SYNCHRONIZATION OF NONUNIFORM KURAMOTO OSCILLATORS FOR POWER GRIDS WITH GENERAL CONNECTIVITY AND DAMPINGS

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ABSTRACT. We consider the synchronization problem of swing equations, a second-order nonuniform Kuramoto model, with *general connectivity and dampings*. This is motivated by its relevance to the dynamics of power grids. As an important topic in power grids, people have been paying special attention to the transient stability which concerns the system's ability to reach an acceptable synchronism after a transient disturbance. For this concern, an important problem is to determine whether the post-fault state is located in the basin of attraction of synchronous states (sync basin). Recently this issue is becoming more and more challenging since the highly stochastic renewable power sources exert more transient disturbances on the power grids with increasing size and complexity. In [*Proc. Natl. Acad. Sci. USA*, 110 (2013), pp. 2005-2010], it was pointed out that the sync basin is an important unsolved problem. In [*SIAM J. Control Optim.*, 52 (2014), pp. 2482-2511], an explicit estimate on the region of attraction of coupled oscillators with homogenous damping for network with diameter less than 2 was obtained. However, it turns out that these assumptions are too restrictive in many real situations. The purpose of this work is to study the emergence of synchronization and give an estimate for sync basin for the nonuniform Kurmaoto model on *connected graphs* with *general dampings*, which is the most general setting for a connected power grid. Our strategy is based on the gradient-like formulation and energy estimate.

1. INTRODUCTION

General background.- The synchronization of large populations of weakly coupled oscillators is very common in nature, and it has been extensively studied in various scientific communities such as physics [1, 31], biology [16], sociology [30], etc. The scientific interest in this topic can be traced back to Christiaan Huygens' report on coupled pendulum clocks [21]. However, its rigorous mathematical treatment was done by Winfree [38] and Kuramoto [24] only several decades ago. Since then, the Kuramoto model became a paradigm for synchronization and various extensions have been explored in scientific communities such as applied mathematics [5, 7, 8], engineering and control theory [9, 12, 13, 14], physics [1, 29, 31] and biology [16].

In the present work, we consider the synchronization of second-order nonuniform Kuramoto oscillators which is a typical model for power grids. As a complex large-scale system, the power grid has rich nonlinear dynamics, and its synchronization and transient stability are very important in real applications. Roughly speaking, the transient stability is concerned

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with the ability of a power network to settle into an acceptable steady state following a large disturbance. In recent years, renewable energy has fascinated not only the scientific community but also the industry. It is believed that the future power generations will rely increasingly on renewables such as wind and solar power. These renewable power sources are highly stochastic; thus, an increasing number of transient disturbances will act on increasingly complex power grids. As a consequence, it is a challenge to study complex power networks and their transient stability.

The Kuramoto models.- The classic Kuramoto model is given by the following equations:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \tag{1.1}$$

where θ_i and $\Omega_i \in \mathbb{R}$ are the phase and natural frequency of i-th oscillator, respectively. This model has been studied in many literature and a central problem is to look for conditions on the parameters and/or initial phase configurations leading to the existence or emergence of phase-locked states, see for example, [5, 9, 18, 22, 25, 36]. Another well-known model of first-order coupled oscillators is the non-uniform Kuramoto model

$$d_i\dot{\theta}_i = \Omega_i + \sum_{j=1}^{N} a_{ij}\sin(\theta_j - \theta_i), \tag{1.2}$$

where $d_i > 0$ and $\Omega_i \in \mathbb{R}$. The coefficients $a_{ij} \geq 0$ represents the coupling (influence) from j-th oscillator to i-th oscillators. We can define an associated underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$ such that $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{W} = \{(i,j) : a_{ij} > 0\}$. In [11, 12, 23], the synchronization condition for (1.2) was considered by using differential inequalities for the phase diameter (difference), spectrum in algebraic graph theory or cutest projections.

Second-order coupled oscillators.- The inertial effect was first conceived by Ermentrout [16] to explain the slow synchronization of certain biological systems; mathematically, incorporating the inertial effect is simply adding the second-order term $m\ddot{\theta}_i$ to (1.1) where $m > 0$ is the inertia constant. For mathematical results on this inertial model we refer to [6, 7, 8]. The bistability of patterns of synchrony in two coupled populations of inertial Kuramoto oscillators was considered in [2].

The standpoint of this paper is that the power grids can be described as a network of nonuniform oscillators with inertia. In the past decades, the relevance of coupled oscillators to the power grids has been reported in many literature and the study on certain properties of power grids through coupled oscillators has fascinated the scientific community in physics, engineering and applied mathematics [12, 15, 17, 26, 28, 33]. Precisely, a model of synchronous motors for power grids can be written as a second-order nonuniform Kuramoto model

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \Omega_i + \sum_{j=1}^{N} a_{ij}\sin(\theta_j - \theta_i), \qquad m_i, d_i > 0,$$

where the coupling between oscillators is symmetric (i.e., $a_{ij} = a_{ji}$) according to its physical background (see Subsection 2.1). This system is referred as the so-called "swing equations" in engineering community. If $\frac{m_i}{d_i} = \frac{m_j}{d_j}$ for all $i \neq j$, the system is said to have *homogeneous* dampings. The swing equations have been used in many references to study the dynamics of the power grids, for example, the paper [26] studies the coupled swing model with the assumption of homogeneous dampings. The connections between first and second-order models can be found by the topological conjugacy argument in [11] or singular perturbation approach in [12].

Motivations and problems.- The transient stability, in terms of power grids, is concerned with the system's ability to reach an acceptable synchronism after a major disturbance such as short circuit caused by lightning, large noises in power injections to network, a cyber attack at some generators in the network, or abrupt changes in environment. Then a fundamental problem, as pointed in the survey [35], is: *whether the post-fault state (when the disturbance is cleared) is located in the sync basin.* Thus, a relevant issue is to estimate the sync basin or find some criteria to determine that a given state is located in the basin.

In literature for transient stability, the approaches fall into two main categories: time-domain simulation method and direct method. The former (see [10] for example) directly adopts numerical integration to solve the state equations of power systems which requires a large amount of calculations and is thus hardly realized for large-scale power systems. Another problem with computational methods is that they usually do not give intuition about the role of different parameters of the system in the sync basin of the model. Thus they are not suitable for design problems in power networks. The latter, so-called direct method [3, 4], has been a celebrated approach for the transient stability analysis of power network with a few nodes. Another tool is based on the singular perturbation theory by which the second-order dynamics can be approximated by the first-order dynamics when the system is sufficiently strongly over-damped, i.e., the ratio d_i/m_i is sufficiently large [12]. In this literature the authors derived algebraic conditions that relate the synchronization to the underlying network topology. Unfortunately there is no formula in [12] to check whether a system with given inertia and damping coefficients is so strongly over-damped that the result can be applied. In recent literature [15], the authors pointed out that finding sync basin is an important unsolved problem. In the survey paper [13], it was also pointed out that the transient dynamics of second-order oscillator networks is a challenging open problem.

As far as the authors know, the explicit estimate on the sync basin for second-order nonuniform Kuramoto model is very rare. In [26], the authors considered a model with nice connectivity and homogeneous dampings; more precisely, the underlying graph has a diameter less than or equal to 2, and the ratios d_i/m_i for all i are equal. In [12], it is required that the system is sufficiently over-damped and therefore the basin for second-order model is approximated by that of the first-order model. In this paper, we will consider the swing equations with general connectivity and dampings.

Contributions.- The main contribution of this paper is to *estimate the sync basin for lossless power grids in a general setting: (1) the underlying graph is only connected, while completeness or denseness is not required; (2) the dampings can be homogeneous or inhomogeneous and the over-damped property is not required.* To the best of the authors' knowledge, this is the first rigorous study on this problem for such a general setting. We use a direct analysis on the dynamics of second-order nonuniform Kuramoto model and derive an *explicit* formula to estimate the sync basin.

Among the rigorous analysis of Kuramoto oscillators, a typical method is to study the dynamics of phase difference, for example, [5, 7, 9, 11, 26]. However, this approach is available only when the dampings are uniform or homogeneous and the underlying graph has a diameter less than or equal to 2. Thus, it fails for the current case. In this paper we will use the gradient-like formulation and energy method. Departing from the (physical) energy for the direct method in [3, 4, 35], we will construct a virtual energy functional which enables us to derive the boundedness of the trajectory. Then we can use Łojasiewicz's theory to derive the convergence.

Organization of paper.- In Section 2, we present the models, main result and some discussions. In Section 3, we give a proof to the main result. In Section 4 we present some simulations. Finally, Section 5 is devoted to a conclusion.

Notation:
$\langle \cdot, \cdot \rangle$— standard inner product in $\mathbb{R}^N$,
$\| \cdot \|$— Euclidean norm in $\mathbb{R}^N$,
$d(\mathcal{G})$— diameter of the graph $\mathcal{G}$,
$\ell^\infty(\mathbb{R}^+, \mathbb{R}^N) = \left\{ f : \mathbb{R}^+ \to \mathbb{R}^N \mid f \text{ is bounded} \right\}$, $\mathbb{R}^+ := \{t \in \mathbb{R} : t \geq 0\}$,
$\ell^{1,\infty}(\mathbb{R}^+, \mathbb{R}^N) = \left\{ f : \mathbb{R}^+ \to \mathbb{R}^N \mid f \text{ is differentiable}, f, f' \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N) \right\}$.

2. Preliminaries and main result

In this section, we present the model of power grids as a second-order Kuramoto-type model and introduce its gradient-like flow formulation. A preliminary inequality and our main result are also presented.

