Existence of Steady Multiple Vortex Patches to the Vortex-wave System

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Abstract

In this paper, we prove the existence of steady multiple vortex patch solutions to the vortex-wave system in a planar bounded domain. The construction is performed by solving a certain variational problem for the vorticity and studying the asymptotic behavior as the vorticity strength goes to infinity.

Keywords: vortex-wave system, vortex patch, Euler equations, Kirchhoff-Routh function, variational problem

1. Introduction

The evolution of an incompressible inviscid fluid is described by the following Euler equations

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{1.1}$$

where $\mathbf{v}$ is the velocity field and P is the pressure. In the planar case $\mathbf{v} = (v_1, v_2)$ and we introduce the scalar vorticity of the fluid as follows

$$\omega := \partial_1 v_2 - \partial_2 v_1. \tag{1.2}$$

By taking the curl on both sides of the first equation of (1.1), we obtain the following vorticity form of the Euler equations

$$\partial_t\omega + \mathbf{v}\cdot\nabla\omega = 0. \tag{1.3}$$

In the whole plane, the velocity can be recovered from the vorticity via the Biot-Savart law, that is,

$$\mathbf{v}(x,t) = K*\omega(x,t) := -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{J(x-y)}{|x-y|^2}\omega(y,t)dy, \tag{1.4}$$

where $J(a,b) := (b,-a)$ denotes clockwise rotation through $\pi/2$ of the planar vector $(a,b)\in\mathbb{R}^2$, and $K(x) = -\frac{1}{2\pi}\frac{Jx}{|x|^2}$ is called the Biot-Savart kernel. (1.3) and (1.4) mean that the vorticity ω is transported by a divergence-free velocity field induced by itself.

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In some cases the vorticity is sharply concentrated in N small disjoint regions, where N is a positive integer, then its time evolution can be approximately described by the point vortex model (see [22] for example), an ODE system that can be written as follows

$$\frac{dx_i}{dt} = \sum_{j=1, j\neq i}^{N} \kappa_j K(x_i - x_j), \quad i = 1, \cdots, N, \tag{1.5}$$

where x_i is the position of the i-th point vortex and κ_i is the corresponding vorticity strength. According to the point vortex model, for each point vortex located at x_i, the velocity it induces is $\kappa_i K(\cdot - x_i)$, and it moves in the velocity field induced by all the other $N-1$ point vortices. The point vortex model and its relation with the Euler equations have been analyzed extensively; see [13, 23, 27, 30] for example.

Now it is natural to consider the mixed problem, that is, the vorticity consists of a continuously distributed part and a finite number of concentrated vortices. Marchioro and Pulvirenti [25] first studied this problem and they called it the vortex-wave system. In the whole plane the vortex-wave system can be written as follows

$$\begin{cases} \partial_t \omega + \mathbf{v} \cdot \nabla \omega = 0, \\ \frac{dx_i}{dt} = K * \omega(x_i, t) + \sum_{j=1, j\neq i}^{N} \kappa_j K(x_i - x_j), \ i = 1, \cdots, N, \\ \mathbf{v} = K * \omega + \sum_{j=1}^{k} \kappa_j K(\cdot - x_j). \end{cases} \tag{1.6}$$

Throughout this paper we call ω the background vorticity. System (1.6) means that the background vorticity is transported by the velocity field induced by itself (the term $K*\omega$) and the N point vortices (the term $\sum_{j=1}^{k} \kappa_j K(\cdot - x_j)$), and each point vortex moves by the velocity induced by the background vorticity (the term $K * \omega(x_i, t)$) and all the other $N-1$ point vortices (the term $\sum_{j\neq i, j=1}^{N} \kappa_j K(\cdot - x_j)$). By constructing Lagrangian paths Marchioro and Pulvirenti in [25] proved an existence theorem for initial background vorticity belonging to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. More existence and uniqueness results can be found in [4, 20, 21, 26].

In this paper, we will be focusing on the steady vortex-wave system in a bounded domain, the precise form of which will be given in the next section. In this case the Kirchhoff-Routh function (defined by (2.2) in the next section) plays an essential role. On the one hand, it is easy to see that any critical point of the Kirchhoff-Routh function is a stationary point of the vortex model, and it has been shown in [13] that if this critical point is non-degenerate, then there exists a family of steady vortex patch solutions (that is, the vorticity is a piecewise constant function) of the Euler equations shrinking to this critical point. Similar results can also be found in [12, 27, 28, 31]. On the other hand, it is proved in [11] that if there is a family of steady vortex patch solutions of the Euler equations shrinks to some point, then this point must be in the interior of the domain and must be a critical point of the Kirchhoff-Routh function. In this way, the Kirchhoff-Routh function establishes connection between the Euler equations and the vortex model.

