On 2D steady Euler flows with small vorticity near the boundary

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Abstract

In this paper, we investigate steady incompressible Euler flows with nonvanishing vorticity in a planar bounded domain. Let q be a harmonic function that corresponds to an irrotational flow. This paper proves that if q has k isolated local extremum points on the boundary, then there exist two kinds of steady Euler flows with small vorticity supported near these k points. For the first kind, near each *maximum* point the vorticity is *positive* and near each *minimum* point the vorticity is *negative*. For the second kind, near each *minimum* point the vorticity is *positive* and near each *maximum* point the vorticity is *negative*. Moreover, near these k points, the flow is characterized by a semilinear elliptic equation with a given profile function in terms of the stream function. The results are achieved by solving a certain variational problem for the vorticity and studying the limiting behavior of the extremizers.

Keywords: Steady Euler Flow, Vorticity, Variational Problem, Limiting Behavior
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1. Introduction

Let D be a bounded and smooth domain in $\mathbb{R}^2$. We consider the following steady incompressible Euler system in D

$$\begin{cases}(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla P, \quad x=(x^1,x^2)\in D,\\ \nabla\cdot\mathbf{v}=0,\end{cases} \tag{1.1}$$

with the boundary condition

$$\mathbf{v}\cdot\mathbf{n}=g, \quad x\in\partial D, \tag{1.2}$$

where $\mathbf{v}=(v^1,v^2)$ is the velocity field, P is the scalar pressure, and $\mathbf{n}$ is the unit outward normal to ∂D. Of course, the given function g is supposed to satisfy the following compatibility condition

$$\int_{\partial D} g\,d\sigma=0, \tag{1.3}$$

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where $d\sigma$ denotes the area unit on ∂D. This system describes the time-independent motion of an ideal fluid (like water) in D with unit density, and the boundary condition (1.2) means that the rate of mass flow across ∂D per unit area is g.

The scalar vorticity of the fluid $\omega = \partial_{x^1} v^2 - \partial_{x^2} v^1$ is the signed magnitude of the vorticity vector curl$\mathbf{v}$ and is a very fundamental quantity in the study of planar fluids. Taking the curl on both sides of the momentum equation (the first equation of (1.1)), we get

$$\mathbf{v} \cdot \nabla \omega = 0. \tag{1.4}$$

On the other hand, $\mathbf{v}$ can be recovered from ω. More precisely, when D is simply-connected, for given g satisfying (1.3), there is a unique $\mathbf{v}$ satisfying

$$\begin{cases} curl\mathbf{v} = \omega, & x \in D, \\ \nabla \cdot \mathbf{v} = 0, & x \in D, \\ \mathbf{v} \cdot \mathbf{n} = g & x \in \partial D. \end{cases} \tag{1.5}$$

In fact, since D is simply-connected and $\mathbf{v}$ is divergence-free, applying Green's theorem we can define a function ψ, called the stream function, such that $\mathbf{v} = (\partial_{x^2}\psi, -\partial_{x^1}\psi)$. To simplify the notation, in this paper we will use $\mathbf{b}^\perp$ to denote the clockwise rotation through $\pi/2$ for any planar vector $\mathbf{b}$, and $\nabla^\perp f$ to denote $(\nabla f)^\perp$ for any C^1 function defined on D. In this way, we have $\mathbf{v} = \nabla^\perp \psi$. Now it is easy to see that

$$\begin{cases} -\Delta \psi = \omega, & x \in D, \\ \nabla^\perp \psi \cdot \mathbf{n} = g, & x \in \partial D. \end{cases} \tag{1.6}$$

Let q be a solution of the problem

$$\begin{cases} -\Delta q = 0, & x \in D, \\ \nabla^\perp q \cdot \mathbf{n} = g, & x \in \partial D, \end{cases} \tag{1.7}$$

Then up to a constant the solution of (1.6) is given by

$$\psi = q + \mathcal{G}\omega,$$

where $\mathcal{G}\omega(x) := \int_D G(x,y)\omega(y)dy$, and G is the Green's function for $-\Delta$ in D with zero Dirichlet boundary condition. Therefore the unique solution to (1.5) is

$$\mathbf{v} = \nabla^\perp (q + \mathcal{G}\omega). \tag{1.8}$$

Taking into account (1.4), we get the following vorticity form of the Euler system

$$\nabla^\perp (q + \mathcal{G}\omega) \cdot \nabla \omega = 0. \tag{1.9}$$

In the rest of this paper, we assume that $q \in C^2(D) \cap C^1(\overline{D})$.

In many cases, the vorticity is not C^1 (not even continuous), so (1.9) has to be interpreted in weak sense. Here we only consider the case $\omega \in L^\infty(D)$, which is enough for later use.

Definition 1.1. Let $\omega \in L^\infty(D)$. If there holds

$$\int_D \omega \nabla^\perp (q + \mathcal{G}\omega) \cdot \nabla \phi dx = 0, \ \forall\, \phi \in C_c^\infty(D), \tag{1.10}$$

then ω is called a weak solution to the vorticity equation (1.9).

In the above definition, by a density argument, it is easy to check that "$\phi \in C_c^\infty(D)$" can be replaced by "$\phi \in W_0^{1,1}(D)$".

From now on, we will be focusing on solving the vorticity equation (1.9). Obviously $\omega \equiv 0$ automatically satisfies (1.10). Our aim in this paper is to investigate solutions with nonvanishing vorticity. This is a very interesting and important topic in the study of incompressible flows. There are already many results in the literature in this respect. See [8, 9, 10, 11, 13, 15, 16, 17, 21, 23, 24] and the references therein.

Roughly speaking, the steady Euler flows obtained in previous works can be divided into two types: the desingularization type and the perturbation type. For solutions of desingularization type, the vorticity is sufficiently supported in a finite number of disjoint regions of small diameter, and the integral of the vorticity in each of these small regions is a given nonzero real number. In other words, the vorticity is almost a finite sum of Dirac measures. Moreover, the limiting position of the support of the vorticity is completely determined by the geometry of the domain D and the boundary flux g. For example, if (1.9) has a family of solutions $\{\omega^\lambda\}$ with $\lambda > 0$ being a parameter, having the form

$$\omega^\lambda = \sum_{i=1}^k \omega_i^\lambda, \ \int_D \omega_i^\lambda dx = \kappa_i, \ supp(\omega_i^\lambda) \subset B_{r^\lambda}(\tilde{x}_i),$$

where $\kappa_1, \cdots, \kappa_k$ are k nonzero real numbers, $\tilde{x}_i \in D$, and $r^\lambda \to 0^+$ as $\lambda \to +\infty$, then according to the point vortex model (see [19] or [20] for example), $(\tilde{x}_1, \cdots, \tilde{x}_k)$ should be a critical point of the Kirchhoff-Routh function $\mathcal{W}_k$, defined by

$$\mathcal{W}_k(x_1, \cdots, x_k) = -\sum_{1 \le i < j \le k} \kappa_i \kappa_j G(x_i, x_j) + \frac{1}{2} \sum_{i=1}^k \kappa_i^2 h(x_i, x_i) - \sum_{i=1}^k \kappa_i q(x_i),$$

where $x_i \in D$ and $x_i \neq x_j$ if $i \neq j$, and $h(\cdot,\cdot)$ is the regular part of the Green's function, that is,

$$h(x,y) := -\frac{1}{2\pi} \ln|x-y| - G(x,y), \ x, y \in D.$$

In the past decades, solutions of desingularization type have been constructed via various methods. Related references include [8, 9, 10, 11, 15, 21, 24].

The other type of solutions, the perturbation type, is to construct a new solution with "small" vorticity near a given irrotational flow. Here by "small" we mean that the support and the integral of the vorticity are both small. Below are several works related to this issue. In

[17], Li–Yan–Yang proved existence of C^1 solutions to (1.9) with small *positive* vorticity near a given isolated local *maximum* point of q on ∂D, moreover, the vorticity satisfies

$$\omega^\lambda = \lambda f(\mathcal{G}\omega^\lambda + q), \tag{1.11}$$

where λ is a large positive parameter, and the profile function f satisfies some regularity and growing conditions. Note that in [17] the authors used opposite sign for q. In [16], Li–Peng used a reduction procedure to construct solutions with small *positive* vorticity supported near any given finite collection of isolated local *maximum* points of q on ∂D, and near each of these points the vorticity satisfies (1.11) with $f(s) = s_+^p$, $1 < p < +\infty$, $s_+ = \max\{s, 0\}$. In [13], Cao–Wang–Zhan considered the vortex patch case, that is, the vorticity has a jump discontinuity. They proved that for any finite collection of isolated local extremum (maximum or minimum) points of q on ∂D, there are two kinds of solutions. For the first kind, near each *maximum* point of q there is a small *positive* patch and near each *minimum* point of q there is a small *negative* patch. For the second kind, near each *minimum* point of q there is a small *positive* patch and near each *maximum* point of q there is a small *negative* patch. They obtained these solutions by solving some variational problems for the vorticity and studying the limiting behavior of the extremizers as the circulation vanishes. Recently Cao–Wang–Zhan in [14] modified the method in [13] to obtain more general Euler flows with small *positive* vorticity supported near a finite number of isolated local *maximum* points of q, and the stream function of the flow near each of these maximum points satisfies a semilinear elliptic equation with a given increasing profile function satisfying some mild conditions.