2.1. **Models.** A mathematical model for a lossless network-reduced power system [4, 15] can be defined by the following swing equations:

$$m_i \ddot{\theta}_i + d_i \dot{\theta}_i = P_{m,i} + \sum_{j=1}^N |V_i| \cdot |V_j| \cdot \Im(Y_{ij}) \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N, \quad t > 0. \tag{2.1}$$

Here θ_i and $\dot{\theta}_i$ are the rotor angle and frequency of the i-th generator, respectively. The parameters $P_{m,i} > 0$, $|V_i| > 0$, $m_i > 0$, and $d_i > 0$ are the effective power input, voltage level, generator inertia constant, and damping coefficient of the i-th generator, respectively. For $Y = (Y_{ij})$ we denote the symmetric nodal admittance matrix, and $\Im(Y_{ij})$ represents the susceptance of the transmission line between i and j. If the power network is subject to energy loss due to the transfer conductance, then it should be depicted by a phase shift in each coupling term [12]. We refer to [12, 15, 32] for more details or the derivation of (2.1) from physical principles. Let us denote $\Omega_i = P_{m,i}$ and $a_{ij} = |V_i| \cdot |V_j| \cdot \Im(Y_{ij})$. Then the system (2.1) becomes a second-order nonuniform Kuramoto oscillators

$$m_i \ddot{\theta}_i + d_i \dot{\theta}_i = \Omega_i + \sum_{j=1}^N a_{ij} \sin(\theta_j - \theta_i), \quad \text{with } a_{ij} = a_{ji}. \tag{2.2}$$

Here, the coupling between oscillators is symmetric since Y is a symmetric matrix.

We acknowledge that the above model includes only generators, while a real power network should contain both generators and loads. In power flow, loads can be modeled by different ways, for example, a system of first-order Kuramoto oscillators [15] or algebraic equations. On the other side, the loads of the system can be modeled as constant impedance loads, then one can use the Kron reduction to obtain the so-called "network-reduced" models where the loads are now involved in the transfer admittance [14, 37]. Thus, as a network-reduced model, (2.2) becomes an often studied mathematical model for power grids. On the other hand, the Kron reduction was studied in [14] and some properties are given. As an example, the New England power grid in [14] has a complete underlying graph after reduction. However, this is not always the case. For example, the Northern European power grid in [28] does not meet the nice connectivity required in [12, 26] after the Kron reduction; this can be seen by looking into the power flow chart in [28, Fig.4] together with the topological properties of Kron reduction in [14, Theorem III.4]. Therefore, it is worthy to consider the network-reduced model (2.2) with general connectivity.

Next, we recall some definitions for complete synchronization of coupled oscillators.

Definition 2.1. *(Synchronization and phase-locked states) Let $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$ be an ensemble of phases of Kuramoto oscillators.*

(1) The Kuramoto ensemble asymptotically exhibits complete frequency synchronization if and only if

$$\lim_{t\to\infty} |\omega_i(t) - \omega_j(t)| = 0, \quad \forall\, i \neq j.$$

Here, $\omega_i(t) := \dot\theta_i(t)$ is the frequency of i-th oscillator at time t.

(2) The Kuramoto ensemble asymptotically exhibits phase-locked state if and only if the relative phase differences converge to some constant asymptotically:

$$\lim_{t\to\infty} (\theta_i(t) - \theta_j(t)) = \theta_{ij}^{\infty}, \qquad \forall\, i \neq j.$$

2.2. **A macro-micro decomposition.** We notice that the system (2.2) can be rewritten as a system of first-order ODEs:

$$\begin{aligned} \dot\theta_i &= \omega_i, \qquad i = 1, 2, \dots, N, \quad t > 0, \\ \dot\omega_i &= \frac{1}{m_i}\left[-d_i\omega_i + \Omega_i + \sum_{j=1}^{N} a_{ij}\sin(\theta_j - \theta_i) \right]. \end{aligned}$$

We introduce micro-variables as follows:

$$\Omega_s := \frac{\sum_{i=1}^N \Omega_i}{\sum_{i=1}^N d_i}, \qquad \hat\theta_i := \theta_i - \Omega_s\, t.$$

Then we find $\ddot{\hat\theta}_i = \ddot\theta_i$, $\dot{\hat\theta}_i = \dot\theta_i - \Omega_s$, and the system (2.2) can be rewritten as

$$m_i\ddot{\hat\theta}_i + d_i\dot{\hat\theta}_i = \hat\Omega_i + \sum_{j=1}^{N} a_{ij}\sin(\hat\theta_j - \hat\theta_i) \quad \text{with} \quad \hat\Omega_i := \Omega_i - d_i\Omega_s, \tag{2.3}$$

where the "micro" natural frequencies $\hat\Omega_i$ sum to zero:

$$\sum_{i=1}^{N} \hat\Omega_i = 0.$$

In particular, if $\Omega_i/d_i = \Omega_j/d_j$ for all $i, j = 1, 2, \dots, N$, then we have $\hat\Omega_i = 0$ for each i and the equation (2.3) reduces to a system of coupled oscillators with identical natural frequencies:

$$m_i\ddot{\hat\theta}_i + d_i\dot{\hat\theta}_i = \sum_{j=1}^{N} a_{ij}\sin(\hat\theta_j - \hat\theta_i).$$

Note that the ensemble of micro-variables $(\hat{\theta}_1, \dots, \hat{\theta}_N)$ is a phase shift of the original ensemble $(\theta_1, \dots, \theta_N)$, thus, they share asymptotic properties as long as we concern the synchronization or phase-locking behavior. Moreover, the equations for θ_i and $\hat{\theta}_i$, i.e., (2.2) and (2.3), have the same form. Thus, we may consider (2.3) instead of (2.2) when we concern the synchronization problem. These observations enable us to assume, without loss of generality, the natural frequencies in (2.2) satisfy

$$\sum_{i=1}^{N} \Omega_i = 0. \tag{2.4}$$

Remark 2.1. *In* [8]*, the second-order model with uniform coefficients was considered:*

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i). \tag{2.5}$$

By introducing averages and fluctuations

$$\Omega_c = \frac{1}{N}\sum_{i=1}^{N} \Omega_i, \quad \tilde{\Omega}_i = \Omega_i - \Omega_c, \quad \theta_c = \frac{1}{N}\sum_{i=1}^{N} \theta_i, \quad \tilde{\theta}_i = \theta_i - \theta_c,$$

we can derive that

$$m\ddot{\tilde{\theta}}_i + \dot{\tilde{\theta}}_i = \tilde{\Omega}_i + \sum_{j=1}^{N} a_{ij} \sin(\tilde{\theta}_j - \tilde{\theta}_i),$$

where the "micro" terms satisfy:

$$\sum_{i=1}^{N} \tilde{\Omega}_i = 0, \quad \sum_{i=1}^{N} \tilde{\theta}_i = 0.$$

Therefore, in [8] *we can assume without loss of generality that* $\sum_{i=1}^{N} \theta_i = 0$*. We notice that this relation relies on the uniformity of the inertia and damping parameters, and plays an important role in the estimate there. Unfortunately, in this paper we cannot assume this relation for* (2.2)*. This is a main difference between the uniform case* (2.5) *and non-uniform case* (2.2)*, which means that the approach does not work for the present case. More precisely, the energy function in* [8]*, cannot work by simply replacing* (m, d) *by* (m_i, d_i)*. In this paper, we will construct a modified energy to overcome this difficulty.*

2.3. **An inequality on connected graphs.** Consider a symmetric and connected network, which is associated to a graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$. We note that the underlying network of power grids (2.2) is *undirected*, i.e., the adjacency matrix $A = \{a_{ij}\}$ is symmetric. We say a graph $\mathcal{G}$ is connected if for any pair of nodes $i, j \in \mathcal{V}$, there exists a shortest path from i to j, say

$$i = p_1 \to p_2 \to p_3 \to \dots \to p_{d_{ij}} = j, \quad (p_k, p_{k+1}) \in \mathcal{W}, \quad k = 1, 2, \dots, d_{ij} - 1.$$

In order for the complete synchronization of (2.2), we assume that the underlying graph $\mathcal{G}$ is *connected*. The following result, which connects the total deviations and the partial deviations along the edges in a connected graph, will be useful in the energy estimate. For its proof, we refer to [8].

Lemma 2.1. *(Total and partial phase deviations) Suppose that the graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$ is connected and let θ_i be the phase of the i-th oscillator. Then, for any ensemble of phases $(\theta_1, \dots, \theta_N)$, we have*

$$L_* \sum_{l,k=1}^{N} |\theta_l - \theta_k|^2 \leq \sum_{(l,k)\in\mathcal{W}} |\theta_l - \theta_k|^2 \leq \sum_{l,k=1}^{N} |\theta_l - \theta_k|^2,$$

where the positive constant L_* is given by

$$L_* := \frac{1}{1 + d(\mathcal{G})|\mathcal{W}^c|}, \quad \text{with} \quad d(\mathcal{G}) := \max_{1\leq i,j\leq N} d_{ij}. \tag{2.6}$$

Here $\mathcal{W}^c$ is the complement of edge set $\mathcal{W}$ in $\mathcal{V}\times\mathcal{V}$ and $|\mathcal{W}^c|$, denotes its cardinality.

Remark 2.2. *(1) L_* has a strictly positive lower bound as*

$$L_* \geq \frac{1}{1 + d(\mathcal{G})N^2}.$$

(2) If $\mathcal{G}$ is a complete graph, then obviously we have $L_* = 1$.

