A note on steady vortex flows in two dimensions

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Abstract

In this note, we give a general criterion for steady vortex flows in a planar bounded domain. More specifically, we show that if the stream function satisfies “locally” a semilinear elliptic equation with monotone or Lipschitz nonlinearity, then the flow must be steady.

Keywords: Euler equations, steady vortex flow, semilinear elliptic equation, stream function

1. Introduction and Main Result

Let $D \subset \mathbb{R}^2$ be a simply connected and bounded domain with a smooth boundary $\partial D$. The motion of an incompressible nonviscous fluid of unit density in $D$ is governed by the following Euler dynamical equations:

\[
\begin{aligned}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\nabla P, \quad (x, t) \in D \times \mathbb{R}^+, \\
\nabla \cdot \mathbf{v} &= 0, \\
\mathbf{v}(x, 0) &= \mathbf{v}_0(x), \quad x \in D, \\
\mathbf{v} \cdot n &= 0, \quad (x, t) \in \partial D \times \mathbb{R}^+,
\end{aligned}
\]  

(1.1)

where $\mathbf{v} = (v_1, v_2)$ is the velocity field, $P$ is the scalar pressure and $\mathbf{n}$ denotes the outward unit normal of $\partial D$. Here we impose the impermeability condition on the boundary.

Define the vorticity of the fluid as $\omega := \partial_1 v_2 - \partial_2 v_1$. Since $\mathbf{v}$ is divergence-free, there is a function $\psi$, called the stream function, such that $\mathbf{v} = J \nabla \psi := (\partial_2 \psi, -\partial_1 \psi)$, where $J(a, b) = (b, -a)$ denotes clockwise rotation through $\frac{\pi}{2}$ for any planar vector $(a, b)$. It is easy to see that $\omega$ and $\psi$ satisfy the following Poisson’s equation:

\[-\Delta \psi = \omega.
\]

By the impermeability boundary condition, we deduce that $\psi$ is a constant on $\partial D$. Without loss of generality, by adding a suitable constant we assume that $\psi$ vanishes on the boundary.
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Thus $\psi$ can be determined by $\omega$ in the following way

$$\psi(x,t) = G\omega(x,t) := \int_D G(x,y)\omega(y,t)dy, \ x \in D,$$

where $G$ is the Green function for $-\Delta$ in $D$ with zero Dirichlet boundary condition.

Taking the curl on both sides of the first equation in (1.1), we obtain

$$\partial_t \omega + J \nabla G \omega \cdot \nabla \omega = 0,$$

which is a nonlinear transport equation for $\omega$ and is usually called the vorticity equation.

In this paper, we are concerned with the steady vorticity equation, that is,

$$J \nabla G \omega \cdot \nabla \omega = 0.$$  \hspace{1cm} (1.3)

To motivate the definition of weak solutions to the steady vorticity equation, we multiply any $\phi \in C^\infty_0(D)$ on both sides of (1.3) and integrate by parts formally to obtain

$$\int_D \omega J \nabla G \omega \cdot \nabla \phi dx = 0.$$  \hspace{1cm} (1.4)

It is not difficult to check that the integral in (1.4) makes sense if $\omega \in L^{4/3}(D)$. In fact, for $\omega \in L^2(D)$, by $L^p$ estimate we have $G\omega \in W^{2,\frac{4}{3}}(D)$, thus $G\omega \in W^{1,4}(D)$ by Sobolev embedding, therefore the integral in (1.4) makes sense by Hölder’s inequality.