In this paper, we continue the study of steady Euler flows of perturbation type. Our main aim is to improve the result in [14], that is, we will show that any finite collection of isolated local extremum (not only maximum) points of q on the boundary generates a family of solutions with small vorticity. As in [13], we will construct two kinds of solutions. For the first kind, the *positive* part of the vorticity is supported near *maximum* points of q and the *negative* part of the vorticity is supported near *minimum* points of q. For the second kind, the *positive* part of the vorticity is supported near *minimum* points of q and the *negative* part of the vorticity is supported near *maximum* points of q. Moreover, near each extremum point the solution can be characterized by a semilinear elliptic equation with a given monotone profile function satisfied by the stream function. The precise statements of our main results will be given in Section 2.

Our strategy of constructing these solutions is mainly based on the method developed in [13] and [14], that is, we will maximize or minimize a suitable functional over some admissible class of vorticity. This is usually called the vorticity method, which was firstly established by Arnold in 1960s and later developed by many authors. See [2, 3, 4, 5, 6, 11, 23] and the references listed therein.

It is also worth mentioning that for the three dimensional case, Alber [1] and Tang–Xin [22] also studied existence of steady Euler flows near a given irrotational flow with some assumptions on the boundary flux.

This paper is organized as follows. In Section 2, we give some notations needed in the sequel and state our main results (Theorem 2.1 and Theorem 2.2). In Section 3, we prove Theorem

2.1 by solving a maximization problem and studying the limiting behavior of the maximizers. In Section 4, we sketch the proof of Theorem 2.2.

2. Main results

2.1. Notations

Throughout this paper, we assume that D is a bounded and simply-connected domain with smooth boundary. For any measurable function f defined on D, $supp(f)$ denotes the *essential support* of f, that is, the complement of the union of all open sets in which f vanishes. Obviously $supp(f)$ is always a compact set in $\mathbb{R}^2$. For any $s \in \mathbb{R}$, $\{x \in D \mid f(x) > s\}$ is abbreviated as $\{f > s\}$. For any subset $A \subset D$, I_A denotes the characteristic function of A, that is, $I_A(x) = 1$ if $x \in A$, and $I_A(x) = 0$ if $x \in D \setminus A$. For any $s \in \mathbb{R}$, $s_+ := \max\{s, 0\}$ and $s_- := \max\{-s, 0\}$.

Let k be a fixed positive integer and α be a fixed positive real number. Define

$$\mathbb{V}_k^\alpha := \{\vec{\kappa} \in \mathbb{R}^k \mid \vec{\kappa} = (\kappa_1, \cdots, \kappa_k),\ \kappa_i \neq 0 \text{ for } 1 \leq i \leq k,\ \frac{|\kappa_i|}{|\kappa_j|} \leq \alpha \text{ for } 1 \leq i, j \leq k\}. \tag{2.1}$$

We also define the norm of $\vec{\kappa} = (\kappa_1, \cdots, \kappa_k) \in \mathbb{V}_k^\alpha$ by

$$\|\vec{\kappa}\| := \sum_{i=1}^{k} |\kappa_i|.$$

It is easy to check that for any $\vec{\kappa} = (\kappa_1, \cdots, \kappa_k) \in \mathbb{V}_k^\alpha$, there holds

$$\frac{1}{k\alpha}\|\vec{\kappa}\| \leq |\kappa_i| \leq \|\vec{\kappa}\|,\ \forall\, i \in \{1, \cdots, k\}. \tag{2.2}$$

As mentioned in Section 1, the stream function of the flow we obtain in this paper satisfies "locally" a semilinear elliptic equation with a given profile function. Here we give the conditions imposed on these profiles functions. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function, where $\mathbb{R}_+$ is the set of all nonnegative real numbers. We make the following two assumptions on f.

(H1) $f(0) = 0$, f is continuous and strictly increasing on $\mathbb{R}_+$;

(H2) There exists $\delta_0 \in (0, 1)$ such that

$$\int_0^s f(r)dr \leq \delta_0 f(s)s,\ \forall\, s \geq 0.$$

By (H1)(H2), it is easy to check that $\lim_{s\to+\infty} f(s) = +\infty$, therefore f has an inverse function on $[0, +\infty)$, denoted by f^{-1}. By using the identity $\int_0^s f(r)dr + \int_0^{f(s)} f^{-1}(r)dr = sf(s)$ for all $s \geq 0$, one can easily check that (H2) is in fact equivalent to

(H2)′ There exists $\delta_1 \in (0,1)$ such that

$$F(s) \geq \delta_1 s f^{-1}(s),\ \forall\, s \geq 0,$$

where $F(s) = \int_0^s f^{-1}(r)dr, \forall\, s \geq 0$.

For instance, for any $p \in (0, +\infty)$, $f(s) = s^p$ satisfies (H1)(H2). In this case, $f^{-1}(s) = s^{1/p}$ and $F(s) = \frac{p}{p+1} s^{1+1/p}$.

Except for the profile function, we also need some assumptions on the L^∞ norm of the vorticity as the circulation vanishes. Let $\Lambda : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function. We make the following two assumptions.

(A1) $\lim_{s\to 0} \frac{\Lambda(s)}{s} = +\infty$;

(A2) There exists some $\gamma_0 > 0$ such that $\lim_{s\to 0} \Lambda(s) s^{\gamma_0} = 0$.

For example,

$$\Lambda(s) = \begin{cases} 1, & \text{if } s > 0 \\ -1, & \text{if } s < 0 \end{cases} \tag{2.3}$$

satisfies (A1)(A2). Note that by (A1), if $|s|$ is sufficiently small, we have $\Lambda(s) > 0$ if s is positive and $\Lambda(s) < 0$ if s is negative.

2.2. Main Results

As mentioned in Section 1, our aim in this paper is to prove existence of steady Euler flows with nonvanishing vorticity near a given irrotational flow.

Throughout this paper, m, n, k are three fixed positive integers satisfying $m + n = k$. We assume that $q \in C^2(D) \cap C^1(\overline{D})$ is a harmonic function (which corresponds to an irrotational flow), and q has m isolated local maximum points $\{\bar{x}_i, \cdots, \bar{x}_m\}$ and n isolated local minimum points $\{\bar{x}_{m+1}, \cdots, \bar{x}_k\}$ on ∂D. Since these extremum points are all isolated, we can choose a small positive number r_0 such that $\overline{D_i} \cap \overline{D_j} = \varnothing$ if $1 \leq i < j \leq k$, where $D_i := B_{r_0}(\bar{x}_i) \cap D$, and $\bar{x}_i$ is the unique maximum (if $1 \leq i \leq m$) or minimum (if $m+1 \leq i \leq k$) point of q over $\overline{D_i}$.

Our first theorem shows that there exists a family of steady Euler flows in which the positive part of the vorticity is supported near $\{\bar{x}_1, \cdots, \bar{x}_m\}$, and the negative part of the vorticity is supported near $\{\bar{x}_{m+1}, \cdots, \bar{x}_k\}$.

Theorem 2.1. *Let α be a fixed positive number and $\mathbb{V}_k^\alpha$ be defined by (2.1). Let $\Lambda_i : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $i = 1 \cdots, k$, be k functions satisfying (A1)(A2), $f_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1 \cdots, k$, be k functions satisfying (H1)(H2). Then there exists $\delta_0 > 0$, such that for any $\vec{\kappa} = (\kappa_1, \cdots, \kappa_k) \in \mathbb{V}_k^\alpha$ with $\|\vec{\kappa}\| < \delta_0$, $\kappa_i > 0$ if $1 \leq i \leq m$ and $\kappa_i < 0$ if $m+1 \leq i \leq k$, there exists a weak solution $\omega^{\vec{\kappa}}$ to (1.9) satisfying*

$$supp(\omega^{\vec{\kappa}}) \subset \cup_{i=1}^k \overline{D}_i,\ \int_{D_i} \omega^{\vec{\kappa}} dx = \kappa_i \text{ for } 1 \leq i \leq k, \tag{2.4}$$

$$\omega^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i\left((\mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}})_+ \right) \text{ in } D_i \text{ if } 1 \leq i \leq m, \tag{2.5}$$

$$\omega^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i\left((\mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}})_- \right) \text{ in } D_i \text{ if } m+1 \leq i \leq k, \tag{2.6}$$

where each $\mu_i^{\vec{\kappa}}$ *is a real number depending on* $\vec{\kappa}$. *Furthermore, for each* $i \in \{1, \cdots, k\}$,

$$\mu_i^{\vec{\kappa}} \to q(\bar{x}_i), \quad supp(\omega_i^{\vec{\kappa}}) \subset B_{o(1)}(\bar{x}_i) \tag{2.7}$$

as $\|\vec{\kappa}\| \to 0^+$.