2.4. **Main result.** Based on Subsections 2.1 and 2.2, the network-reduced model (2.2) for power grids can be transformed to (2.2) together with the restriction (2.4), i.e.,

$$\begin{aligned} & m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \Omega_i + \sum_{j=1}^{N} a_{ij}\sin(\theta_j - \theta_i), \\ & \sum_{i=1}^{N}\Omega_i = 0, \quad a_{ij} = a_{ji}, \end{aligned} \tag{2.7}$$

with initial data

$$\theta(0) = \theta_0 = (\theta_{10}, \dots, \theta_{N0}), \quad \omega(0) = \omega_0 = (\omega_{10}, \dots, \omega_{N0}).$$

We denote $\theta := (\theta_1, \dots, \theta_N)$, $\omega := (\omega_1, \dots, \omega_N)$, and $\Omega := (\Omega_1, \dots, \Omega_N)$.

Before we introduce the main result, for readability we list some important notation used here or hereafter in the following table.

name	notation	name	notation
average phase	$\theta_c = \frac{1}{N}\sum_{i=1}^{N}\theta_i$	weighted sum of phases	$\theta_s = \sum_{i=1}^{N} d_i\theta_i$
average frequency	$\omega_c = \frac{1}{N}\sum_{i=1}^{N}\omega_i$	weighted sum of frequencies	$\omega_s = \sum_{i=1}^{N} m_i\omega_i$
extremal damping	$d_u = \max_{1\leq i\leq N} d_i$ $d_\ell = \min_{1\leq i\leq N} d_i$	extremal inertia	$m_u = \max_{1\leq i\leq N} m_i$ $m_\ell = \min_{1\leq i\leq N} m_i$
fluctuation of damping	$\hat{d}_i := d_i - \frac{1}{N}\sum_{i=1}^{N} d_i$	fluctuation of inertia	$\hat{m}_i := m_i - \frac{1}{N}\sum_{i=1}^{N} m_i$
extremal coupling strength	$a_u = \max\{a_{ij} : (j,i)\in\mathcal{W}\}$ $a_\ell = \min\{a_{ij} : (j,i)\in\mathcal{W}\}$	-	-

Next, we introduce our main hypotheses on the parameters and initial configurations below.

(**H1**) The underlying graph $\mathcal{G}$ is connected.

(**H2**) Let $D_0 \in (0,\pi)$ be given. The parameters satisfy

$$2\mathcal{R}_0 a_\ell L_* N > \lambda, \tag{2.8}$$

where L_* is given in Lemma 2.1, and

$$\mathcal{R}_0 := \frac{\sin D_0}{D_0}, \qquad \lambda := \frac{\sqrt{tr(\hat{D}^2)}}{\sqrt{N}} + \frac{2\sqrt{tr(\hat{M}^2)}}{\sqrt{N}} \tag{2.9}$$

with $\hat{D} = diag(\hat{d}_1, \hat{d}_2, \dots, \hat{d}_N)$, $\hat{M} = diag(\hat{m}_1, \hat{m}_2, \dots, \hat{m}_N)$, and $tr(\cdot)$ denoting the trace of a matrix.

(**H3**) For some $\varepsilon, \mu > 0$ with

$$\frac{\varepsilon}{\mu} < \frac{d_\ell}{2m_u + \lambda}, \quad \text{and} \quad \frac{2\sqrt{2}C_1 \max\{\varepsilon,\mu\}\|\Omega\|}{C_\ell C_0} < \frac{1}{2}D_0, \tag{2.10}$$

the initial data satisfy

$$\sqrt{\frac{\widetilde{\mathcal{E}}[\theta_0,\omega_0]}{C_0}} < \frac{1}{2}D_0, \tag{2.11}$$

where

$$\begin{aligned}
&C_0 := \min\left\{\frac{\mu m_\ell}{2}, \varepsilon d_\ell + 2\mu N a_\ell L_* \frac{1-\cos D_0}{D_0^2} - \frac{2\varepsilon^2 m_u}{\mu}\right\},\\
&C_1 := \max\left\{\frac{3\mu m_u}{2}, \varepsilon d_u + \frac{2\varepsilon^2 m_u d_u}{\mu d_\ell} + \mu N a_u\right\},\\
&C_\ell := \min\{2\mu d_\ell - 2\varepsilon m_u, 2\varepsilon\mathcal{R}_0 a_\ell L_* N\} - \varepsilon\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathcal{E}}[\theta_0,\omega_0] :=&\varepsilon\sum_{i=1}^N d_i(\theta_{i0}-\theta_{c0})^2 + 2\varepsilon\sum_{i=1}^N m_i(\theta_{i0}-\theta_{c0})\omega_{i0}\\
&+\mu\sum_{i=1}^N m_i\omega_{i0}^2 + \mu\sum_{i,j=1}^N a_{ij}\left(1-\cos(\theta_{i0}-\theta_{j0})\right)
\end{aligned} \tag{2.12}$$

with

$$\theta_{c0} := \frac{1}{N}\sum_{i=1}^N \theta_{i0}.$$

The main result of this paper is as follows.

Theorem 2.1. *(Main theorem for synchronization) Suppose that the hypotheses* (**H1**)-(**H3**) *hold and let $\theta(t)$ be the global solution to system (2.7). Then we have:*

(i) $|\theta_i(t) - \theta_j(t)| \le D_0$ for all $i, j = 1, 2, \dots, N$ and for all $t \ge 0$, where $D_0 \in (0,\pi)$ is the constant assigned in (**H2**).

(ii) $\theta(t)$ asymptotically exhibits complete frequency synchronization, more precisely,

$$\lim_{t\to\infty} \dot{\theta}_i(t) = 0, \qquad i = 1, 2, \dots, N.$$

(iii) $\theta(t)$ asymptotically exhibits phase-locked state.

Remark 2.3. *(1) The connectedness of the underlying graph is indeed necessary for synchronization. Otherwise, the oscillators from different components, in general, cannot be expected to synchronize.*
(2) The positivity of $\widetilde{\mathcal{E}}[\theta,\omega]$ *will be proved in Lemma 3.7.*

2.5. **Discussion and Comparison.** We first explain about the accessibility of the hypotheses (**H1**)-(**H3**). The connectivity of $\mathcal{G}$, i.e., (**H1**), guarantees the positivity of the constant L_*. The hypothesis (**H2**) can hold true when the variances of inertia and damping are small. Furthermore, the hypothesis (**H2**) together with the condition $\frac{\varepsilon}{\mu} < \frac{d_\ell}{2m_u+\lambda}$ guarantees that $C_0 > 0$ and $C_\ell > 0$. Finally, the second condition in (2.10) can be fulfilled when the size of (micro) natural frequencies $\|\Omega\|$ are small. In view of the macro-micro decomposition in Subsection 2.2, the smallness of $\|\Omega\|$ means that the ratios $P_{m,i}/d_i$ for all i are similar.

Compared to [12, 26], the advantage of our main result lies in at least two aspects. First, we study the power grid systems with general damping and inertia coefficients. However, the highly over-damped property is required in [12], and the analysis in [26] is strictly limited to the case of homogeneous dampings. Second, we are dealing with the system with most general network topology since the hypothesis (**H1**) is necessary for synchronization. In contrast, a crucial condition in [26] is that the underlying graph should have a diameter less than or equal to 2.

In [8], the authors considered the synchronization of second-order model with uniform inertia and damping (see (2.5) in Remark 2.1). Obviously, the current system (2.2) (or equivalently, (2.7)) covers the above model as a special case. If we apply Theorem 2.1 for (2.5), we find $\lambda = 0$ since $d_i = 1$ and $m_i = m$ for all $i \in \{1, 2, \ldots, N\}$. Hence, the assumption (**H2**) must hold true ; in this sense, Theorem 2.1 clearly improves the result in [8, Theorem 3.4].

Let us turn to our result. First, we would like to mention that Theorem 2.1 gives an estimate for the sync basin of (2.7), through an *explicit* formula. Supposing we are given a transient state of a power system in terms of transient phase differences and frequencies, it could be a hard problem to quickly determine whether this state is located in the sync basin, in particular when we are working with a large network. Nevertheless, by Theorem 2.1 we can predicate that the transient state is going to synchronize if it meets the framework in (**H1**)-(**H3**). Let us assume that the parameters m_i, d_i, Ω_i and a_{ij} are given, and the coefficients D_0, ε and μ are suitably pre-assigned. By Theorem 2.1, if the transient state is located in the set

$$\left\{(\theta,\omega) : \sqrt{\frac{\widetilde{\mathcal{E}}[\theta,\omega]}{C_0}} < \frac{1}{2}D_0\right\}, \tag{2.13}$$

then the system will be synchronized. Although the set looks complicated, it is still easy to operate since it can be fast-speed tested by carrying out simple operations with a computer.

Second, we could observe some interesting points from the statement of Theorem 2.1. We notice that the parametric condition (**H2**) becomes more flexible when we increase the constant L_* and/or decrease the constant λ. Recalling (2.6) we see that if one decreases the diameter of the graph or increase the number of arcs, then the value of L_* becomes larger and the parametric condition is relaxed. On the other hand, by (2.9), decreasing the variances of d_i and m_i also helps to relax this condition.

On the other hand, we acknowledge that our rigorous estimate is conservative, in particular for graphs with small $\frac{a_l}{a_u}$. Let $\mathcal{G}$ be a connected graph which satisfies the hypothesis (**H2**). If we put a redundant edge with a small weight, we realize that the condition (**H2**) and the initial condition (2.11) may become harder to satisfy. Actually, it will break finally as the weight of the redundant edge decreases to some small enough value. Therefore, our estimate is sensitive to the network "structure" in the sense that some connection is removed or added. In despite if this point, our estimate is continuously dependent on the network "strength" when the network structure is fixed. Another point is that the condition (**H2**) depends on the number of oscillators N. If we increase N, the condition (**H2**) becomes easier to satisfy, and smaller a_l can be allowed.