Definition 1.1. We call $\omega \in L^{4/3}(D)$ a weak solution to the steady vorticity equation (1.3) if it satisfies

$$\int_D \omega J \nabla G \omega \cdot \nabla \phi dx = 0, \ \forall \phi \in C^\infty_0(D).$$  \hspace{1cm} (1.5)

In the past several decades, various methods have been proposed to construct steady vortex flows. The most commonly used method is to investigate the following semilinear elliptic problem

$$\begin{cases}
-\Delta \psi = f(\psi), & x \in D, \\
\psi = 0, & x \in \partial D,
\end{cases}$$  \hspace{1cm} (1.6)

where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. It is obvious that

$$J \nabla \psi \cdot \nabla f(\psi) = J \nabla \psi \cdot (f'(\psi) \nabla \psi) = 0 \text{ a.e. in } D,$$

which means that any solution to (1.6) corresponds to a steady vortex flow in classical sense with $\psi$ as the stream function. See [6][8][15][14] and the references listed therein. Another efficient way to construct steady vortex flows is called the vorticity method, which was first established by Arnold [1][2]. See also [3][4][9][10][11][12][16]. Roughly speaking, the vorticity method states that any steady vortex flow is equivalent to a critical point of the kinetic energy subject to
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some appropriate constraints for the vorticity. For example, Turkington in [16] considered the maximization of the kinetic energy

\[ E(\omega) := \frac{1}{2} \int_D \int_D G(x, y) \omega(x) \omega(y) dx \, dy \]

in the admissible class

\[ K_\lambda(D) := \{ \omega \in L^\infty(D) \mid 0 \leq \omega \leq \lambda \text{ a.e. in } D, \int_D \omega(x) dx = 1 \} . \]

(1.7)

Turkington proved the existence of a maximizer and showed that any maximizer \( \omega^\lambda \) must be of the form

\[ \omega^\lambda = \lambda I_{A^\lambda}, \quad A^\lambda = \{ x \in D \mid G\omega^\lambda(x) > \mu^\lambda \}, \]

(1.8)

where \( I_{A^\lambda} \) denotes the characteristic function of \( A^\lambda \), and \( \mu^\lambda \) is a Lagrange multiplier depending on \( \lambda \). To show that \( \omega^\lambda \) is a steady solution to the vorticity equation (1.3), one can not use the form (1.8) anymore, since the nonlinearity here is a Heaviside function with discontinuity at \( \mu^\lambda \) and the regularity of \( \partial A^\lambda \) is unknown (still an open question). To show that \( \omega^\lambda \) satisfies (1.5), Turkington used the fact that \( \omega^\lambda \) is an energy maximizer in \( K_\lambda(D) \). See [16] for the detailed proof.

In [5], Burton proved that if \( \omega \) belongs to \( L^{4/3}(D) \) and satisfies \( \omega = f(G\omega) \) a.e. in \( D \), where \( f : \mathbb{R} \to \mathbb{R} \cup \{ \pm \infty \} \) is a monotone function, then \( \omega \) is a weak solution to (1.3). By Burton’s result, in order to obtain a steady vortex flow from (1.8), we do not need additional information about the energy of \( \omega^\lambda \).

Another example of steady vortex flows with vorticity concentrated in multiple separated regions is given in [7]. Therein the authors studied the following elliptic problem

\[
\begin{aligned}
-\Delta \psi &= \lambda \sum_{i=1}^k I_{A_i}, \quad x \in D, \\
A_i &= B_\delta(x_{0,i}) \cap \{ x \in D \mid \psi(x) > \kappa_i \},
\end{aligned}
\]

(1.9)

where \( I_{A_i} \) denotes the characteristic function of \( A_i \), \( \lambda \) is a given positive number, \( \kappa_i \) is a real number depending on \( \lambda \), \( x_{0,i} \) is a given point in \( D \), \( \delta \) is a very small positive number such that \( B_\delta(x_{0,i}) \subset D \) and \( B_\delta(x_{0,i}) \cap B_\delta(x_{0,j}) = \emptyset \) if \( i \neq j \). The authors constructed a solution to (1.9) for sufficiently large \( \lambda \), such that each \( A_i \) is a simply connected domain bounded by a \( C^1 \) closed curve and is strictly contained in \( B_\delta(x_{0,i}) \) (or equivalently, \( \text{dist}(A_i, \partial B_\delta(x_{0,i})) > 0 \)). To show that \( \psi \) satisfies (1.5), one can integrate by parts directly since each \( \partial A_i \) is \( C^1 \) and \( \psi \) is continuous across \( \partial A_i \).