Our purpose in the present paper is to extend the result in [13] to the vortex-wave system. To be more precise, we prove that for any given strict local minimum point of the Kirchhoff-Routh function, there exists a family of steady vortex patch solutions to the vortex-wave system that shrinks to this point. Here by vortex patch solution of the vortex-wave system we mean that the background vorticity is a piecewise constant function.

The method we use in this paper to construct steady solutions is called the vorticity method, which was first established by Arnold [1] and further developed by many authors [5, 7, 9, 17, 18, 28, 29]. Roughly speaking, the vorticity method is to maximize the kinetic energy of the fluid under some suitable constraints for the vorticity. For the Euler equations, the kinetic energy of the fluid with bounded vorticity is always finite, but for the vortex-wave system the kinetic energy is infinite due to the presence of point vortices. To overcome this difficulty, we drop the infinite self-energy term for each point vortex. We refer the interested reader to [15] where the energy of the vortex-wave system with a single point vortex was calculated rigorously .

It is worth mentioning that the construction in [13] was based on a finite dimensional reduction argument. The advantage of the method in [13] is that solutions concentrating at a given saddle point of the Kirchhoff-Routh function can be constructed. However, non-degenerate condition of this saddle point is required in this situation. Using the vorticity method, we are able to construct solutions concentrating at a given strict local minimum point of the Kirchhoff-Routh function, even if this point is degenerate. Another advantage of the vorticity method is that we can analyze the energy of the solution, which is helpful to prove nonlinear stability; see [6, 14] for example.

This paper is organized as follows. In Section 2, we give the formulation of the vortex-wave system in a bounded domain and state the main result. In Section 3, we solve a maximization problem for the vorticity and study the asymptotic behavior of the maximizers. In Section 4 we prove the main result.

2. Main result

2.1. Notation

Let $D \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary. The Green's function for $-\Delta$ in D with zero Dirichlet data on ∂D can be written as follows

$$G(x,y)=\frac{1}{2\pi}\ln\frac{1}{|x-y|}-h(x,y),\ \ x,y\in D, \tag{2.1}$$

where h is called the regular part of the Green's function. Define

$$H(x):=\frac{1}{2}h(x,x),$$

which is usually called the Robin function. Let k be a positive integer, $\kappa_i \in \mathbb{R}\setminus\{0\}$, $i=1,\cdots,k$. Define the Kirchhoff-Routh function as follows

$$\mathcal{K}_k(x_1,\cdots,x_k)=-\sum_{i\neq j,1\leq i,j\leq k}\kappa_i\kappa_j G(x_i,x_j)+\sum_{i=1}^{k}\kappa_i^2 h(x_i,x_i), \tag{2.2}$$

where $x_i \in D$ and $x_i \neq x_j$ if $i \neq j$. Note that if $k = 1$, then $\mathcal{K}_1 = 2\kappa_1^2 H$.

Throughout this paper we will use the following notation. For any function g, $supp(g)$ denotes the support of g. For any real number a, $sgn(a)$ denotes the sign of a, that is,

$$sgn(a) := \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{if } a = 0, \\ -1, & \text{if } a < 0. \end{cases} \tag{2.3}$$

For any Lebesgue measurable set $A \subset \mathbb{R}^2$, $|A|$ denotes the two-dimensional Lebesgue measure of A; I_A denotes the characteristic function of A, that is, $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ elsewhere; $\overline{A}$ denotes the closure of A in the Euclidean topology; and $diam(A)$ denotes the diameter of A, that is,

$$diam(A) = \sup_{x,y \in A} |x - y|. \tag{2.4}$$

2.2. Vortex-wave system in a bounded domain

We consider an incompressible steady flow confined in D with impermeability boundary condition. The evolution of the velocity field $\mathbf{v} = (v_1, v_2)$ and the pressure P is described by the following Euler equations

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P, & \text{in } D \times (0, +\infty), \\ \nabla \cdot \mathbf{v} = 0, & \text{in } D \times (0, +\infty), \\ \mathbf{v} \cdot \nu = 0, & \text{on } \partial D \times (0, +\infty), \end{cases} \tag{2.5}$$

where ν is the outward unit normal of ∂D.

Taking the curl on both sides of the first equation of (2.5), we get the vorticity equation

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega = 0. \tag{2.6}$$

Since $\mathbf{v}$ is divergence-free, there is a function ψ, called the stream function, such that

$$\mathbf{v} = J\nabla\psi = (\partial_2\psi, -\partial_1\psi). \tag{2.7}$$

By the definition of ω (recall (1.2)), it is easy to see that

$$-\Delta\psi = \omega. \tag{2.8}$$

The impermeability boundary condition given in the third equation of (2.5) implies that ψ is a constant on each connected component of ∂D. Since D is simply connected, after suitably adding a constant to ψ we can assume

$$\psi(x, t) = 0, \quad x \in \partial D. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\psi(x,t) = (-\Delta)^{-1}\omega(x,t) := \int_D G(x,y)\omega(y,t)dy, \quad x \in D. \tag{2.10}$$