Our second theorem shows that the set $\{\bar{x}_1, \cdots, \bar{x}_k\}$ also gives rise to a family of steady Euler flows in which the *negative* part of the vorticity is supported near $\{\bar{x}_i, \cdots, \bar{x}_m\}$, and the *positive* part of the vorticity is supported near $\{\bar{x}_{m+1}, \cdots, \bar{x}_k\}$,

Theorem 2.2. *Let* α *be a positive number and* $\mathbb{V}_k^\alpha$ *be defined by* (2.1). *Let* $\Lambda_i : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $i = 1 \cdot \cdot, k$, *be* k *functions satisfying (A1)(A2),* $f_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1 \cdot \cdot, k$, *be* k *functions satisfying (H1)(H2). Then there exists* $\delta_0 > 0$, *such that for any* $\vec{\kappa} = (\kappa_1, \cdots, \kappa_k) \in \mathbb{V}_k^\alpha$ *with* $\|\vec{\kappa}\| < \delta_0$, $\kappa_i < 0$ *if* $1 \le i \le m$ *and* $\kappa_i > 0$ *if* $m+1 \le i \le k$, *there exists a weak solution* $\omega^{\vec{\kappa}}$ *to* (1.9) *satisfying*

$$supp(\omega^{\vec{\kappa}}) \subset \cup_{i=1}^k \overline{D}_i, \ \int_{D_i} \omega^{\vec{\kappa}} dx = \kappa_i \ for \ 1 \le i \le k, \tag{2.8}$$

$$\omega^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i\left((\mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}})_+\right) \ in \ D_i \ if \ 1 \le i \le m, \tag{2.9}$$

$$\omega^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i\left((\mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}})_-\right) \ in \ D_i \ if \ m+1 \le i \le k, \tag{2.10}$$

where each $\mu_i^{\vec{\kappa}}$ *is a real number depending on* $\vec{\kappa}$. *Furthermore, for each* $i \in \{1, \cdots, k\}$,

$$\mu_i^{\vec{\kappa}} \to q(\bar{x}_i), \quad supp(\omega_i^{\vec{\kappa}}) \subset B_{o(1)}(\bar{x}_i) \tag{2.11}$$

as $\|\vec{\kappa}\| \to 0^+$.

Remark 2.3. In Theorem 2.1, in each D_i the vorticity is a non-decreasing function of the stream function; while in Theorem 2.2, in each D_i the vorticity is a non-increasing function of the stream function. This is due to the variational nature of these solutions. See Burton–Mcleod [7] for a general discussion.

Remark 2.4. For solutions of perturbation type, the location of the support of the vorticity is not arbitrary, but determined by q. To see this, let us consider a simple example. Let $\{w^\kappa\}$ be a family of solutions to (1.9) with κ being a small positive parameter, satisfying

$$0 \le w^\kappa \le C \text{ a.e. in } D, \ \int_D w^\kappa dx = \kappa, \ supp(w^\kappa) \subset B_{r^\kappa}(\tilde{x}_0), \tag{2.12}$$

where C is a positive number not depending on κ, $\tilde{x}_0 \in \overline{D}$, and $r^\kappa \to 0^+$ as $\kappa \to 0^+$. By (1.10), w^κ satisfies

$$\int_D w^\kappa \nabla^\perp (\mathcal{G}w^\kappa + q) \cdot \nabla \phi dx = 0$$

for any $\phi \in W_0^{1,1}(D)$. If $\tilde{x}_0$ is in the interior of D, then we choose $\phi(x) = \chi(x)\mathbf{b} \cdot x$, where $\mathbf{b}$ is an arbitrary planar vector and $\chi \in C_c^\infty(D)$, $\chi \equiv 1$ near x_0. Letting $\kappa \to 0^+$ it is easy to see that

$\mathbf{b}\cdot\nabla^{\perp}q(\tilde{x}_0)=0$, which implies $\nabla q(\tilde{x}_0)=\mathbf{0}$. If $\tilde{x}_0\in\partial D$, then we can choose $\phi(x)=d(x)$, where $d(x):=\mathrm{dist}(x,\partial D)$. Note that d is Lipschitz continuous and vanishes on ∂D. Letting $\kappa\to 0^+$ we deduce that $\nabla^{\perp}q(\tilde{x}_0)\cdot\mathbf{n}(\tilde{x}_0)=0$. This example indicates that for solutions of perturbation type, there are only two possibilities for the limiting location of the support of vorticity. For the interior case, to our knowledge, there is no result concerning the existence of solutions with small vorticity supported near a critical point of q in the literature. For the boundary case, a local maximum or minimum point of q on ∂D obviously satisfies $\nabla^{\perp}q\cdot\mathbf{n}=0$, which is exactly the situation we are concerned with in this paper.

3. Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1. The strategy is to solve a suitable maximization problem for the vorticity and studying the limiting behavior of the maximizers.

3.1. Maximization Problem

For brevity, throughout this section we denote

$$\mathbb{S}_k^{\alpha}:=\{\vec{\kappa}=(\kappa_1,\cdots,\kappa_k)\in\mathbb{V}_k^{\alpha}\mid \kappa_i>0 \text{ if } 1\leq i\leq m,\ \kappa_i<0 \text{ if } m+1\leq i\leq k\}.$$

For any $\vec{\kappa}\in\mathbb{S}_k^{\alpha}$, define

$$\begin{aligned}\mathcal{M}^{\vec{\kappa}}:=&\{\omega\in L^{\infty}(D)\mid supp(\omega)\subset\cup_{i=1}^{k}\overline{D}_i,\ 0\leq\omega\leq\Lambda_i(\kappa_i) \text{ a.e. in } D_i \text{ if } 1\leq i\leq m,\\ &\Lambda_i(\kappa_i)\leq\omega\leq 0 \text{ a.e. in } D_i \text{ if } m+1\leq i\leq k,\ \int_{D_i}\omega dx=\kappa_i \text{ for } 1\leq i\leq k\}.\end{aligned}\tag{3.1}$$

By (A1), it is easy to check that $\mathcal{M}^{\vec{\kappa}}$ is not empty if $\|\vec{\kappa}\|$ is sufficiently small. Also note that for any $\omega\in\mathcal{M}^{\vec{\kappa}}$, there holds $0\leq\frac{\omega}{\Lambda_i(\kappa_i)}\leq 1$ a.e. in D_i for any $1\leq i\leq k$. We consider the maximization of the following functional over $\mathcal{M}^{\kappa}$

$$\begin{aligned}\mathcal{P}(\omega):=&\frac{1}{2}\int_D\omega\mathcal{G}\omega dx+\int_D q\omega dx-\sum_{i=1}^{m}\int_{D_i}\Lambda_i(\kappa_i)F_i\left(\frac{\omega}{\Lambda_i(\kappa_i)}\right)dx\\ &+\sum_{i=m+1}^{k}\int_{D_i}\Lambda_i(\kappa_i)F_i\left(\frac{\omega}{\Lambda_i(\kappa_i)}\right)dx,\end{aligned}\tag{3.2}$$

where $F_i(s)=\int_0^s f_i^{-1}(r)dr,\forall\, s\geq 0$. To make it brief, we denote

$$\mathcal{E}(\omega):=\frac{1}{2}\int_D\omega\mathcal{G}\omega dx,\tag{3.3}$$

$$\mathcal{Q}(\omega):=\int_D q\omega dx,\tag{3.4}$$

$$\mathcal{F}_i(\omega):=\int_{D_i}\Lambda_i(\kappa_i)F_i\left(\frac{\omega}{\Lambda_i(\kappa_i)}\right)dx,\ 1\leq i\leq k,\tag{3.5}$$

thus

$$\mathcal{P}(\omega)=\mathcal{E}(\omega)+\mathcal{Q}(\omega)-\sum_{i=1}^{m}\mathcal{F}_i(\omega)+\sum_{i=m+1}^{k}\mathcal{F}_i(\omega),\ \forall\,\omega\in\mathcal{M}^{\vec{\kappa}}. \tag{3.6}$$

Since F_i is a convex function in $[0,+\infty)$, it is easy to check that $\mathcal{F}_i$ is convex if $1\leq i\leq m$, and is concave if $m+1\leq i\leq k$.

Lemma 3.1. *For fixed $\vec{\kappa}$, $\mathcal{P}$ is bounded from above attains a maximum value over $\mathcal{M}^{\vec{\kappa}}$.*

Proof. First we show that $\mathcal{P}$ is bounded from above over $\mathcal{M}^{\vec{\kappa}}$. For any $\omega\in\mathcal{M}^{\vec{\kappa}}$, we have

$$\mathcal{E}(\omega)\leq\frac{1}{2}\|\omega\|^2_{L^\infty(D)}\|G(\cdot,\cdot)\|_{L^1(D\times D)}\leq\frac{1}{2}\left(\sum_{i=1}^{k}|\Lambda_i(\kappa_i)|\right)^2\|G(\cdot,\cdot)\|_{L^1(D\times D)}, \tag{3.7}$$

$$\mathcal{Q}(\omega)=\sum_{i=1}^{m}\int_{D_i}q\omega dx+\sum_{i=m+1}^{k}\int_{D_i}q\omega dx\leq\sum_{i=1}^{m}\kappa_iq(\bar{x}_i)+\sum_{i=m+1}^{k}\kappa_iq(\bar{x}_i)=\sum_{i=1}^{k}\kappa_iq(\bar{x}_i), \tag{3.8}$$

$$\mathcal{F}_i(\omega)\geq 0\ \text{ if }\ 1\leq i\leq m,\quad \mathcal{F}_i(\omega)\leq 0\ \text{ if }\ m+1\leq i\leq k, \tag{3.9}$$

which implies that $\mathcal{P}$ is bounded from above for fixed $\vec{\kappa}$.