Notice that in (1.9) the vorticity \( \omega = -\Delta \psi \) is no longer a function of the stream function \( \psi \), since the \( k \) Lagrange multipliers \( \kappa_i, \ldots, \kappa_k \) may be different numbers. However, the vorticity is a function of the stream function “locally”.

Our aim in this note is to give a general criterion for solutions of the steady vorticity equation (1.3), that is, if the stream function satisfies “locally” a semilinear elliptic equation with monotone or Lipschitz nonlinearity, then the corresponding flow must be steady.
Before stating the theorem, we give some notations for clarity. We will use $\text{supp}(f)$ to denote the support of some function $f$, and the distance between two planar sets $A$ and $B$ is defined by

$$\text{dist}(A, B) := \inf_{x \in A, y \in B} |x - y|.$$ 

Let $\delta > 0$ be a positive number, the notation $A_\delta$ denotes the $\delta$-neighbourhood in $D$ of some planar set $A$, or equivalently,

$$A_\delta := \{x \in D \mid \text{dist}(x, A) < \delta\}.$$ 

**Theorem 1.2.** Let $k$ be a positive integer. Suppose that $\omega \in L^{4/3}(D)$ satisfies

$$\omega = \sum_{i=1}^{k} \omega_i, \quad \min_{1 \leq i < j \leq k} \{\text{dist}(\text{supp}(\omega_i), \text{supp}(\omega_j))\} > 0, \quad \omega_i = f^i(G\omega), \ a.e. \ in \ \text{supp}(\omega_i)_\delta \quad (1.10)$$

for some $\delta > 0$, where each $f^i$ is either monotone from $\mathbb{R}$ to $\mathbb{R} \cup \{\pm \infty\}$ or Lipschitz continuous from $\mathbb{R}$ to $\mathbb{R}$, then $\omega$ is a weak solution to the steady vorticity equation $(1.3)$.

**Remark 1.3.** In the above theorem, if $f_i$ is Lipschitz continuous from $\mathbb{R}$ to $\mathbb{R}$, then $\omega_i$ must be bounded since $G\omega \in L^\infty(D)$ by $L^p$ estimate and Sobolev embedding.

**Remark 1.4.** Examples of steady vortex flows satisfying $(1.10)$ can also be found in [12], where each $f^i : \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$ is an unknown nondecreasing function.

## 2. Proof of the Main Result

In this section, we give the proof of Theorem 1.2. The basic idea is to approximate each $f^i$ by a sequence of bounded Lipschitz functions.

For a function $f : \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$, we shall use the following notation in the rest of this paper for convenience:

$$C_f := \{s \in \mathbb{R} \mid f \text{ is continuous at } s\},$$

$$D_f := \{s \in \mathbb{R} \mid f \text{ is not continuous at } s\}.$$ 

**Lemma 2.1.** Suppose that $f : \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$ is a monotone function, then there exists a sequence of bounded and smooth functions $\{f_n\}$ such that