3. Proof of main result: convergence to phase-locked states

In this section, we give the proof of the main result, Theorem 2.1. Our main strategy can be summarized as follows. First, we present a gradient formulation of the system (2.7) and introduce some related estimates (see Subsection 3.1); they tell that the boundedness of $\theta(t)$ implies the emergence of phase-locking. Second, in order to prove the boundedness of $\theta(t)$, we construct a virtual energy functional and present several properties (see Subsection 3.2). Departing from [8, 19], our energy functional $\widetilde{\mathcal{E}}(t)$ involves the fluctuation of phases around their average $\theta_c(t)$. Finally, we combine the above estimates to prove Theorem 2.1 (see Subsection 3.3).

3.1. A gradient-like flow formulation.

In this part we present a formulation of system (2.2) as a second-order gradient-like flow if the adjacency matrix $A = (a_{ij})$ is symmetric. For the classic Kuramoto model, the potential for the gradient flow was first introduced in [34], which can be extended to the Kuramoto model with symmetric interactions.

Let $M := diag\{m_1, \ldots, m_N\}$ and $D := diag\{d_1, \ldots, d_N\}$. The following result was presented in [8]; we sketch the proof here for reader's convenience.

Lemma 3.1. *The system* (2.2) *is a second-order gradient-like system with a real analytical potential f, i.e.,*

$$M\ddot{\theta} + D\dot{\theta} = \nabla f(\theta), \tag{3.1}$$

if and only if the adjacency matrix $A = (a_{ij})$ is symmetric.

Proof. (i) Suppose that the matrix A is symmetric, i.e., $a_{ij} = a_{ji}$. We define $f : \mathbb{R}^N \to \mathbb{R}$ as

$$f(\theta) := \sum_{k=1}^{N} \Omega_k \theta_k + \frac{1}{2} \sum_{k,l=1}^{N} a_{kl} \cos(\theta_k - \theta_l). \tag{3.2}$$

It is clearly analytic in θ, and system (2.2) is a second-order gradient-like system (3.1) with the potential f defined in (3.2).

(ii) We now assume that the system (2.2) is a gradient system with an analytic potential f, i.e.,

$$\frac{\partial f(\theta)}{\partial \theta_i} = \Omega_i + \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N.$$

Then the potential f must satisfy $\dfrac{\partial^2 f}{\partial \theta_k \partial \theta_l} = \dfrac{\partial^2 f}{\partial \theta_l \partial \theta_k}$ for $l \neq k$. This concludes $a_{lk} = a_{kl}$ for all $l, k \in \{1, 2, \ldots, N\}$. □

We next present a convergence result for the second-order gradient-like system on $\mathbb{R}^N$:

$$M\ddot{\theta} + D\dot{\theta} = \nabla f(\theta), \quad \theta \in \mathbb{R}^N, \quad t \geq 0. \tag{3.3}$$

Note that the set of equilibria $\mathcal{S}$ coincides with the set of critical points of the potential f:

$$\mathcal{S} := \{\theta \in \mathbb{R}^N : \ \nabla f(\theta) = 0\}.$$

Based on the celebrated theory of Łojasiewicz [27], a convergence result of the gradient-like system with uniform inertia was established in [20]; as a slight extension the following result was given in [26].

Lemma 3.2. [26] *Assume that f is analytic and let $\theta = \theta(t)$ be a global solution of* (3.3)*. If $\theta(\cdot) \in \ell^{1,\infty}(\mathbb{R}^+, \mathbb{R}^N)$, i.e., $\theta(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$ and $\dot{\theta}(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$, then there exists an equilibrium $\theta_e \in \mathcal{S}$ such that*

$$\lim_{t\to+\infty} \Big\{ \|\dot{\theta}(t)\| + \|\theta(t) - \theta_e\| \Big\} = 0.$$

Remark 3.1. *By Lemma 3.2, if we can prove the boundedness of position (or phase) and velocity (or frequency) for the solution of* (3.3)*, then we immediately obtain their convergences.*

Before we proceed, we first clarify that the Kuramoto oscillators are treated, in this paper, as a dynamic system on the whole space $\mathbb{R}^N$. Indeed, one can consider it as a system on the N-torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ since the coupling function $\sin(\cdot)$ is 2π-periodic. However, in order to apply the Łojasiewicz's theory, we should treat (2.2) as a system on $\mathbb{R}^N$. For more details on Łojasiewicz's theory and applications, please refer to [8, 20, 25, 26].

By Lemma 3.1, the system (2.2) is a special case of general second-order gradient-like systems. This enables us to obtain *a priori* result for convergence of system (2.2) by using Lemma 3.2. However, when we consider (2.2) for general natural frequencies with $\sum_{i=1}^N \Omega_i \neq 0$, we cannot expect $\theta(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$, since the right hand side of (2.2) sums to $\sum_{i=1}^N \Omega_i \neq 0$. This is why we need to apply the macro-micro decomposition and introduce the micro-variables in Section 2.2, which allows us to assume without loss of any generality that $\sum_{i=1}^N \Omega_i = 0$. In the following context, we will work with system (2.7).

Lemma 3.3. *(Boundedness implies convergence) Let $\theta = \theta(t)$ be a solution to* (2.7) *in $\ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$. Then there exists an equilibrium θ^∞ such that*

$$\lim_{t\to\infty} \{\|\dot{\theta}(t)\| + \|\theta(t) - \theta^\infty\|\} = 0.$$

Proof. By Lemma 3.1, the system (2.7) is a gradient like system (3.3). By Lemma 3.2, to show the desired result it suffices to show that

$$\dot{\theta}(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N).$$

Next we verify this relation.

It follows from (2.7) that ω_i satisfies

$$m_i\dot{\omega}_i + d_i\omega_i = \Omega_i + \sum_{j=1}^N a_{ij} \sin(\theta_j - \theta_i) \leq |\Omega_i| + \sum_{j=1}^N a_{ij}.$$

Note that ω_i is an analytic function of t. This implies that the zero-set $\{t : \omega_i(t) = 0\}$ is countable and finite in any finite time-interval, i.e., $|\omega_i(t)|$ is piecewise differentiable and continuous. We multiply the above relation by $\mathrm{sgn}(\omega_i)$ and divide it by $m_i > 0$ to get

$$\frac{d|\omega_i|}{dt} + \frac{d_i}{m_i}|\omega_i| \le \frac{1}{m_i}\left(|\Omega_i| + \sum_{j=1}^N a_{ij}\right), \quad \text{a.e. } t \ge 0.$$

We now use Gronwall inequality and continuity of $|\omega_i|$ to obtain that for all $t > 0$,

$$|\omega_i(t)| \le |\omega_i(0)|e^{-\frac{d_i}{m_i}t} + \frac{1}{d_i}\left(|\Omega_i| + \sum_{j=1}^N a_{ij}\right)\left(1 - e^{-\frac{d_i}{m_i}t}\right) \le |\omega_i(0)| + \frac{1}{d_i}\left(|\Omega_i| + \sum_{j=1}^N a_{ij}\right).$$

□

By Lemma 3.3, to prove the emergence of phase-locking for (2.7), it suffices to show $\theta(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$. In the following subsection we will construct an energy functional and present some estimates, which lead to the boundedness of phase.

3.2. **Construction of the energy functional $\widetilde{\mathcal{E}}$.** The proof of boundedness of phase along the trajectory will rely on a virtual energy and its estimate. This method was employed in [8] to study the second-order model with uniform parameters, i.e., (2.5). In [19] this approach was also used to study the practical synchronization of second-order oscillators with heterogeneous dynamics and uniform parameters.

Inspired by the approach in [8, 19], we first consider a temporal energy functional $\mathcal{E}$:

$$\mathcal{E}[\theta,\omega] := \underbrace{\varepsilon\sum_{i=1}^N d_i\theta_i^2 + 2\varepsilon\sum_{i=1}^N m_i\theta_i\omega_i + \mu\sum_{i=1}^N m_i\omega_i^2}_{\mathcal{E}_1[\theta,\omega]} + \underbrace{\mu\sum_{i,j=1}^N a_{ij}\left(1-\cos(\theta_i-\theta_j)\right)}_{\mathcal{E}_2[\theta]}, \tag{3.4}$$

where ε and μ are positive constants. In order for the boundedness of phase for (2.7), we need to derive some basic properties for $\mathcal{E}[\theta,\omega]$ and some estimates on the evolution of $\mathcal{E}[\theta,\omega]$ along the flow (2.7).

First we can find the equivalence between $\mathcal{E}_1[\theta,\omega]$ and $\|\theta\|^2 + \|\omega\|^2$.