For brevity we introduce the notation

$$\partial(f,g) := \nabla f \cdot J\nabla g = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g,$$

then the vorticity equation (2.6) can be written as

$$\begin{cases} \partial_t \omega + \partial(\omega,\psi) = 0, \\ \psi = (-\Delta)^{-1}\omega. \end{cases} \tag{2.11}$$

If the vorticity is a Dirac delta measure (also called a point vortex) located at $x \in D$, i.e., $\omega = \delta(x)$, then formally the velocity field it induces is

$$J\nabla(-\Delta)^{-1}\delta(x) = J\nabla G(x,\cdot) = \frac{1}{2\pi}\frac{J(x-\cdot)}{|x-\cdot|^2} - J\nabla h(x,\cdot).$$

Note that this velocity field is singular at x. Due to symmetry, we formally drop the term $\frac{1}{2\pi}\frac{J(x-\cdot)}{|x-\cdot|^2}$, that is, we assume that the velocity at x is $-J\nabla h(x,\cdot)\big|_x = -J\nabla H(x)$, then the evolution of this point vortex is described by the following ODE:

$$\frac{dx}{dt} = -J\nabla H(x). \tag{2.12}$$

Similarly, the evolution of l point vortices can be described by the following ODE system:

$$\frac{dx_i}{dt} = -\kappa_i J\nabla H(x_i) + \sum_{j=1, j\neq i}^{l} \kappa_j J\nabla_{x_i} G(x_j, x_i), \quad i = 1,\cdots,l, \tag{2.13}$$

where κ_i is the vorticity strength of the i-th point vortex. System (2.13) is also called the Kirchhoff-Routh equation. It is easy to see that the Kirchhoff-Routh function $\mathcal{K}_l$ is exactly the Hamiltonian of the system.

Now we consider the mixed problem, that is, the vorticity consists of a continuously distributed part ω and l point vortices x_i with strength κ_i, $i = 1,\cdots,l$, then the evolution of ω and x_i will obey the following equations

$$\begin{cases} \partial_t \omega + \partial\left(\omega, \psi + \sum_{j=1}^{l} \kappa_j G(x_j,\cdot)\right) = 0 \\ \frac{dx_i}{dt} = J\nabla\left(\psi + \sum_{j=1, j\neq i}^{l} \kappa_j G(x_j,\cdot) - \kappa_i H\right)(x_i), \ i = 1,\cdots,l, \\ \psi = (-\Delta)^{-1}\omega. \end{cases} \tag{2.14}$$

(2.14) is called the vortex-wave system in D.

In this paper, we confine ourselves to the stationary case, that is, we consider the following system of equations

$$\begin{cases}\partial\left(\omega, \psi + \sum_{j=1}^{l}\kappa_j G(x_j,\cdot)\right) = 0,\\ \nabla\left(\psi + \sum_{j=1, j\neq i}^{l}\kappa_j G(x_j,\cdot) - \kappa_i H\right)(x_i) = 0,\ i = 1,\cdots,l,\\ \psi = (-\Delta)^{-1}\omega.\end{cases} \tag{2.15}$$

Since we are going to deal with vortex patch solutions which are discontinuous, it is necessary to give the weak formulation of (2.15).

Definition 2.1. Let $\omega \in L^\infty(D)$, $x_i \in D, i = 1,\cdots,l$, then $(\omega, x_1,\cdots,x_l)$ is called a weak solution to (2.15) if it satisfies

$$\begin{cases}\int_D \omega(x)\partial\left(\psi(x) + \sum_{j=1}^{l}\kappa_j G(x_j,x), \phi(x)\right)dx = 0,\ \forall \phi \in C_c^\infty(D),\\ \nabla\left(\psi(x) + \sum_{j=1,j\neq i}^{l}\kappa_j G(x_j,x) - \kappa_i H(x)\right)\Big|_{x=x_i} = 0,\ i = 1,\cdots,l,\end{cases} \tag{2.16}$$

where $\psi = (-\Delta)^{-1}\omega$.

Remark 2.2. Note that since $\omega \in L^\infty(D)$, by L^p estimate $\psi \in W^{2,p}(D)$ for any $1 < p < +\infty$, then by Sobolev embedding $\psi \in C^{1,\alpha}(\overline{D})$ for any $0 < \alpha < 1$.

Remark 2.3. Definition 2.1 can be derived formally from (2.15) by integration by parts; see [15] for the detailed calculations.