Now we prove that $\mathcal{P}$ attains its supremum. Let $\{\omega_n\}$, $n=1,2,\cdots$, be a sequence of $\mathcal{M}^{\vec{\kappa}}$ satisfying

$$\lim_{n\to+\infty}\mathcal{P}(\omega_n)=\sup_{\omega\in\mathcal{M}^{\vec{\kappa}}}\mathcal{P}(\omega). \tag{3.10}$$

Since $\{\omega_n\}$ is bounded in $L^\infty(D)$, there is a subsequence, still denoted by $\{\omega_n\}$, such that $\{\omega_n\}$ converges to $\omega^{\vec{\kappa}}$ weakly star in $L^\infty(D)$ for some $\omega^{\vec{\kappa}}\in L^\infty(D)$. It is not hard to check that $\omega^{\vec{\kappa}}\in\mathcal{M}^{\vec{\kappa}}$ (See also Lemma 3.1 in [13]). Below we show that $\omega^{\vec{\kappa}}$ is a maximizer. First for $\mathcal{E}$ and $\mathcal{Q}$, it is not hard to verify that

$$\lim_{n\to+\infty}\mathcal{E}(\omega_n)=\mathcal{E}(\omega^{\vec{\kappa}}),\quad \lim_{n\to+\infty}\mathcal{Q}(\omega_n)=\mathcal{Q}(\omega^{\vec{\kappa}}). \tag{3.11}$$

Here we used the definition of weak star convergence and the fact that $\mathcal{G}\omega_n\to\mathcal{G}\omega^{\vec{\kappa}}$ in $C(\overline{D})$ by Sobolev embedding and L^p estimate. For $\mathcal{F}_i$, $1\leq i\leq m$, since $\mathcal{F}_i$ is convex, one can check that $\mathcal{F}_i$ is lower semicontinuous in the weak star topology of $L^\infty(D)$ (see Lemma 2.1 in [14]), that is,

$$\mathcal{F}_i(\omega^{\vec{\kappa}})\leq\liminf_{n\to+\infty}\mathcal{F}_i(\omega_n),\quad 1\leq i\leq m, \tag{3.12}$$

Similarly,

$$\mathcal{F}_i(\omega^{\vec{\kappa}})\geq\limsup_{n\to+\infty}\mathcal{F}_i(\omega_n),\quad m+1\leq i\leq k. \tag{3.13}$$

From (3.11), (3.12) and (3.13), we obtain

$$\mathcal{P}(\omega^{\vec{\kappa}})\geq\limsup_{n\to+\infty}\mathcal{P}(\omega_n),$$

which together with (3.10) gives the desired result. $\square$

Note that uniqueness of the maximizer is open. Next we study the profile of a fixed maximizer.

Lemma 3.2. *Let $\omega^{\vec{\kappa}}$ be a maximizer of $\mathcal{P}$ over $\mathcal{M}^{\vec{\kappa}}$. Set $\omega_i^{\vec{\kappa}}=\omega^{\vec{\kappa}}I_{D_i}$, $i=1,\cdots,k$. Then*

$$\omega_i^{\vec{\kappa}}=\Lambda_i(\kappa_i)I_{\{\phi^{\vec{\kappa},i}\geq f_i^{-1}(1)\}\cap D_i}+\Lambda_i(\kappa_i)f_i(\phi_+^{\vec{\kappa},i})I_{\{0<\phi^{\vec{\kappa},i}<f_i^{-1}(1)\}\cap D_i},\ \ 1\leq i\leq m, \tag{3.14}$$

$$\omega_i^{\vec{\kappa}}=\Lambda_i(\kappa_i)I_{\{\phi^{\vec{\kappa},i}\leq -f_i^{-1}(1)\}\cap D_i}+\Lambda_i(\kappa_i)f_i(\phi_-^{\vec{\kappa},i})I_{\{-f_i^{-1}(1)<\phi^{\vec{\kappa},i}<0\}\cap D_i},\ \ m+1\leq i\leq k, \tag{3.15}$$

where each $\mu_i^{\vec{\kappa}}$ is a real number depending on $\vec{\kappa}$, and

$$\phi^{\vec{\kappa},i}:=\mathcal{G}\omega^{\vec{\kappa}}+q-\mu_i^{\vec{\kappa}}.$$

Proof. First we prove (3.14). For any fixed index i, $1\leq i\leq m$, define

$$\omega=\sum_{j=1,j\neq i}^{k}\omega_j^{\vec{\kappa}}+\omega_i,$$

where $\omega_i\in\mathcal{C}_i$ and

$$\mathcal{C}_i:=\{w\in L^\infty(D)\mid supp(w)\subset D_i,\ 0\leq w\leq\Lambda_i(\kappa_i)\text{ a.e. in }D_i,\ \int_D wdx=\kappa_i\}. \tag{3.16}$$

It is clear that $\omega\in\mathcal{M}^{\vec{\kappa}}$. Besides, it is easy to check that $\mathcal{M}^{\vec{\kappa}}$ is a convex set, so $\omega_s:=\omega^{\vec{\kappa}}+s(\omega-\omega^{\vec{\kappa}})\in\mathcal{M}^{\vec{\kappa}}$ for any $s\in[0,1]$. Therefore we have

$$\left.\frac{d\mathcal{P}(\omega_s)}{ds}\right|_{s=0^+}\leq 0 \tag{3.17}$$

On the other hand, we can calculate directly to obtain

$$\left.\frac{d\mathcal{P}(\omega_s)}{ds}\right|_{s=0^+}=\int_{D_i}\left(\mathcal{G}\omega^{\vec{\kappa}}+q-f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)\right)(\omega_i-\omega_i^{\vec{\kappa}})dx. \tag{3.18}$$

Therefore we get from (3.17) and (3.18) that

$$\int_{D_i}\left(\mathcal{G}\omega^{\vec{\kappa}}+q-f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)\right)\omega_i dx\leq\int_{D_i}\left(\mathcal{G}\omega^{\vec{\kappa}}+q-f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)\right)\omega_i^{\vec{\kappa}}dx.$$

Notice that $\omega_i\in\mathcal{C}_i$ is arbitrary, so $\omega_i^{\vec{\kappa}}$ is in fact a maximizer of the linear functional

$$\mathcal{I}_i(w):=\int_{D_i}\left(\mathcal{G}\omega^{\vec{\kappa}}+q-f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)\right)wdx$$

over $\mathcal{C}_i$. By an adaption of the bathtub principle (see Theorem 1.14 in [18]), $\omega_i^{\vec{\kappa}}$ must be of the form

$$\omega_i^{\vec{\kappa}}=\Lambda_i(\kappa_i)I_{\left\{\mathcal{G}\omega^{\vec{\kappa}}+q-f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)>\mu_i^{\vec{\kappa}}\right\}\cap D_i}+hI_{\left\{\mathcal{G}\omega^{\vec{\kappa}}+q-f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)=\mu_i^{\vec{\kappa}}\right\}\cap D_i},$$

where $\mu_i^{\vec{\kappa}}$ is a real number depending on $\vec{\kappa}$, and h is a measurable function such that $\omega_i^{\vec{\kappa}} \in \mathcal{C}_i$ (obviously $0 \le h \le \Lambda_i(\kappa_i)$ a.e. in D_i). Now it is easy to see that

$$
\begin{aligned}
&\mathcal{G}\omega^{\vec{\kappa}} + q - f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) \ge \mu_i^{\vec{\kappa}} \quad \text{on } \{\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i)\} \cap D_i,\\
&\mathcal{G}\omega^{\vec{\kappa}} + q - f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) = \mu_i^{\vec{\kappa}} \quad \text{on } \{0 < \omega_i^{\vec{\kappa}} < \Lambda_i(\kappa_i)\} \cap D_i,\\
&\mathcal{G}\omega^{\vec{\kappa}} + q - f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) \le \mu_i^{\vec{\kappa}} \quad \text{on } \{\omega_i^{\vec{\kappa}} = 0\} \cap D_i.
\end{aligned}
\tag{3.19}
$$

Set $\phi^{\vec{\kappa},i} := \mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}}$. From (3.19) we obtain

$$
\begin{aligned}
\phi^{\vec{\kappa},i} &\ge f_i^{-1}(1) \quad \text{on } \{\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i)\} \cap D_i,\\
\omega_i^{\vec{\kappa}} &= \Lambda_i(\kappa_i) f_i(\phi^{\vec{\kappa},i}) \quad \text{on } \{0 < \omega_i^{\vec{\kappa}} < \Lambda_i(\kappa_i)\} \cap D_i,\\
\phi^{\vec{\kappa},i} &\le 0 \quad \text{on } \{\omega_i^{\vec{\kappa}} = 0\} \cap D_i.
\end{aligned}
\tag{3.20}
$$

Thus (3.14) is verified.