$$|f_n(s)| \leq |f(s)|, \ \forall s \in \mathbb{R},$$

$$\lim_{n \to +\infty} f_n(s) = f(s), \ \forall s \in C_f.$$ 

**Proof.** Without loss of generality we assume that $f$ is nondecreasing and bounded (we can use truncation if $f$ is unbounded).
First we consider the case $f$ is nonnegative. Let $\rho$ be the standard mollifier of one dimension, that is,
\[
\rho(s) = \begin{cases} 
  c_0e^{-\frac{1}{1-s^2}}, & |s| < 1, \\
  0, & |s| \geq 1,
\end{cases}
\]
where $c_0$ is a positive number such that $\int_\mathbb{R} \rho(s)ds = 1$. Note that $\rho \in C^\infty_c(\mathbb{R})$. Define
\[
f^\varepsilon(s) = \int_{-\infty}^{+\infty} \rho_\varepsilon(s - \varepsilon - r)f(r)dr = \int_{s-2\varepsilon}^{s} \rho_\varepsilon(s - \varepsilon - r)f(r)dr,
\]
where $\rho_\varepsilon(s) := \varepsilon^{-1}\rho(s\varepsilon^{-1})$ with $\varepsilon > 0$ as a parameter. It is easy to check that $f^\varepsilon \in C^\infty(\mathbb{R})$.

Since $f$ is nonnegative and nondecreasing, we have
\[
|f^\varepsilon(s)| = \int_{s-2\varepsilon}^{s} \rho_\varepsilon(s - \varepsilon - r)f(r)dr \leq \int_{s-2\varepsilon}^{s} \rho_\varepsilon(s - \varepsilon - r)f(s)dr = |f(s)|, \ \forall s \in \mathbb{R}.
\]
Moreover, for any $s \in C_f$,
\[
|f^\varepsilon(s) - f(s)| = \int_{s-2\varepsilon}^{s} \rho_\varepsilon(s - \varepsilon - r)f(r)dr - f(s) \\
= \int_{s-2\varepsilon}^{s} \rho_\varepsilon(s - \varepsilon - r)(f(r) - f(s))dr \\
\leq \sup_{r \in [s-2\varepsilon,s]} |f(r) - f(s)|
\]
which goes to 0 as $\varepsilon \to 0^+$. Thus we have proved the lemma for nonnegative $f$.

For the case $f$ is non-positive, we can define
\[
f^\varepsilon(s) = \int_{-\infty}^{+\infty} \rho_\varepsilon(s + \varepsilon - r)f(r)dr = \int_{s}^{s+2\varepsilon} \rho_\varepsilon(s + \varepsilon - r)f(r)dr,
\]
then by repeating the above argument we can prove the lemma for non-positive $f$.

When $f$ is a general nondecreasing function, we write $f = f^+ - f^-$, where $f^+(s) := \max\{f(s), 0\}$ and $f^-(s) := -\min\{f(s), 0\}$. According to the above discussion, we can choose two sequences of smooth functions $\{f^+_n\}$ and $\{f^-_n\}$ such that
\[
|f^+_n(s)| \leq |f^+(s)|, \ |f^-_n(s)| \leq |f^-(s)|, \ \forall s \in \mathbb{R},
\]
\[
\lim_{n \to +\infty} f^+_n(s) = f^+(s), \ \lim_{n \to +\infty} f^-_n(s) = f^-(s), \ \forall s \in C_f.
\]
Here we used the fact that $f^+$ and $f^-$ are both continuous on $C_f$. The lemma is proved by choosing $f_n = f^+_n - f^-_n$. 

\[ \square \]
Proof of Theorem 1.2: For \( i = 1, \cdots, k \), if \( f^i \) is a monotone function, by Lemma 2.1 we can choose a sequence of bounded and smooth functions \( \{ f^i_n \} \) such that

\[
|f^i_n(s)| \leq |f^i(s)|, \quad \forall s \in \mathbb{R}, \tag{2.1}
\]

\[
\lim_{n \to +\infty} f^i_n(s) = f^i(s), \quad \forall s \in C_f. \tag{2.2}
\]

If \( f^i \) is Lipschitz continuous, we can also choose a sequence of bounded Lipschitz functions \( \{ f^i_n \} \) satisfying (2.1)(2.2) by using truncation.

