function $t \to \mathrm{Tr}(A^\alpha_t)$ is continuous in t for $\alpha \in (0,1)$.

Moreover, for $\alpha \in (0,1)$,

$$\begin{aligned}|\mathrm{Tr}(A_{t+\delta}^{2\alpha}) - \mathrm{Tr}(A_t^{2\alpha})| &\leq \|A_{t+\delta}^{2\alpha} - A_t^{2\alpha}\|_1 \\ &= \|A_{t+\delta}^{\alpha}A_{t+\delta}^{\alpha} - A_t^{\alpha}A_t^{\alpha}\|_1 \\ &\leq \|A_{t+\delta}^{\alpha}(A_{t+\delta}^{\alpha} - A_t^{\alpha})\|_1 + \|(A_{t+\delta}^{\alpha} - A_t^{\alpha})A_t^{\alpha}\|_1 \\ &\leq (\|A_{t+\delta}^{\alpha}\|_\infty + \|A_t^{\alpha}\|_\infty)\|A_{t+\delta}^{\alpha} - A_t^{\alpha}\|_1 \\ &\leq c\varepsilon,\end{aligned}$$

where $c = b(\|A_{t+\delta}^{\alpha}\|_\infty + \|A_t^{\alpha}\|_\infty) < \infty$. Thus $\mathrm{Tr}(A_t^{2\alpha})$ is continuous in t for $\alpha \in (0,1)$, in other words, $\mathrm{Tr}(A_t^{\alpha})$ is continuous in t for $\alpha \in (0,2)$. By iterating the above steps, we may prove that $\mathrm{Tr}(A_t^{\alpha})$ is continuous in t when $\alpha \in (0,2^n)$, $n \in \mathbb{N}$. Therefore, $\mathrm{Tr}(A_t^{\alpha})$ is continuous in t for each $\alpha > 0$, which completes our proof. □

Remark. Our results can be applied to roughly estimating the values of some computing complex α-fidelities. They are also widely used in quantum information and quantum computation, such as, optimal quantum control [8].

For $\alpha \in (0,1) \cup (1,\infty)$, we note that the sandwiched quantum α-Rényi relative entropy is defined as follows [12]

$$S_\alpha(\rho\|\sigma) = \begin{cases}\alpha' \log F_\alpha(\rho,\sigma), \ \mathit{supp}(\rho) \subseteq \mathit{supp}(\sigma), \\ \infty, \ \mathit{otherwise}.\end{cases} \tag{3.12}$$

For two probability distributions $\tilde{p} = \{\tilde{p}_1, \ldots, \tilde{p}_d\}$ and $\tilde{q} = \{\tilde{q}_1, \ldots, \tilde{q}_d\}$, the classical α-Rényi relative entropy is given by [25]

$$S_\alpha(\tilde{p}\|\tilde{q}) = \alpha' \log F_\alpha^C(\tilde{p},\tilde{q}). \tag{3.13}$$

Immediately, we have the following corollary.

Corollary 3.3. *For two states $\rho, \sigma \in D(H)$, where σ is full-ranked, and all $\alpha \in (0,1) \cup (1,\infty)$, we have*

$$\max_{U \in U(H)} S_\alpha(\rho\|U_\sigma) = S_\alpha^C(\lambda^\downarrow(\rho)\|\lambda^\uparrow(\sigma)), \tag{3.14}$$

and

$$\min_{U \in U(H)} S_\alpha(\rho\|U_\sigma) = S_\alpha^C(\lambda^\downarrow(\rho)\|\lambda^\downarrow(\sigma)). \tag{3.15}$$

Moreover, the set $\{S_\alpha(\rho\|U_\sigma) : U \in U(H)\}$ is identical to the interval

$$\left[S_\alpha^C(\lambda^\downarrow(\rho)\|\lambda^\downarrow(\sigma)), S_\alpha^C(\lambda^\downarrow(\rho)\|\lambda^\uparrow(\sigma))\right].$$

4 Conclusion

In this paper, we studied the quantum α-fidelity between unitary orbits of two quantum states. We constructed explicit formulas for the corresponding extremal values. Interestingly, the extremal values are obtained when the two states ρ and σ satisfy certain commuting relation. Namely, $[\rho, U\sigma U^*] = 0$ for some unitary U. We also found that the values of the α-fidelity fill out a closed interval between the minimum and the maximum values. As a corollary, the extremum values of the quantum sandwiched α-Rényi relative entropy between two unitary orbits are described by the corresponding classical ones.

As well as the quantum fidelity, the quantum α-fidelity is also a 'distance' measure of two states. In this paper, our motivation is to study the asymptotic behavior of the evolution of quantum states by using such distance measures. We hope our work could shed some light on related studies.

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