Next we give the proof of (3.15), which is very similar to the one of (3.14). Let $m+1 \le i \le k$ be a fixed index. Define

$$
\mathcal{D}_i := \{w \in L^\infty(D) \mid supp(\omega_i) \subset D_i,\ \Lambda_i(\kappa_i) \le \omega_i \le 0 \text{ a.e. in } D_i,\ \int_D \omega_i dx = \kappa_i\}. \tag{3.21}
$$

Then a similar calculation gives

$$
\int_{D_i} \left(\mathcal{G}\omega^{\vec{\kappa}} + q + f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)\right) w dx \le \int_{D_i} \left(\mathcal{G}\omega^{\vec{\kappa}} + q + f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right)\right) \omega_i^{\vec{\kappa}} dx
$$

for any $w \in \mathcal{D}_i$. By bathtub principle, $\omega_i^{\vec{\kappa}}$ must be of the form

$$
\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i) I_{\left\{\mathcal{G}\omega^{\vec{\kappa}} + q + f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) < \mu_i^{\vec{\kappa}}\right\} \cap D_i} + h I_{\left\{\mathcal{G}\omega^{\vec{\kappa}} + q + f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) = \mu_i^{\vec{\kappa}}\right\} \cap D_i},
$$

where h satisfies $\Lambda_i(\kappa_i) \le h \le 0$ a.e. in D_i. This gives

$$
\begin{aligned}
&\mathcal{G}\omega^{\vec{\kappa}} + q + f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) \le \mu_i^{\vec{\kappa}} \quad \text{on } \{\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i)\} \cap D_i,\\
&\mathcal{G}\omega^{\vec{\kappa}} + q + f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) = \mu_i^{\vec{\kappa}} \quad \text{on } \{\Lambda_i(\kappa_i) < \omega_i^{\vec{\kappa}} < 0\} \cap D_i,\\
&\mathcal{G}\omega^{\vec{\kappa}} + q + f_i^{-1}\left(\frac{\omega_i^{\vec{\kappa}}}{\Lambda_i(\kappa_i)}\right) \ge \mu_i^{\vec{\kappa}} \quad \text{on } \{\omega_i^{\vec{\kappa}} = 0\} \cap D_i.
\end{aligned}
\tag{3.22}
$$

We still denote $\phi^{\vec{\kappa},i} := \mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}}$. By (3.22),

$$\begin{aligned} \phi^{\vec{\kappa},i} &\le -f_i^{-1}(1) &&\text{on } \{\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i)\} \cap D_i,\\ \omega_i^{\vec{\kappa}} &= \Lambda_i(\kappa_i) f_i(-\phi^{\vec{\kappa},i}) &&\text{on } \{\Lambda_i(\kappa_i) < \omega_i^{\vec{\kappa}} < 0\} \cap D_i,\\ \phi^{\vec{\kappa},i} &\ge 0 &&\text{on } \{\omega_i^{\vec{\kappa}} = 0\} \cap D_i, \end{aligned} \tag{3.23}$$

from which (3.15) clearly follows.

□

3.2. Limiting behavior

Now we turn to analyzing the limiting behavior $\omega^{\vec{\kappa}}$ as $\|\vec{\kappa}\| \to 0^+$. Our final purpose is to show that the support of $\omega_i^{\vec{\kappa}}$ "shrinks" to $\bar{x}_i$ for each $i \in \{1, \cdots, k\}$ as $\|\vec{\kappa}\| \to 0^+$. To this end, the key ingredient is to estimate the Lagrange multiplier $\mu_i^{\vec{\kappa}}$.

We begin with a lemma that is frequently used later on. For convenience, we shall use C to denote various positive numbers not depending on $\vec{\kappa}$, $o(1)$ to denote various quantities that go to zero as $\|\vec{\kappa}\| \to 0^+$, and $o(\|\vec{\kappa}\|)$ to denote $\|\vec{\kappa}\| o(1)$.

Lemma 3.3. *As $\|\vec{\kappa}\| \to 0^+$, there holds*

$$\sup_{\omega \in \mathcal{M}^{\vec{\kappa}}} \|\mathcal{G}\omega\|_{W^{1,2}(D)} = o(1), \quad \sup_{\omega \in \mathcal{M}^{\vec{\kappa}}} \|\mathcal{G}\omega\|_{L^\infty(D)} = o(1).$$

Proof. By Sobolev embedding, it is sufficient to prove that there exists some $p \in (1, +\infty)$ such that

$$\sup_{\omega \in \mathcal{M}^{\vec{\kappa}}} \|\mathcal{G}\omega\|_{W^{2,p}(D)} = o(1). \tag{3.24}$$

Let $p \in (1, +\infty)$ be an index to be determined later. For any $\omega \in \mathcal{M}^{\vec{\kappa}}$, by L^p estimate we have

$$\|\mathcal{G}\omega\|_{W^{2,p}(D)} \le C\|\omega\|_{L^p(D)} \le C\|\omega\|_{L^1(D)}^{1/p} \|\omega\|_{L^\infty(D)}^{1-1/p} \le \|\vec{\kappa}\|^{1/p} \left(\sum_{i=1}^k |\Lambda_i(\kappa_i)| \right)^{1-1/p}. \tag{3.25}$$

Taking (2.2) into consideration, we obtain

$$\|\mathcal{G}\omega\|_{W^{2,p}(D)} \le C \left(\sum_{i=1}^k \left(|\kappa_i|^{\frac{1}{p-1}} |\Lambda_i(\kappa_i)| \right) \right)^{1-1/p}. \tag{3.26}$$

Recalling (A2), by choosing $p = 1 + \gamma_0^{-1}$ we get the desired result.

□

Now we begin to estimate the Lagrange multiplier $\mu_i^{\vec{\kappa}}$. The following bound is straightforward.

Lemma 3.4. *As* $\|\vec{\kappa}\| \to 0^+$, *we have*

$$\mu_i^{\vec{\kappa}} \le q(\bar{x}_i) + o(1), \quad 1 \le i \le m, \tag{3.27}$$

$$\mu_i^{\vec{\kappa}} \ge q(\bar{x}_i) + o(1), \quad m+1 \le i \le k. \tag{3.28}$$

Proof. We only prove (3.27), since the proof of (3.28) is similar. For any fixed index i, $1 \le i \le m$, it is obvious by (3.14) that $\{\mathcal{G}\omega^{\vec{\kappa}} + q - \mu^{\vec{\kappa}} > 0\} \cap D_i$ is not empty. Consequently

$$\mu^{\vec{\kappa}} \le \|\mathcal{G}\omega^{\vec{\kappa}}\|_{L^\infty(D_i)} + \sup_{D_i} q \le q(\bar{x}_i) + o(1).$$

Here we used Lemma 3.3 and the fact that $\bar{x}_i$ is a maximum point of q over $\overline{D}_i$. □

The proofs of the inverse inequalities of (3.27) and (3.28) are a little involved. We begin with the following energy estimate.

Lemma 3.5. *As* $\|\vec{\kappa}\| \to 0^+$, *there holds*

$$\mathcal{P}(\omega^{\vec{\kappa}}) = \sum_{i=1}^{k} \kappa_i q(\bar{x}_i) + o(\|\vec{\kappa}\|).$$

Proof. First it is clear that

$$\begin{aligned}\mathcal{P}(\omega) &= \mathcal{E}(\omega) + \mathcal{Q}(\omega) - \sum_{i=1}^{m} \mathcal{F}_i(\omega) + \sum_{i=m+1}^{k} \mathcal{F}_i(\omega) \\ &\le \mathcal{E}(\omega) + \mathcal{Q}(\omega) \\ &\le \sum_{i=1}^{k} \kappa_i q(\bar{x}_i) + o(\|\vec{\kappa}\|).\end{aligned} \tag{3.29}$$

Here we used (3.8), (3.9) and Lemma 3.3. To finish the proof, it suffices to show that

$$\mathcal{P}(\omega^{\vec{\kappa}}) \ge \sum_{i=1}^{k} \kappa_i q(\bar{x}_i) + o(\|\vec{\kappa}\|). \tag{3.30}$$

The idea is to choose a suitable test function to compare the energy. Since D is a smooth domain, ∂D satisfies the interior sphere condition at each $\bar{x}_i \in \partial D$, $1 \le i \le k$. As a result, for $\|\vec{\kappa}\|$ sufficiently small, there exists a disc $B_{\varepsilon_i}(x_i^{\vec{\kappa}}) \subset D$, where $|x_i^{\vec{\kappa}} - \bar{x}_i| = \varepsilon_i$ and ε_i satisfies $\pi\varepsilon_i^2 = \sqrt{\kappa_i/\Lambda_i(\kappa_i)}$. Note that by (A1), $\varepsilon_i \to 0^+$ as $\|\vec{\kappa}\| \to 0^+$. Define

$$\tilde{\omega} = \sum_{i=1}^{m} \sqrt{\kappa_i \Lambda_i(\kappa_i)} I_{D \cap B_{\varepsilon_i}(x_i^{\vec{\kappa}})} - \sum_{i=m+1}^{k} \sqrt{\kappa_i \Lambda_i(\kappa_i)} I_{D \cap B_{\varepsilon_i}(x_i^{\vec{\kappa}})}.$$