Lemma 3.4. *Let ε and μ be constants with $\frac{\varepsilon}{\mu} < \frac{d_\ell}{2m_u}$. Then we have the following relation:*

$$\bar{C}_0(\|\theta\|^2 + \|\omega\|^2) \le \mathcal{E}_1[\theta,\omega] \le \bar{C}_1(\|\theta\|^2 + \|\omega\|^2), \quad \forall\, \theta,\omega \in \mathbb{R}^N,$$

where $\bar{C}_0$ and $\bar{C}_1$ are positive constants (independent of (θ,ω)) given by

$$\bar{C}_0 := \min\left\{\frac{\mu m_\ell}{2}, \varepsilon d_\ell - \frac{2\varepsilon^2 m_u}{\mu}\right\} \quad \textit{and} \quad \bar{C}_1 := \max\left\{\frac{3\mu m_u}{2}, \varepsilon d_u + \frac{2\varepsilon^2 m_u d_u}{\mu d_\ell}\right\}.$$

Proof. In (3.4), the cross term $\theta_i\omega_i$ can be estimated by Young's inequality:

$$|\theta_i\omega_i| \le \frac{\varepsilon}{\mu}\theta_i^2 + \frac{\mu}{4\varepsilon}\omega_i^2.$$

Then, we have

$$2\varepsilon m_i|\theta_i\omega_i| \le 2\frac{\varepsilon^2}{\mu}m_i\theta_i^2 + \frac{\mu m_i}{2}\omega_i^2 \le 2\frac{\varepsilon^2}{\mu}\frac{m_u}{d_\ell}d_i\theta_i^2 + \frac{\mu m_i}{2}\omega_i^2.$$

Therefore,

$$\mu\sum_{i=1}^N \frac{m_i}{2}\omega_i^2 + \left(\varepsilon d_\ell - \frac{2\varepsilon^2 m_u}{\mu}\right)\sum_{i=1}^N \theta_i^2 \le \mathcal{E}_1[\theta,\omega] \le \mu\sum_{i=1}^N \frac{3m_i}{2}\omega_i^2 + \left(\varepsilon d_u + \frac{2\varepsilon^2 m_u d_u}{\mu d_\ell}\right)\sum_{i=1}^N \theta_i^2.$$

This gives the desired result. □

For the term $\mathcal{E}_2[\theta]$, we cannot expect the equivalence relation between $\mathcal{E}_2[\theta]$ and $\|\theta\|^2$. Instead, we have the following lemma.

Lemma 3.5. *Let $D_0 \in (0,\pi)$ and suppose that*

$$\max_{1\le i,j\le N} |\theta_i - \theta_j| \le D_0.$$

Then we have the following relation:

$$\hat{C}_0\|\theta-\theta_c\|^2 \le \mathcal{E}_2[\theta] \le \mu N a_u \|\theta-\theta_c\|^2,$$

where $\hat{C}_0 = 2\mu N a_\ell L_* \frac{1-\cos D_0}{D_0^2}$ and $\theta-\theta_c$ is understood as the vector $\theta-\theta_c := (\theta_1,\dots,\theta_N) - (\theta_c,\dots,\theta_c)$.

Proof. Since $\theta_i - \theta_j \in [-D_0, D_0]$, we have

$$\frac{1-\cos D_0}{D_0^2}|\theta_i-\theta_j|^2 \le 1-\cos(\theta_i-\theta_j) \le \frac{1}{2}|\theta_i-\theta_j|^2. \tag{3.5}$$

Here, the left inequality relies on the fact that $x \mapsto \frac{1-\cos x}{x^2}$ is an even function which is monotonically decreasing on $(0,\pi)$. We then use (3.5) to estimate $\mathcal{E}_2[\theta]$ as follows:

$$\begin{aligned}
\mathcal{E}_2[\theta] &\le \mu \sum_{i,j=1}^N a_u(1-\cos(\theta_i-\theta_j)) \le \frac{\mu a_u}{2}\sum_{i,j=1}^N |\theta_i-\theta_j|^2 = \mu N a_u \|\theta-\theta_c\|^2,\\
\mathcal{E}_2[\theta] &\ge \mu \sum_{(i,j)\in\mathcal{W}} a_\ell(1-\cos(\theta_i-\theta_j)) \ge \mu a_\ell \frac{1-\cos D_0}{D_0^2}\sum_{(i,j)\in\mathcal{W}} |\theta_i-\theta_j|^2\\
&\ge \mu a_\ell L_* \frac{1-\cos D_0}{D_0^2}\sum_{i,j=1}^N |\theta_i-\theta_j|^2 = 2\mu N a_\ell L_* \frac{1-\cos D_0}{D_0^2}\|\theta-\theta_c\|^2.
\end{aligned}$$

Here we used the relation

$$\sum_{i,j=1}^N |\theta_i-\theta_j|^2 = 2N\|\theta-\theta_c\|^2. \tag{3.6}$$

□

Next we turn to consider the evolution of $\mathcal{E}[\theta,\omega]$ along the flow (2.7). The following lemma will be useful.

Lemma 3.6. *Let $D_0 \in (0,\pi)$ and suppose that the phase configuration $\{\theta_i\}_{i=1}^N$ satisfies*

$$\max_{1\le i,j\le N} |\theta_i - \theta_j| \le D_0.$$

Then we have

$$\sum_{(i,j)\in\mathcal{W}} a_{ij}\sin(\theta_j-\theta_i)(\theta_j-\theta_i) \ge 2\mathcal{R}_0 a_\ell L_* N\|\theta-\theta_c\|^2,$$

where $\mathcal{R}_0$ is given by $\mathcal{R}_0 := \frac{\sin D_0}{D_0}$.

Proof. It follows from the assumption

$$\max_{1\le i,j\le N} |\theta_j - \theta_i| \le D_0 < \pi,$$

and the simple relation

$$x \sin x \ge \mathcal{R}_0 x^2 \quad \text{for} \quad x \in [-D_0, D_0],$$

that

$$\begin{aligned}\sum_{(i,j)\in\mathcal{W}} a_{ij}\sin(\theta_j-\theta_i)(\theta_j-\theta_i) &\ge \mathcal{R}_0 \sum_{(i,j)\in\mathcal{W}} a_{ij}|\theta_j-\theta_i|^2\\ &\ge \mathcal{R}_0 a_\ell L_* \sum_{1\le i,j\le N} |\theta_j-\theta_i|^2 = 2\mathcal{R}_0 a_\ell L_* N\|\theta-\theta_c\|^2.\end{aligned}$$

Here we used the relation (3.6), and L_* is the positive constant defined in Lemma 2.1. □

Now we derive a differential inequality for $\mathcal{E}[\theta,\omega]$ along the flow (2.7). We recall that the system (2.7) can be rewritten as

$$\begin{aligned}&\dot\theta_i = \omega_i,\\ &\dot\omega_i = \frac{1}{m_i}\left[-d_i\omega_i + \Omega_i + \sum_{j=1}^N a_{ij}\sin(\theta_j-\theta_i)\right],\\ &\sum_{i=1}^N \Omega_i = 0, \qquad a_{ij} = a_{ji}.\end{aligned} \tag{3.7}$$

For notational simplicity, let's denote

$$\mathcal{E}(t) := \mathcal{E}[\theta(t),\omega(t)],$$

where $(\theta(t),\omega(t))$ is a solution to the system (2.7) or (3.7).

Proposition 3.1. *Let $D_0 \in (0,\pi)$ and $\{\theta_i\}_{i=1}^N$ be any smooth solution to the system* (2.7)*. Suppose that for some $T_0 > 0$,*

$$\max_{t\in[0,T_0]}\max_{1\le i,j\le N} |\theta_i(t)-\theta_j(t)| \le D_0.$$

Then, for any $\varepsilon,\mu>0$ with $\frac{\varepsilon}{\mu} < \frac{d_\ell}{m_u}$, we have

$$\frac{d}{dt}\mathcal{E}(t) + \bar C_\ell \mathcal{D}(t) \le 2\max\{\varepsilon,\mu\}\|\Omega\|\left(\|\theta-\theta_c\| + \|\omega\|\right), \quad \textit{for } t\in[0,T_0], \tag{3.8}$$

where $\mathcal{D}(t) := \mathcal{D}[\theta(t),\omega(t)]$ and $\bar C_\ell$ are defined by

$$\mathcal{D}[\theta,\omega] := \|\omega\|^2 + \|\theta-\theta_c\|^2 \quad \textit{and} \quad \bar C_\ell := \min\{2\mu d_\ell - 2\varepsilon m_u, 2\varepsilon\mathcal{R}_0 a_\ell L_* N\}.$$

Proof. The proof is divided into three steps.

• **Step A.-** We multiply $2\omega_i$ on both sides of the second equation in $(3.7)_2$, sum it over i, and then use the symmetry of a_{ij} to obtain

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N m_i\omega_i^2 &= -2\sum_{i=1}^N d_i\omega_i^2 + 2\sum_{i=1}^N \Omega_i\omega_i + 2\sum_{i,j=1}^N a_{ij}\sin(\theta_j-\theta_i)\omega_i \\ &= -2\sum_{i=1}^N d_i\omega_i^2 + 2\sum_{i=1}^N \Omega_i\omega_i - \sum_{i,j=1}^N a_{ij}\sin(\theta_j-\theta_i)(\omega_j-\omega_i).\end{aligned}$$

On the other hand,

$$\frac{d}{dt}\mathcal{E}_2[\theta] = \mu\sum_{i,j=1}^N a_{ij}\sin(\theta_i-\theta_j)(\omega_i-\omega_j).$$

So we have

$$\frac{d}{dt}\left(\mu\sum_{i=1}^N m_i\omega_i^2 + \mathcal{E}_2[\theta]\right) = -2\mu\sum_{i=1}^N d_i\omega_i^2 + 2\mu\sum_{i=1}^N \Omega_i\omega_i \le -2\mu\sum_{i=1}^N d_i\omega_i^2 + 2\mu\|\Omega\|\|\omega\|.$$

This yields

$$\frac{d}{dt}\left(\mu\langle M\omega,\omega\rangle + \mathcal{E}_2[\theta]\right) \le -2\mu d_\ell\|\omega\|^2 + 2\mu\|\Omega\|\|\omega\|. \tag{3.9}$$

• **Step B.-** We now multiply $2\theta_i$ on both sides of $(3.7)_2$ to obtain

$$2m_i\left(\frac{d\omega_i}{dt}\right)\theta_i = -d_i\frac{d}{dt}\theta_i^2 + 2\Omega_i\theta_i + 2\sum_{j=1}^N a_{ij}\sin(\theta_j-\theta_i)\theta_i.$$