Since \( \omega \in L^{4/3}(D) \), by \( L^p \) estimate we have \( G\omega \in W^{2,4/3}(D) \), then by Sobolev embedding we obtain \( G\omega \in W^{1,4}(D) \). By the chain rule for Sobolev functions (see [13], 4.22), it is easy to verify that

\[
J\nabla G\omega \cdot \nabla (f^i_n(G\omega)) = (f^i_n)'(G\omega)J\nabla G\omega \cdot \nabla G\omega = 0 \quad \text{for a.e. } x \in \text{supp}(\omega_i)_\delta, \tag{2.3}
\]

where we used the fact \( J\nabla G\omega \cdot \nabla G\omega \equiv 0 \). Since \( |f^i_n(s)| \leq |f^i(s)| \) for each \( s \in \mathbb{R} \) and \( n = 1, 2, \cdots \), we deduce that

\[
\text{supp}(f^i_n(G\omega)) \cap \text{supp}(\omega_i)_\delta \subset \text{supp}(\omega_i).
\]

Define \( \omega_n = \sum_{i=1}^{k} f^i_n(G\omega)I_{\text{supp}(\omega_i)_\delta} \). Taking into account (1.10) and (2.3), we can easily check that \( \omega_n \) belongs to \( W^{1,4}(D) \) and satisfies

\[
|\omega_n| \leq \sum_{i=1}^{k} |f^i_n(G\omega)I_{\text{supp}(\omega_i)_\delta}| \leq \sum_{i=1}^{k} |f^i(G\omega)I_{\text{supp}(\omega_i)_\delta}| = |\omega| \quad \text{a.e. } x \in D, \tag{2.4}
\]

\[
J\nabla G\omega \cdot \nabla \omega_n = 0 \quad \text{a.e. in } D. \tag{2.5}
\]

Therefore we obtain

\[
\int_D \omega_n J\nabla G\omega \cdot \nabla \phi dx = 0, \quad \forall \phi \in C_c^\infty(D). \tag{2.6}
\]

Now we claim that

\[
\lim_{n \to +\infty} \omega_n = \omega \quad \text{a.e. in } D. \tag{2.7}
\]

In fact, it suffices to show that for each \( i \)

\[
\lim_{n \to +\infty} f^i_n(G\omega(x)) = f^i(G\omega(x)) \quad \text{for a.e. } x \in \text{supp}(\omega_i)_\delta.
\]

For \( x \in (G\omega)^{-1}(C_f) \), by (2.2) we have \( f^i_n(G\omega(x)) \to f^i(G\omega(x)) \). So we need just consider the case \( x \in (G\omega)^{-1}(D_f) \). Since each \( f^i \) is either monotone or Lipschitz continuous, the set \( D_f \) is countable, thus it suffices to show that for each \( s \in D_f \), there holds

\[
\lim_{n \to +\infty} f^i_n(G\omega(x)) = f^i(G\omega(x)) \quad \text{for a.e. } x \in (G\omega)^{-1}(s).
\]

To show this, first we use the property of Sobolev functions (see [13], 4.22) to obtain

\[
\omega = -\Delta G\omega = 0 \quad \text{a.e. on } (G\omega)^{-1}(s),
\]
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then by (2.4) we have

$$\omega_n = 0 \text{ a.e. on } (G\omega)^{-1}(s),$$

therefore

$$\lim_{n \to +\infty} \omega_n = \omega \text{ a.e. on } (G\omega)^{-1}(s).$$

Combining (2.4), (2.6) and (2.7), we are able to apply the dominated convergence theorem to obtain

$$\int_D \omega J \nabla G \omega \cdot \nabla \phi dx = 0,$$

which is the desired result.

Remark 2.2. According to the proof of Theorem 1.2, we need only impose the following two abstract conditions on $f^i$:

1. $D_{f^i}$ is a countable set;

2. there exist a sequence of bounded Lipschitz functions $\{f^i_n\}$ and a constant $C > 0$ such that

$$|f^i_n(s)| \leq C|f^i(s)|, \quad \forall s \in \mathbb{R},$$

$$\lim_{n \to +\infty} f^i_n(s) = f^i(s), \quad \forall s \in C_f.$$  

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References