Recalling (A1), we can easily verify that $\tilde{\omega} \in \mathcal{M}^{\vec{\kappa}}$ if $\|\vec{\kappa}\|$ is sufficiently small. Since $\omega^{\vec{\kappa}}$ is a maximizer, we have

$$\mathcal{P}(\omega^{\vec{\kappa}}) \geq \mathcal{P}(\tilde{\omega}). \tag{3.31}$$

On the other hand, it is easy to check that

$$\mathcal{E}(\tilde{\omega}) \geq 0, \quad \mathcal{Q}(\tilde{\omega}) = \sum_{i=1}^{k} \kappa_i q(\bar{x}_i) + o(1). \tag{3.32}$$

For each $\mathcal{F}_i$, a direct calculation gives

$$|\mathcal{F}_i(\tilde{\omega})| = |\kappa_i| \sqrt{\frac{\Lambda_i(\kappa_i)}{\kappa_i}} F_i\left(\sqrt{\frac{\kappa_i}{\Lambda_i(\kappa_i)}}\right), \ 1 \leq i \leq k. \tag{3.33}$$

Recalling (A1) and taking into the fact that $\lim_{s\to 0^+} F_i(s)/s = 0$, we get

$$|\mathcal{F}_i(\tilde{\omega})| = o(\|\vec{\kappa}\|),$$

which together with (3.31) and (3.32) leads to (3.30). Thus the lemma is proved. □

Lemma 3.6. *As $\|\vec{\kappa}\| \to 0^+$, for each $i \in \{1, \cdots, k\}$, there holds*

$$\mathcal{F}_i(\omega^{\vec{\kappa}}) = o(\|\vec{\kappa}\|). \tag{3.34}$$

Proof. First by Lemma 3.5, we have

$$\mathcal{E}(\omega^{\vec{\kappa}}) + \mathcal{Q}(\omega^{\vec{\kappa}}) - \sum_{i=1}^{m} \mathcal{F}_i(\omega^{\vec{\kappa}}) + \sum_{i=m+1}^{k} \mathcal{F}_i(\omega^{\vec{\kappa}}) = \sum_{i=1}^{k} \kappa_i q(\bar{x}_i) + o(\|\vec{\kappa}\|). \tag{3.35}$$

On the other hand,

$$\mathcal{E}(\omega^{\vec{\kappa}}) = o(\|\vec{\kappa}\|), \quad \mathcal{Q}(\omega^{\vec{\kappa}}) \leq \sum_{i=1}^{k} \kappa_i q(\bar{x}_i). \tag{3.36}$$

Combining (3.35) and (3.36) we get

$$-\sum_{i=1}^{m} \mathcal{F}_i(\omega^{\vec{\kappa}}) + \sum_{i=m+1}^{k} \mathcal{F}_i(\omega^{\vec{\kappa}}) \geq o(\|\vec{\kappa}\|). \tag{3.37}$$

Since $\mathcal{F}_i \geq 0$ for $1 \leq i \leq m$ and $\mathcal{F}_i \leq 0$ for $m+1 \leq i \leq k$, (3.37) clearly implies (3.34). □

Recall that $\phi^{\vec{\kappa},i} = \mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}}$. We have the following lemma.

Lemma 3.7. *For each $i \in \{1, \cdots, k\}$, there holds*

$$\int_D \omega_i^{\vec{\kappa}} \phi^{\vec{\kappa},i} dx = o(\|\vec{\kappa}\|). \tag{3.38}$$

Proof. First we prove (3.38) for $1 \leq i \leq m$. In this case, by Lemma 3.2 we have

$$\begin{aligned}\int_D \omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx &= \int_{\{0<\omega_i^{\vec\kappa}<\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx + \int_{\{\omega_i^{\vec\kappa}=\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx\\ &= \int_{\{0<\omega_i^{\vec\kappa}<\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx + \int_{\{\omega_i^{\vec\kappa}=\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx\\ &= \int_{D_i}\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx + \int_{\{\omega_i^{\vec\kappa}=\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}(\phi^{\vec\kappa,i}-f_i^{-1}(1))dx\\ &= \int_{D_i}\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx + \int_{D_i}\omega_i^{\vec\kappa}(\phi^{\vec\kappa,i}-f_i^{-1}(1))_+dx. \end{aligned} \tag{3.39}$$

For the first term in (3.39), recalling that f_i satisfies (H2)$'$, we deduce that

$$\begin{aligned}\int_{D_i}\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx &= \Lambda_i(\kappa_i)\int_{D_i}\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx\\ &\leq \frac{\Lambda_i(\kappa_i)}{\delta_1}\int_{D_i}F_i\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx\\ &= \frac{1}{\delta_1}\mathcal{F}_i(\omega^{\vec\kappa})\\ &= o(\|\vec\kappa\|).\end{aligned} \tag{3.40}$$

Here we used Lemma 3.6. Therefore, in order to prove (3.38), it is sufficient to prove

$$\int_{D_i}\omega_i^{\vec\kappa}\zeta_+^{\vec\kappa,i}dx = o(\|\vec\kappa\|), \tag{3.41}$$

where $\zeta^{\vec\kappa,i}=\phi^{\vec\kappa,i}-f_i^{-1}(1)$. To this end, we denote $A_i^{\vec\kappa}=\{\omega_i^{\vec\kappa}(x)=\Lambda_i(\kappa_i)\}\cap D_i$ and calculate as follows

$$\begin{aligned}\int_{D_i}\omega_i^{\vec\kappa}\zeta_+^{\vec\kappa,i}dx =&\Lambda_i(\kappa_i)\int_{A_i^{\vec\kappa}}\zeta_+^{\vec\kappa,i}dx\\ \leq&\Lambda_i(\kappa_i)|A_i^{\vec\kappa}|^{1/2}\left(\int_{D_i}|\zeta_+^{\vec\kappa,i}|^2dx\right)^{1/2} &(3.42)\\ \leq& C\Lambda_i(\kappa_i)|A_i^{\vec\kappa}|^{1/2}\int_{D_i}\left(\zeta_+^{\vec\kappa,i}+|\nabla\zeta_+^{\vec\kappa,i}|\right)dx &(3.43)\\ =&C|A_i^{\vec\kappa}|^{1/2}\int_{D_i}\omega_i^{\vec\kappa}\zeta_+^{\vec\kappa,i}dx + C\Lambda_i(\kappa_i)|A_i^{\vec\kappa}|^{1/2}\int_{D_i}|\nabla\zeta_+^{\vec\kappa,i}|dx. &(3.44)\end{aligned}$$

Here we used Hölder's inequality in (3.42) and Sobolev embedding $W^{1,1}(D_i)\hookrightarrow L^2(D_i)$ in (3.43). By (A1), we have $|A_i^{\vec\kappa}|\to 0^+$ as $\|\vec\kappa\|\to 0^+$, therefore (3.44) implies

$$\int_{D_i}\omega_i^{\vec\kappa}\zeta_+^{\vec\kappa,i}dx \leq C\Lambda_i(\kappa_i)|A_i^{\vec\kappa}|^{1/2}\int_{D_i}|\nabla\zeta_+^{\vec\kappa,i}|dx. \tag{3.45}$$

On the other hand, using Hölder's inequality and Lemma 3.3 we can estimate the righthand of (3.45) as follows

$$\begin{aligned}\Lambda_i(\kappa_i)|A_i^{\vec\kappa}|^{1/2}\int_{D_i}|\nabla\zeta_+^{\vec\kappa,i}|dx &\le \Lambda_i(\kappa_i)|A_i^{\vec\kappa}|\left(\int_{D_i}|\nabla\zeta_+^{\vec\kappa,i}|^2dx\right)^{1/2}\\ &\le C\kappa_i\left(\int_{A_i^{\vec\kappa}}|\nabla q|^2+|\nabla\mathcal{G}\omega^{\vec\kappa}|^2dx\right)^{1/2}\\ &= o(\|\vec\kappa\|),\end{aligned} \tag{3.46}$$

which together with (3.45) gives (3.41).

Next we give the proof of (3.38) for $m+1\le i\le k$. Although the procedure is similar, we present it below for readers' convenience. In this situation we have

$$\begin{aligned}\int_D\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx &= \int_{\{\Lambda_i(\kappa_i)<\omega_i^{\vec\kappa}<0\}\cap D_i}\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx+\int_{\{\omega_i^{\vec\kappa}=\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx\\ &=-\int_{\{\Lambda_i(\kappa_i)<\omega_i^{\vec\kappa}<0\}\cap D_i}\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx+\int_{\{\omega_i^{\vec\kappa}=\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx\\ &=-\int_{D_i}\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx+\int_{\{\omega_i^{\vec\kappa}=\Lambda_i(\kappa_i)\}\cap D_i}\omega_i^{\vec\kappa}(\phi^{\vec\kappa,i}+f_i^{-1}(1))dx\\ &=-\int_{D_i}\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx-\int_{D_i}\omega_i^{\vec\kappa}(\phi^{\vec\kappa,i}+f_i^{-1}(1))_-dx.\end{aligned} \tag{3.47}$$

Taking into account (H2)′ and Lemma 3.6, we can easily verify that

$$0\le\int_{D_i}-\omega_i^{\vec\kappa}f_i^{-1}\left(\frac{\omega_i^{\vec\kappa}}{\Lambda_i(\kappa_i)}\right)dx\le-\frac{1}{\delta_1}\mathcal{F}_i(\omega^{\vec\kappa})=o(\|\vec\kappa\|). \tag{3.48}$$