Summing the above equality over i and using the symmetry of a_{ij} and Lemma 3.6, we find

$$\begin{aligned}2\sum_{i=1}^N m_i\left(\frac{d\omega_i}{dt}\right)\theta_i &= -\frac{d}{dt}\sum_{i=1}^N d_i\theta_i^2 + 2\sum_{i=1}^N\Omega_i\theta_i + 2\sum_{(j,i)\in\mathcal{W}} a_{ij}\sin(\theta_j-\theta_i)\theta_i \\ &= -\frac{d}{dt}\sum_{i=1}^N d_i\theta_i^2 + 2\sum_{i=1}^N\Omega_i\theta_i - \sum_{(j,i)\in\mathcal{W}} a_{ij}\sin(\theta_j-\theta_i)(\theta_j-\theta_i) \\ &= -\frac{d}{dt}\sum_{i=1}^N d_i\theta_i^2 + 2\sum_{i=1}^N\Omega_i(\theta_i-\theta_c) - \sum_{(j,i)\in\mathcal{W}} a_{ij}\sin(\theta_j-\theta_i)(\theta_j-\theta_i) \\ &\le -\frac{d}{dt}\sum_{i=1}^N d_i\theta_i^2 + 2\|\Omega\|\|\theta-\theta_c\| - 2\mathcal{R}_0 a_\ell L_* N\|\theta-\theta_c\|^2,\end{aligned} \tag{3.10}$$

where we used the restriction that

$$\sum_{i=1}^N \Omega_i = 0.$$

On the other hand, the term in the left hand side of relation (3.10) can be rewritten as

$$m_i\frac{d\omega_i}{dt}\theta_i = m_i\frac{d}{dt}(\omega_i\theta_i) - m_i\omega_i^2. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\frac{d}{dt}\left(2\sum_{i=1}^{N} m_i\omega_i\theta_i + \sum_{i=1}^{N} d_i\theta_i^2\right) + 2\mathcal{R}_0 a_\ell L_* N\|\theta-\theta_c\|^2 \le 2\|\Omega\|\|\theta-\theta_c\| + 2\sum_{i=1}^{N} m_i\omega_i^2.$$

Finally, we use the fact

$$\sum_{i=1}^{N} m_i\omega_i^2 \le m_u\|\omega\|^2,$$

to conclude

$$\frac{d}{dt}\left(\langle D\theta,\theta\rangle + 2\langle M\theta,\omega\rangle\right) + 2\mathcal{R}_0 a_\ell L_* N\|\theta-\theta_c\|^2 \le 2\|\Omega\|\|\theta-\theta_c\| + 2m_u\|\omega\|^2. \tag{3.12}$$

• **Step C.-** Taking (3.9) + $\varepsilon\times$ (3.12) yields

$$\frac{d}{dt}\mathcal{E}(t) + 2(\mu d_\ell - \varepsilon m_u)\|\omega\|^2 + 2\varepsilon\mathcal{R}_0 a_\ell L_* N\|\theta-\theta_c\|^2 \le 2\max\{\varepsilon,\mu\}\|\Omega\|\Big(\|\theta-\theta_c\| + \|\omega\|\Big).$$

Then

$$\frac{d}{dt}\mathcal{E}(t) + \bar{C}_\ell\mathcal{D}(t) \le 2\max\{\varepsilon,\mu\}\|\Omega\|\Big(\|\theta-\theta_c\| + \|\omega\|\Big), \quad \text{for } t\in[0,T_0].$$

This is the desired inequality and the proof is completed. □

As in [8], in order to show the boundedness of phases we expect a differential inequality for the energy in the following form

$$\frac{d}{dt}\mathcal{E}(t) + \beta_1\mathcal{E}(t) \le \beta_2\sqrt{\mathcal{E}(t)},$$

where β_1 and β_2 are positive constants. In the case of uniform inertia and damping, without loss of generality we can assume $\theta_c(t) = 0$ for all $t\ge 0$ (see [8, Subsection 2.1] or Remark 2.1). Thus a desired differential inequality immediately follows from (3.8) by invoking Lemmas 3.4 and 3.5 with $\theta_c(t)\equiv 0$. However, as we mentioned in Remark 2.1, for (2.7) we cannot assume $\theta_c(t) = 0$ due to the nonuniform parameters. This means that the functional $\mathcal{E}[\theta,\omega]$ may not be bounded from above by $\mathcal{D}[\theta,\omega]$. Therefore, $\mathcal{E}[\theta,\omega]$ and $\mathcal{D}[\theta,\omega]$ are not equivalent and the relation (3.8) does not produce a desired differential inequality for $\mathcal{E}[\theta,\omega]$. In order to obtain the desired property, we introduce a modified energy functional $\widetilde{\mathcal{E}}$ as follows:

$$\widetilde{\mathcal{E}}[\theta,\omega] := \underbrace{\varepsilon\sum_{i=1}^{N} d_i(\theta_i-\theta_c)^2 + 2\varepsilon\sum_{i=1}^{N} m_i(\theta_i-\theta_c)\omega_i + \mu\sum_{i=1}^{N} m_i\omega_i^2}_{\widetilde{\mathcal{E}}_1[\theta,\omega]} + \underbrace{\mu\sum_{i,j=1}^{N} a_{ij}\left(1-\cos(\theta_i-\theta_j)\right)}_{\mathcal{E}_2[\theta]}.$$

Next we derive the relation between $\widetilde{\mathcal{E}}$ and $\mathcal{E}$, and the relation between $\widetilde{\mathcal{E}}$ and $\mathcal{D}$.

Lemma 3.7. *(Properties of $\widetilde{\mathcal{E}}$)*

(i) The functionals $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ have the following relation:

$$\widetilde{\mathcal{E}} = \mathcal{E} - 2\varepsilon\theta_s\theta_c + \varepsilon\, tr(D)\theta_c^2 - 2\varepsilon\omega_s\theta_c\,. \tag{3.13}$$

(ii) The functional $\widetilde{\mathcal{E}}[\theta,\omega]$ is positive, and it is equivalent with $\mathcal{D}[\theta,\omega]$:

$$C_0\mathcal{D}[\theta,\omega] \le \widetilde{\mathcal{E}}[\theta,\omega] \le C_1\mathcal{D}[\theta,\omega], \quad \forall\,\theta,\omega\in\mathbb{R}^N, \tag{3.14}$$

where C_0 and C_1 are given as in (**H3**).

Proof. (i) The relation between $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ immediately follows from the definition of $\widetilde{\mathcal{E}}$:

$$\begin{aligned}\widetilde{\mathcal{E}} &= \mathcal{E} - 2\varepsilon \sum_{i=1}^N d_i\theta_i\theta_c + \varepsilon \sum_{i=1}^N d_i\theta_c^2 - 2\varepsilon \sum_{i=1}^N m_i\omega_i\theta_c \\ &= \mathcal{E} - 2\varepsilon\theta_s\theta_c + \varepsilon\, tr(D)\theta_c^2 - 2\varepsilon\omega_s\theta_c\,.\end{aligned}$$

(ii) Replacing the term θ by $\theta - \theta_c$ in the proof of Lemma 3.4 yields

$$\frac{\mu m_\ell}{2}\|\omega\|^2 + \left(\varepsilon d_\ell - 2\frac{\varepsilon^2 m_u}{\mu}\right)\|\theta-\theta_c\|^2 \le \widetilde{\mathcal{E}}_1[\theta,\omega] \le \frac{3\mu m_u}{2}\|\omega\|^2 + \left(\varepsilon d_u + \frac{2\varepsilon^2 m_u d_u}{\mu d_\ell}\right)\|\theta-\theta_c\|^2.$$

We combine this relation with Lemma 3.5 to find the desired inequalities. The positivity of $\widetilde{\mathcal{E}}[\theta,\omega]$ immediately follows from the positivity of $\mathcal{D}[\theta,\omega]$. □

We now consider the evolution of the modified energy functional along (2.7) and denote

$$\widetilde{\mathcal{E}}(t) := \widetilde{\mathcal{E}}[\theta(t),\omega(t)].$$

The following lemma will be useful.

Lemma 3.8. *(Conservation property) The weighted sums of phases and frequencies satisfy*

$$\dot\theta_s + \dot\omega_s = 0. \tag{3.15}$$

Proof. This immediately follows from (2.7). Here we used the restriction $\sum_{i=1}^N \Omega_i = 0$. □

Proposition 3.2. *(Estimate for modified energy) Let $D_0 \in (0,\pi)$ and $\{\theta_i\}_{i=1}^N$ be any smooth solution to the system* (2.7)*. Suppose that*

$$2\mathcal{R}_0 a_\ell L_* N > \lambda \quad \textit{with} \quad \lambda = \frac{\sqrt{tr(\hat D^2)}}{\sqrt N} + \frac{2\sqrt{tr(\hat M^2)}}{\sqrt N},$$

and

$$\max_{t\in[0,T_0]} \max_{1\le i,j\le N} |\theta_i(t) - \theta_j(t)| \le D_0, \tag{3.16}$$

for some $T_0 > 0$. Then, for any $\varepsilon, \mu > 0$ with $\frac{\varepsilon}{\mu} < \frac{d_\ell}{2m_u+\lambda}$, we have

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + C_\ell \mathcal{D}(t) \le \frac{2\sqrt2 \max\{\varepsilon,\mu\}\|\Omega\|}{\sqrt{C_0}}\sqrt{\widetilde{\mathcal{E}}(t)}, \quad \textit{for } t\in[0,T_0], \tag{3.17}$$

where C_ℓ is a positive constant given by $C_\ell := \bar C_\ell - \varepsilon\lambda$. Moreover, we have

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \frac{C_\ell}{C_1}\widetilde{\mathcal{E}}(t) \le \frac{2\sqrt2 \max\{\varepsilon,\mu\}\|\Omega\|}{\sqrt{C_0}}\sqrt{\widetilde{\mathcal{E}}(t)}, \quad \textit{for } t\in[0,T_0]. \tag{3.18}$$

Proof. It follows from Proposition 3.1 and (3.13) in Lemma 3.7 that $\widetilde{\mathcal{E}}$ satisfies

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \bar C_\ell\mathcal{D}(t) \le \underbrace{\frac{d}{dt}\left(\varepsilon\, tr(D)\theta_c^2 - 2\varepsilon\theta_s\theta_c - 2\varepsilon\omega_s\theta_c\right)}_{I} + \underbrace{2\max\{\varepsilon,\mu\}\|\Omega\|\left(\|\theta-\theta_c\| + \|\omega\|\right)}_{J}.$$

Using (3.15), we rewrite I as

$$\begin{aligned}
I &= 2\varepsilon\, tr(D)\theta_c\dot\theta_c - 2\varepsilon\dot\theta_s\theta_c - 2\varepsilon\theta_s\dot\theta_c - 2\varepsilon\dot\omega_s\theta_c - 2\varepsilon\omega_s\dot\theta_c \\
&= 2\varepsilon\, tr(D)\theta_c\dot\theta_c - 2\varepsilon\theta_s\dot\theta_c - 2\varepsilon\omega_s\dot\theta_c \quad \left(\because \dot\theta_s + \dot\omega_s = 0\right) \\
&= -2\varepsilon\dot\theta_c\sum_{i=1}^N d_i(\theta_i - \theta_c) - 2\varepsilon\omega_s\dot\theta_c \quad \left(\because \theta_s = \sum_{i=1}^N d_i(\theta_i - \theta_c) + tr(D)\theta_c\right) \\
&= -2\varepsilon\omega_c\sum_{i=1}^N d_i(\theta_i - \theta_c) - 2\varepsilon\omega_s\omega_c \quad \left(\because \dot\theta_c = \omega_c := \frac{1}{N}\sum_{i=1}^N \omega_i\right).
\end{aligned}$$

Note that

$$\sum_{i=1}^N d_i(\theta_i - \theta_c) = \sum_{i=1}^N \hat d_i(\theta_i - \theta_c) \quad \text{and} \quad \omega_s = \sum_{i=1}^N \hat m_i\omega_i + tr(M)w_c.$$

This yields

$$\begin{aligned}
I &= -2\varepsilon\omega_c\sum_{i=1}^N \hat d_i(\theta_i - \theta_c) - 2\varepsilon\left(\sum_{i=1}^N \hat m_i\omega_i + tr(M)w_c\right)\omega_c \\
&\le -2\varepsilon\omega_c\sum_{i=1}^N \hat d_i(\theta_i - \theta_c) - 2\varepsilon\omega_c\sum_{i=1}^N \hat m_i\omega_i.
\end{aligned}$$

On the other hand, we find

$$\begin{aligned}
&\left|2\varepsilon\omega_c\sum_{i=1}^N \hat d_i(\theta_i - \theta_c) + 2\varepsilon\omega_c\sum_{i=1}^N \hat m_i\omega_i\right| \\
&\quad = \left|\frac{2\varepsilon}{N}\left(\sum_{i=1}^N \omega_i\right)\left(\sum_{i=1}^N \hat d_i(\theta_i - \theta_c)\right) + \frac{2\varepsilon}{N}\left(\sum_{i=1}^N \hat m_i\omega_i\right)\left(\sum_{i=1}^N \omega_i\right)\right| \\
&\quad \le \frac{2\varepsilon}{N}\sqrt{N}\|\omega\|\sqrt{tr(\hat D^2)}\|\theta - \theta_c\| + \frac{2\varepsilon}{N}\sqrt{tr(\hat M^2)}\|\omega\|\sqrt{N}\|\omega\| \\
&\quad = \frac{2\varepsilon}{\sqrt{N}}\sqrt{tr(\hat D^2)}\|\omega\|\|\theta - \theta_c\| + \frac{2\varepsilon\sqrt{tr(\hat M^2)}}{\sqrt{N}}\|\omega\|^2 \\
&\quad \le \varepsilon\left(\frac{\sqrt{tr(\hat D^2)}}{\sqrt{N}}\|\omega\|^2 + \frac{\sqrt{tr(\hat D^2)}}{\sqrt{N}}\|\theta - \theta_c\|^2\right) + \frac{2\varepsilon\sqrt{tr(\hat M^2)}}{\sqrt{N}}\|\omega\|^2 \\
&\quad \le \varepsilon\left(\frac{\sqrt{tr(\hat D^2)}}{\sqrt{N}} + \frac{2\sqrt{tr(\hat M^2)}}{\sqrt{N}}\right)\mathcal{D}[\theta, \omega].
\end{aligned}$$

Thus, we have

$$I \le \varepsilon\left(\frac{\sqrt{tr(\hat D^2)}}{\sqrt{N}} + \frac{2\sqrt{tr(\hat M^2)}}{\sqrt{N}}\right)\mathcal{D}[\theta, \omega].$$

For the estimate of J, we obtain

$$\begin{aligned} J &= 2\max\{\varepsilon,\mu\}\|\Omega\|\left(\|\theta-\theta_c\|+\|\omega\|\right) \\ &\le 2\sqrt{2}\max\{\varepsilon,\mu\}\|\Omega\|\sqrt{\|\theta-\theta_c\|^2+\|\omega\|^2} \\ &\le \frac{2\sqrt{2}\max\{\varepsilon,\mu\}\|\Omega\|}{\sqrt{C_0}}\sqrt{\widetilde{\mathcal{E}}(t)}, \end{aligned}$$

where we used the elementary relation $a+b \le \sqrt{2}\sqrt{a^2+b^2}$ for $a,b\ge 0$ and (3.14) in Lemma 3.7. We now combine the above estimates for I and J to see that, for $t\in[0,T_0]$,

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t)+\left(\bar{C}_\ell-\varepsilon\lambda\right)\mathcal{D}(t) \le \frac{2\sqrt{2}\max\{\varepsilon,\mu\}\|\Omega\|}{\sqrt{C_0}}\sqrt{\widetilde{\mathcal{E}}(t)}.$$

This is the desired inequality (3.17). Finally, the last inequality (3.18) immediately follows from (3.14) and (3.17). □

Now we are able to prove the main result of this paper.

3.3. **Proof of Theorem 2.1.** For the sake of notational simplicity, we set

$$y(t) := \sqrt{\widetilde{\mathcal{E}}(t)}, \quad t\ge 0.$$

Define

$$\mathcal{T} := \left\{T\in\mathbb{R}_+ : y(t) < \frac{\sqrt{C_0}}{2}D_0, \quad \forall\, t\in[0,T)\right\}, \quad T^* := \sup\mathcal{T}.$$

Note that by the assumption (2.11),

$$y(0) < \frac{\sqrt{C_0}}{2}D_0.$$

Due to the continuity of y, there exists a positive constant $T>0$ such that $T\in\mathcal{T}$. We now claim that

$$T^* = \infty. \tag{3.19}$$

Suppose the opposite, i.e., T^* is finite. Then, we should have

$$y(T^*) = \frac{\sqrt{C_0}}{2}D_0. \tag{3.20}$$

Note that on the interval $[0,T^*)$, we can derive that

$$\begin{aligned} \max_{1\le i,j\le N}|\theta_i(t)-\theta_j(t)|^2 &\le 4\max_{1\le i\le N}|\theta_i(t)-\theta_c(t)|^2 \le 4\sum_{i=1}^N|\theta_i(t)-\theta_c(t)|^2 \\ &\le 4\mathcal{D}(t) \le \frac{4}{C_0}\widetilde{\mathcal{E}}(t) \le \frac{4}{C_0}\left(\frac{\sqrt{C_0}}{2}D_0\right)^2 = D_0^2, \end{aligned}$$

which means that the condition (3.16) is fulfilled, and Proposition 3.2 can be applied for $[0,T^*]$. By (3.18) we have

$$\frac{dy}{dt} \le \frac{\sqrt{2}\max\{\varepsilon,\mu\}\|\Omega\|}{\sqrt{C_0}} - \frac{C_\ell}{2C_1}y, \quad \text{for} \quad t\in[0,T^*]. \tag{3.21}$$

Solving the above differential inequality (3.21) yields

$$y(t) \le y(0)e^{-\frac{C_\ell}{2C_1}t} + \frac{2\sqrt{2}C_1 \max\{\varepsilon,\mu\}\|\Omega\|}{C_\ell\sqrt{C_0}}\left(1-e^{-\frac{C_\ell}{2C_1}t}\right), \quad \text{for} \quad t\in[0,T^*].$$

Thus we get

$$y(T^*) \le \max\left\{y(0), \frac{2\sqrt{2}C_1 \max\{\varepsilon,\mu\}\|\Omega\|}{C_\ell\sqrt{C_0}}\right\} < \frac{\sqrt{C_0}}{2}D_0,$$

where we used the assumptions (2.10) and (2.11). This contradicts (3.20) and the claim (3.19) is proved, i.e.,

$$\widetilde{\mathcal{E}}(t) < \frac{C_0}{4}D_0^2, \qquad \forall\ t\ge 0.$$

This implies that

$$\max_{1\le i,j\le N}|\theta_i(t)-\theta_j(t)|^2 \le 4\mathcal{D}(t) \le \frac{4}{C_0}\widetilde{\mathcal{E}}(t) < D_0^2, \qquad \forall\ t\ge 0, \tag{3.22}$$

and the desired result (i) in Theorem 2.1 is obtained. On the other hand, we recall the relation (3.15) to get

$$\omega_s(t)+\theta_s(t) = \omega_s(0)+\theta_s(0), \qquad \forall\ t\ge 0.$$

This means that

$$|\theta_s(t)| \le |\omega_s(t)+\theta_s(t)| + |\omega_s(t)| = |\omega_s(0)+\theta_s(0)| + |\omega_s(t)|, \quad \forall\, t\ge 0.$$

We now use the fact that $\omega(\cdot)\in\ell^\infty(\mathbb{R}^+,\mathbb{R}^N)$ in Lemma 3.3 to deduce

$$|\theta_s(t)| \le K_0, \quad \forall\, t\ge 0, \tag{3.23}$$

for some positive constant K_0. Combining the relations (3.22) and (3.23), we see that the trajectory $\theta(\cdot)$ is bounded as a function in time t. We now apply Lemma 3.3 to obtain the desired results (ii) and (iii) in Theorem 2.1. This completes the proof.