Consequently we only need to check that

$$-\int_{D_i}\omega_i^{\vec\kappa}\xi_-^{\vec\kappa,i}dx=o(\|\vec\kappa\|), \tag{3.49}$$

where $\xi^{\vec\kappa,i}=\phi^{\vec\kappa,i}+f_i^{-1}(1)$. Denote $B_i^{\vec\kappa}=\{\omega_i^{\vec\kappa}(x)=\Lambda_i(\kappa_i)\}\cap D_i$. Using Hölder's inequality and Sobolev embedding, we have

$$\begin{aligned}-\int_{D_i}\omega_i^{\vec\kappa}\xi_-^{\vec\kappa,i}dx &= -\Lambda_i(\kappa_i)\int_{B_i^{\vec\kappa}}\xi_-^{\vec\kappa,i}dx\\ &\le -\Lambda_i(\kappa_i)|B_i^{\vec\kappa}|^{1/2}\left(\int_{D_i}|\xi_-^{\vec\kappa,i}|^2dx\right)^{1/2} &(3.50)\\ &\le -C\Lambda_i(\kappa_i)|B_i^{\vec\kappa}|^{1/2}\int_{D_i}\left(\xi_-^{\vec\kappa,i}+|\nabla\xi_-^{\vec\kappa,i}|\right)dx &(3.51)\\ &= -C|A_i^{\vec\kappa}|^{1/2}\int_{D_i}\omega_i^{\vec\kappa}\xi_-^{\vec\kappa,i}dx-C\Lambda_i(\kappa_i)|B_i^{\vec\kappa}|^{1/2}\int_{D_i}|\nabla\xi_-^{\vec\kappa,i}|dx. &(3.52)\end{aligned}$$

Since $|B_i^{\vec\kappa}|\to 0^+$ as $\|\vec\kappa\|\to 0^+$, we deduce from (3.52) that

$$-\int_{D_i}\omega_i^{\vec\kappa}\xi_-^{\vec\kappa,i}dx\leq -C\Lambda_i(\kappa_i)|A_i^{\vec\kappa}|^{1/2}\int_{D_i}|\nabla\xi_-^{\vec\kappa,i}|dx. \tag{3.53}$$

But

$$\begin{aligned}-\Lambda_i(\kappa_i)|B_i^{\vec\kappa}|^{1/2}\int_{D_i}|\nabla\xi_-^{\vec\kappa,i}|dx &\leq -\Lambda_i(\kappa_i)|B_i^{\vec\kappa}|\left(\int_{D_i}|\nabla\xi_-^{\vec\kappa,i}|^2dx\right)^{1/2}\\ &\leq C|\kappa_i|\left(\int_{B_i^{\vec\kappa}}|\nabla q|^2+|\nabla\mathcal{G}\omega^{\vec\kappa}|^2dx\right)^{1/2}\\ &=o(\|\vec\kappa\|).\end{aligned} \tag{3.54}$$

From (3.53) and (3.54) we get (3.49). Thus the proof is completed.

□

Now we are in a position to derive the desired estimate for each $\mu_i^{\vec\kappa}$.

Lemma 3.8. $\mu_i^{\vec\kappa}=q(\bar x_i)+o(1)$.

Proof. By Lemma 3.4, we only need to show

$$\mu_i^{\vec\kappa}\geq q(\bar x_i)+o(1),\quad 1\leq i\leq m, \tag{3.55}$$

$$\mu_i^{\vec\kappa}\leq q(\bar x_i)+o(1),\quad m+1\leq i\leq k. \tag{3.56}$$

Notice that

$$\begin{aligned}\mathcal{P}(\omega^{\vec\kappa})&=\mathcal{E}(\omega^{\vec\kappa})+\mathcal{Q}(\omega^{\vec\kappa})-\sum_{i=1}^m\mathcal{F}_i(\omega^{\vec\kappa})+\sum_{i=m+1}^k\mathcal{F}_i(\omega^{\vec\kappa})\\ &=\sum_{i=1}^k\int_D\omega_i^{\vec\kappa}\phi^{\vec\kappa,i}dx+\sum_{i=1}^k\kappa_i\mu_i^{\vec\kappa}+o(\|\vec\kappa\|)\\ &=\sum_{i=1}^k\kappa_i\mu_i^{\vec\kappa}+o(\|\vec\kappa\|).\end{aligned} \tag{3.57}$$

Here we used Lemma 3.6 and Lemma 3.7. From (3.57) and Lemma 3.5 we get

$$\sum_{i=1}^k\kappa_i\mu_i^{\vec\kappa}=\sum_{i=1}^k\kappa_iq(\bar x_i)+o(\|\vec\kappa\|). \tag{3.58}$$

Now (3.55) and (3.56) are only easy consequences of (3.58) and Lemma 3.4. In fact, if $1 \le i \le m$, then $\kappa_i > 0$, thus by (3.58) and Lemma 3.4 we have

$$\begin{aligned}
\mu_i^{\vec{\kappa}} &= \frac{1}{\kappa_i}\left(\sum_{j=1}^{k}\kappa_j\mu_j^{\vec{\kappa}} - \sum_{j=1,j\neq i}^{k}\kappa_j\mu_j^{\vec{\kappa}}\right)\\
&= \frac{1}{\kappa_i}\left(\sum_{j=1}^{k}\kappa_j q(\bar{x}_j) + o(\|\vec{\kappa}\|) - \sum_{j=1,j\neq i}^{k}\kappa_j\mu_j^{\vec{\kappa}}\right)\\
&\ge \frac{1}{\kappa_i}\left(\sum_{j=1}^{k}\kappa_j q(\bar{x}_j) + o(\|\vec{\kappa}\|) - \sum_{j=1,j\neq i}^{k}\kappa_j q(\bar{x}_j)\right)\\
&= q(\bar{x}_i) + o(1).
\end{aligned}$$

Similarly, for $m+1 \le i \le k$ we have

$$\begin{aligned}
\mu_i^{\vec{\kappa}} &= \frac{1}{\kappa_i}\left(\sum_{j=1}^{k}\kappa_j\mu_j^{\vec{\kappa}} - \sum_{j=1,j\neq i}^{k}\kappa_j\mu_j^{\vec{\kappa}}\right)\\
&= \frac{1}{\kappa_i}\left(\sum_{j=1}^{k}\kappa_j q(\bar{x}_j) + o(\|\vec{\kappa}\|) - \sum_{j=1,j\neq i}^{k}\kappa_j\mu_j^{\vec{\kappa}}\right)\\
&\le \frac{1}{\kappa_i}\left(\sum_{j=1}^{k}\kappa_j q(\bar{x}_j) + o(\|\vec{\kappa}\|) - \sum_{j=1,j\neq i}^{k}\kappa_j q(\bar{x}_j)\right)\\
&= q(\bar{x}_i) + o(1).
\end{aligned}$$

Thus the lemma is proved.

□

Lemma 3.9. *If $\|\vec{\kappa}\|$ is sufficiently small, then $\{\phi^{\vec{\kappa},i} \ge f_i^{-1}(1)\} \cap D_i = \varnothing$ for each $i \in \{1,\cdots,k\}$, and consequently $\omega_i^{\vec{\kappa}}$ has the form*

$$\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i(\phi_+^{\vec{\kappa},i}) \quad \textit{if} \quad 1 \le i \le m, \tag{3.59}$$

$$\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i(\phi_-^{\vec{\kappa},i}) \quad \textit{if} \quad m+1 \le i \le k. \tag{3.60}$$

Proof. We only prove the case $1 \le i \le m$. Notice that $f_i^{-1}(1)$ is a positive number not depending on $\vec{\kappa}$. By Lemma 3.8 and the fact that $\|\mathcal{G}\omega^{\vec{\kappa}}\|_{L^\infty(D)} = o(1)$ (recall Lemma 3.3), there exists some $\delta_0 > 0$, such that for any $\|\vec{\kappa}\| < \delta_0$, there holds

$$\phi_i^{\vec{\kappa}} = \mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}} \le \frac{1}{2} f^{-1}(1) \ \text{ in } D_i.$$

Thus $\{\phi^{\vec{\kappa},i} \ge f_i^{-1}(1)\} \cap D_i$ is empty if $\|\vec{\kappa}\| < \delta_0$.

□

Lemma 3.10. *For each $i \in \{1, \cdots, k\}$, supp$(\omega_i^{\vec{\kappa}})$ shrinks to $\bar{x}_i$ as $\|\vec{\kappa}\| \to 0^+$, or more precisely, for any $\epsilon > 0$, there exists some $\delta > 0$, such that for any $\vec{\kappa} \in \mathbb{S}_k^\alpha$, $\|\vec{\kappa}\| < \delta$, we have supp$(\omega_i^{\vec{\kappa}}) \subset B_\epsilon(\bar{x}_i)$.*

Proof. We only prove the case $1 \le i \le m$. It suffices to show that $\{\phi^{\vec{\kappa},i} > 0\} \cap D_i$ shrinks to $\bar{x}_i$. We prove this by contradiction. Suppose that there exist $\epsilon_0 > 0$, $\{\vec{\kappa}_n\} \subset \mathbb{S}_k^\alpha$, $\{x_n\} \subset \{\phi^{\vec{\kappa}_n,i} > 0\} \cap D_i$, $n = 1, \cdots$, such that $\|\vec{\kappa}_n\| < 1/n$, but $|x_n - \bar{x}_i| \ge \epsilon_0$ for each n. Since $\bar{x}_i$ is the unique maximum point of q over $\overline{D}_i$, by continuity of q, there exists some $\epsilon_1 > 0$ such that $\sup_n q(x_n) < q(\bar{x}_i) - \epsilon_1$. Thus we have

$$0 < \mathcal{G}\omega^{\vec{\kappa}_n}(x_n) + q(x_n) - \mu_i^{\vec{\kappa}_n} \le \mathcal{G}\omega^{\vec{\kappa}_n}(x_n) + q(\bar{x}_i) - \epsilon_1 - \mu_i^{\vec{\kappa}_n}.$$

Letting $n \to +\infty$, we get

$$\limsup_{n \to +\infty} \mu_i^{\vec{\kappa}_n} \le q(\bar{x}_i) - \epsilon_1,$$

which obviously contradicts Lemma 3.8. □

3.3. Proof of Theorem 2.1

Having made enough preparations, now we are ready to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. We only need to prove that $\omega^{\vec{\kappa}}$ is a weak solution to the vorticity equation, that is, $\omega^{\vec{\kappa}}$ satisfies (1.10), if $\|\vec{\kappa}\|$ is sufficiently small, since other assertions in Theorem 2.1 have been verified in the last subsection.

Noting that by Lemma 3.10, the support of each $\omega_i^{\vec{\kappa}}$ is away from the boundary of D_i, thus we can apply Theorem 1.2 in [12] to show that $\omega^{\vec{\kappa}}$ is a weak solution. However, we prefer to prove this statement directly on account of the variational nature of $\omega^{\vec{\kappa}}$.

Let $\phi \in C_c^\infty(D)$. Consider the following ODE

$$\begin{cases} \frac{d\Phi_s(x)}{ds} = -\nabla^\perp \phi(\Phi_s(x)), \quad s \in \mathbb{R} \\ \Phi_0(x) = x. \end{cases} \tag{3.61}$$

Since $\nabla^\perp \phi$ is a smooth vector field with compact support, (3.61) admits a unique smooth solution for any $x \in D$. Thus (3.61) gives rise to a family of transformations $\{\Phi_s\}_{s\in\mathbb{R}}$ from D to D. Since $\nabla^\perp\phi$ is divergence-free, it is easy to see that Φ_s is area-preserving, that is, for any measurable set $A \subset D$, there holds $|\{\Phi_s(x) \mid x \in A\}| = |A|$, $\forall\, s \in \mathbb{R}$. Define $\omega_s(x) := \omega^{\vec{\kappa}}(\Phi_s(x))$. By Lemma 3.10 and the continuity of Φ_s, we have *supp*$(\omega_s) \subset \cup_{i=1}^k D_i$ if $|s|$ is sufficiently small. Thus it is easy to see that $\omega_s \in \mathcal{M}^{\vec{\kappa}}$ if $|s|$ is sufficiently small. As $s \to 0$ one can check that (see also (1.13) in [23])

$$\mathcal{P}(\omega_s) = \mathcal{P}(\omega^{\vec{\kappa}}) + s\int_D \omega^{\vec{\kappa}} \nabla^\perp(\mathcal{G}\omega^{\vec{\kappa}} + q) \cdot \nabla\phi dx + o(s), \tag{3.62}$$

which also implies that $\mathcal{P}(\omega_s)$ is differentiable at $s = 0$. On the other hand, since $\omega^{\vec{\kappa}}$ is a maximizer, we see that $s = 0$ is a local maximum point of $\mathcal{P}(\omega_s)$, which together with (3.62) gives

$$\int_D \omega^{\vec{\kappa}} \nabla^\perp(\mathcal{G}\omega^{\vec{\kappa}} + q) \cdot \nabla\phi dx = 0$$

if $\|\vec{\kappa}\|$ is sufficiently small.

□

4. Proof of Theorem 2.2

In this section, we sketch the proof of Theorem 2.2. As in Section 3, we need to solve a suitable minimization problem for the vorticity and studying the limiting behavior of the minimizers. Although the procedure is parallel to the one of Theorem 2.1, some details must be taken into careful consideration.

Set

$$\mathbb{T}_k^\alpha := \{\vec{\kappa} = (\kappa_1, \cdots, \kappa_k) \in \mathbb{V}_k^\alpha \mid \kappa_i < 0 \text{ if } 1 \le i \le m,\ \kappa_i > 0 \text{ if } m+1 \le i \le k\}.$$

For any $\vec{\kappa} \in \mathbb{T}_k^\alpha$, define

$$\begin{aligned}\mathcal{N}^{\vec{\kappa}} :=& \{\omega \in L^\infty(D) \mid supp(\omega) \subset \cup_{i=1}^k \overline{D}_i,\ \Lambda_i(\kappa_i) \le \omega \le 0 \text{ a.e. in } D_i \text{ if } 1 \le i \le m, \\ & 0 \le \omega \le \Lambda_i(\kappa_i) \text{ a.e. in } D_i \text{ if } m+1 \le i \le k,\ \int_{D_i} \omega dx = \kappa_i \text{ for } 1 \le i \le k\}.\end{aligned} \tag{4.1}$$

By (A1), $\mathcal{N}^{\vec{\kappa}}$ is not empty if $\|\vec{\kappa}\|$ is sufficiently small, and for any $\omega \in \mathcal{N}^{\vec{\kappa}}$ there holds $0 \le \frac{\omega}{\Lambda_i(\kappa_i)} \le 1$ a.e. in D_i, $1 \le i \le k$.

We consider the minimization problem of $\mathcal{P}$ (defined by (3.2) in Section 3) over $\mathcal{N}^{\vec{\kappa}}$.

The proof of the existence of a minimizer is almost identical to that in the maximization case. What is new here is that the minimizer is fact unique, which is due to the strict convexity of $\mathcal{P}$ over $\mathcal{N}^{\vec{\kappa}}$.

Lemma 4.1. *There is a unique minimizer $\omega^{\vec{\kappa}}$ of $\mathcal{P}$ over $\mathcal{N}^{\vec{\kappa}}$. Let $\omega_i^{\vec{\kappa}} = \omega^{\vec{\kappa}} I_{D_i}$, $1 \le i \le k$, then*

$$\begin{aligned}&\omega^{\vec{\kappa}} = \Lambda_i(\kappa_i) I_{\{\phi_i^{\vec{\kappa}}(x) \ge f_i^{-1}(1)\} \cap D_i} + \Lambda_i(\kappa_i) f_i(\phi_+^{\vec{\kappa},i}) I_{\{0 < \phi^{\vec{\kappa},i} < f_i^{-1}(1)\} \cap D_i} \quad \text{if } 1 \le i \le m, \\ &\omega_i^{\vec{\kappa}} = \Lambda_i(\kappa_i) I_{\{\phi^{\vec{\kappa},i} \le -f_i^{-1}(1)\} \cap D_i} + \Lambda_i(\kappa_i) f_i(\phi_-^{\vec{\kappa},i}) I_{\{-f_i^{-1}(1) < \phi_-^{\vec{\kappa},i}(x) < 0\} \cap D_i} \quad \text{if } m+1 \le i \le k,\end{aligned}$$

where each $\mu_i^{\vec{\kappa}}$ is a real number depending on $\vec{\kappa}$ and $\phi^{\vec{\kappa},i} := \mathcal{G}\omega^{\vec{\kappa}} + q - \mu_i^{\vec{\kappa}}$.

Proceeding as in Section 3, we have the following asymptotic estimates.

Lemma 4.2. *As $\|\vec{\kappa}\| \to 0^+$, we have*

(i) $\mathcal{P}(\omega^{\vec{\kappa}}) = \sum_{i=1}^k \kappa_i q(\bar{x}_i) + o(\|\vec{\kappa}\|)$;

(ii) For each $1 \le i \le k$, $\mathcal{F}_i(\omega^{\vec{\kappa}}) = o(\|\vec{\kappa}\|)$;

(iii) For each $1 \le i \le k$, $\mu_i^{\vec{\kappa}} = q(\bar{x}_i) + o(1)$;

(iv) For each $1 \le i \le k$, $supp(\omega_i^{\vec{\kappa}})$ shrinks to $\bar{x}_i$.

Using Lemma 4.2, we finally have

Lemma 4.3. *If $\|\vec{\kappa}\|$ is sufficiently small, then $\omega^{\vec{\kappa}}$ is a weak solution to the vorticity equation, moreover, $\omega^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i(\phi_+^{\vec{\kappa},i})$ a.e. in D_i for $1 \le i \le m$, and $\omega^{\vec{\kappa}} = \Lambda_i(\kappa_i) f_i(\phi_-^{\vec{\kappa},i})$ a.e. in D_i for $m+1 \le i \le k$.*

The proofs of Lemma 4.1–4.3 are analogous to those in Section 3, therefore we omit them here. Theorem 2.2 is an obvious consequence of these lemmas.

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