Remark 3.2. *If, in addition, $D_0 \le \pi/2$, then the emergent phase-locked state must be confined in an arc with length less than $\pi/2$. Thus, the result in* [26, Theorem 3.1] *holds. Furthermore, by appealing to the approach in* [26] *(see the Step 2 in the proof of Theorem 2.1), we can derive that the convergence to the phase-locked states is exponentially fast.*

4. Numerical simulations

In this section, we present some simulation for three oscillators to visualize the region of attraction. As we see in Theorem 2.1, the estimated region (2.13) depends on the choice of parameters such as D_0 and (ε,μ). Note that the constant D_0 is actually an upper bound of phase differences for the system (see (3.22)), while the pair (ε,μ) produces the energy functional. In general, it is not analytically possible to find the optimal choice of the pair (ε,μ) which leads to a best estimate of the region. In this section, we will make some simulations for a special setting and illustrate how D_0 and (ε,μ) influence the estimated region (2.13). The numerical method is a classical fourth order Runge-Kutta one using the built-in *ode45* Matlab command.

We will consider a model of three oscillators given by

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \Omega_i + \sum_{j=1}^N a_{ij}\sin(\theta_j-\theta_i), \quad i=1,2,3.$$

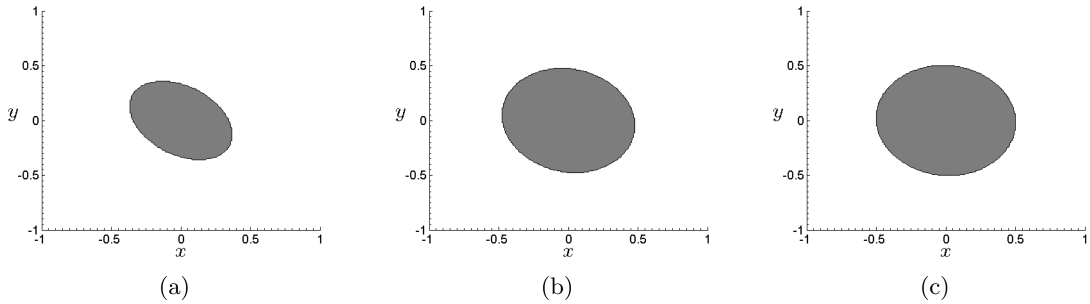

FIGURE 1. The region of attraction for special choices of $D_0 = 0.5\pi$ and admissible (ε, μ) with (a): $\frac{\varepsilon}{\mu} = 1$, (b): $\frac{\varepsilon}{\mu} = \frac{1}{5}$, (c): $\frac{\varepsilon}{\mu} = \frac{1}{9}$.

To reduce the dimension of variables, we assume that the initial frequencies are determined by initial phases in the following way:

$$d_i\omega_i(0) = \Omega_i + \sum_{j=1}^{N} a_{ij} \sin(\theta_j(0) - \theta_i(0)). \tag{4.1}$$

We set the inertias m_i's and non-homogeneous dampings d_i's by using random data which are uniformly distributed with

$$m_i \in (0.10, 0.15), \quad d_i \in (0.30, 0.40), \quad \frac{d_\ell}{2m_u + \lambda} > 1,$$

and set the symmetric coupling as $a_{ij} = 0.1$ for any $(i,j) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise. We consider a network on a line-shaped graph (that is, $1 \leftrightarrow 2 \leftrightarrow 3$).

In order to show the region of attraction in a plane, we introduce the following variables:

$$x = \theta_1 - \theta_2, \quad y = \theta_2 - \theta_3,$$

then $x + y = \theta_1 - \theta_3$ and

$$\theta_1 - \theta_c = \frac{2x+y}{3}, \quad \theta_2 - \theta_c = \frac{-x+y}{3}, \quad \theta_3 - \theta_c = \frac{-x-2y}{3}.$$

Therefore, the initial frequencies in (4.1) and initial energy in (2.12) are fully determined by x and y. We will plot the region (x, y) in the plane to illustrate the region of attraction.

4.1. **Varying** (ε, μ)**.** In this part, we illustrate the region of attraction for varying (ε, μ) by fixing range of phases as $D_0 = 0.5\pi$. For given parameters m_i, d_i and a_{ij}, we can compute the parameters $\mathcal{R}_0, \lambda$ and check (2.8). For simplicity in the simulation, let us consider the following situations:

$$\varepsilon = \mu, \quad \text{or} \quad \varepsilon = \frac{1}{5}\mu, \quad \text{or} \quad \varepsilon = \frac{1}{9}\mu,$$

which allows us to reduce the two-parameter (ε, μ) to one-parameter ε. The regions of attraction, registered by the gray color, are shown in Figure 1.

4.2. **Varying** D_0**.** In Figure 2 and Figure 3, we illustrate the estimated region for different choices of D_0, namely, $D_0 = 0.3\pi$ and $D_0 = 0.7\pi$. We use the same parameters and (ε, μ) as in Figure 1 for a reasonable comparison. Figures 1-3 indicate that the larger choice of D_0 produces a larger region.

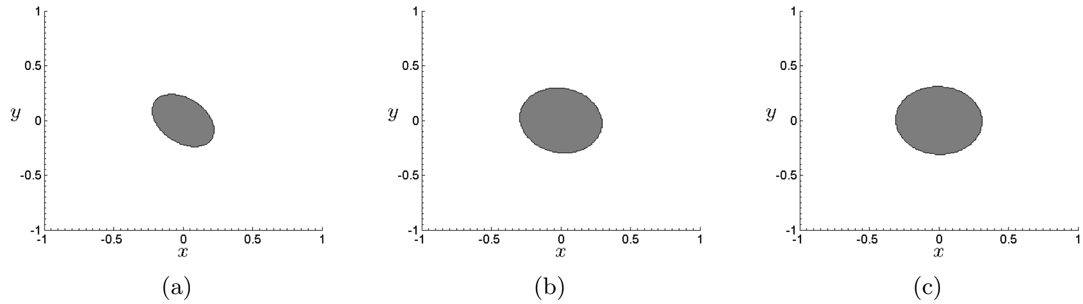

FIGURE 2. The region of attraction for special choices of $D_0 = 0.3\pi$ and admissible (ε, μ) with (a): $\frac{\varepsilon}{\mu} = 1$, (b): $\frac{\varepsilon}{\mu} = \frac{1}{5}$, (c): $\frac{\varepsilon}{\mu} = \frac{1}{9}$.

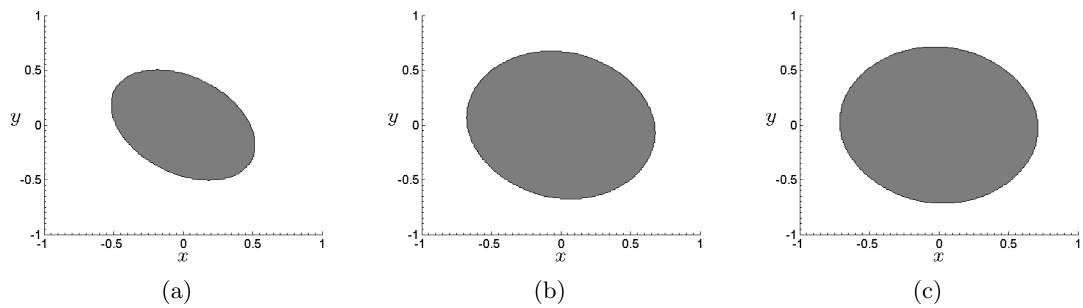

FIGURE 3. The region of attraction for special choices of $D_0 = 0.7\pi$ and admissible (ε, μ) with (a): $\frac{\varepsilon}{\mu} = 1$, (b): $\frac{\varepsilon}{\mu} = \frac{1}{5}$, (c): $\frac{\varepsilon}{\mu} = \frac{1}{9}$.

5. Conclusion

In this paper, we studied the synchronization of second-order nonuniform Kuramoto oscillators (swing equations) for power grids on connected networks with general dampings. We employed the energy method and gradient-like flow approach to obtain a sufficient condition for the synchronization, which gives an explicit estimate for sync basin. In view of the potential application in engineering, the quantitative improvement of the estimate, including the parametric condition and the basin, would be an interesting future problem. The heterogeneity of the parameters and/or general connectivity mean that the method of studying the phase difference cannot work well, while our estimate gives a way to overcome these difficulties.

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