2.3. Main result

Our main result in this paper is the following theorem:

Theorem 2.4. *Let k, p, l be positive integers such that $p + l = k$, and $\kappa_i, i = 1,\cdots,k$, be k real numbers such that $\kappa_i \neq 0$. Suppose that $(\bar{x}_1,\cdots,\bar{x}_k)$ is a strict local minimum point of $\mathcal{K}_k$ defined by (2.2), where $\bar{x}_i \in D$ and $\bar{x}_i \neq \bar{x}_j$ for $i \neq j$. Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, (2.16) has a solution $(\omega^\lambda, x_{p+1}^\lambda,\cdots,x_k^\lambda)$ satisfying*

$$\omega^\lambda = \sum_{i=1}^{p}\omega_i^\lambda,\quad \int_D \omega_i^\lambda(x)dx = \kappa_i,\quad \omega_i^\lambda = sgn(\kappa_i)\lambda I_{A_i^\lambda},\quad i = 1,\cdots,p, \tag{2.17}$$

where A_i^λ has the form

$$A_i^\lambda = \left\{x \in D \;\middle|\; sgn(\kappa_i)\left(\psi^\lambda(x) + \sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda, x)\right) > c_i^\lambda\right\} \cap B_\delta(\bar{x}_i) \tag{2.18}$$

for some $c_i^\lambda > 0$ and $\delta > 0$ (δ does not depend on λ). Moreover,

$$diam(A_i^\lambda) \leq C\lambda^{-\frac{1}{2}},\ \lim_{\lambda\to+\infty}\left|\frac{1}{\kappa_i}\int_D x\omega_i^\lambda(x)dx - \bar{x}_i\right| = 0,\quad i = 1,\cdots,p, \tag{2.19}$$

$$\lim_{\lambda\to+\infty}\left|x_j^\lambda-\bar{x}_j\right|=0,\quad j=p+1,\cdots,k, \tag{2.20}$$

where C is a positive number not depending on λ.

Remark 2.5. By (2.19), we see that A_i^λ shrinks to $\bar{x}_i$ as $\lambda\to+\infty$, that is,

$$\lim_{\lambda\to+\infty}\sup_{x\in A_i^\lambda}|x-\bar{x}_i|=0,\quad i=1,\cdots,p.$$

Consequently $\overline{A_i^\lambda}\subset B_\delta(\bar{x}_i)$ for sufficiently large λ.

3. Variational problem

Let $(\bar{x}_1,\cdots,\bar{x}_k)$ be a strict local minimum point of $\mathcal{K}_k$, where $\bar{x}_i\in D$ and $\bar{x}_i\neq\bar{x}_j$ for $i\neq j$. Without loss of generality, we assume that $(\bar{x}_1,\cdots,\bar{x}_k)$ is the unique minimum point of $\mathcal{K}_k$ on $\overline{B_{\delta_0}(\bar{x}_1)}\times\cdots\times\overline{B_{\delta_0}(\bar{x}_k)}$, where $\delta_0>0$ is a small positive number such that $\overline{B_{\delta_0}(\bar{x}_i)}\subset D$ and $\overline{B_{\delta_0}(\bar{x}_i)}\cap\overline{B_{\delta_0}(\bar{x}_j)}=\varnothing$ for all $i,j=1,\cdots,k$ and $i\neq j$.

Remark 3.1. To our knowledge, there is no general result that guarantees the existence of a strict local minimum point of $\mathcal{K}_k$ for $k\geq 2$. Some special cases are as follows: if $k=1$ and D is convex, by [10] $\mathcal{K}_1$ is a strictly convex function in D, thus has a unique minimum point; if $k\geq 2,\kappa_i>0$ for each i and D is convex, by [19] there does not exist any critical point of $\mathcal{K}_k$; if $k=2$, some examples of strict local minimum points of $\mathcal{K}_2$ are given computationally in [18]. More related results can also be found in [2, 3] and the references therein.

Let λ be a positive real number. Define

$$N_p^\lambda=\left\{\omega\in L^\infty(D)\;\middle|\;\omega=\sum_{i=1}^p\omega_i, supp(\omega_i)\subset B_\delta(\bar{x}_i),\int_D\omega_i(x)dx=\kappa_i, 0\leq sgn(\kappa_i)\omega_i\leq\lambda\right\},$$

where $\delta<\frac{\delta_0}{2}$ is a small positive number to be determined later. Hereafter we assume that λ is sufficiently large such that N_p^λ is not empty.

For $(\omega,x_{p+1},\cdots,x_k)\in N_p^\lambda\times\overline{B_\delta(\bar{x}_{p+1})}\times\cdots\times\overline{B_\delta(\bar{x}_k)}$, define

$$\mathcal{E}(\omega,x_{p+1},\cdots,x_k):=E(\omega)+\sum_{j=p+1}^k\kappa_j\psi(x_j)+\frac{1}{2}\sum_{\substack{i\neq j\\ p+1\leq i,j\leq k}}\kappa_i\kappa_jG(x_i,x_j)-\sum_{j=p+1}^k\kappa_j^2H(x_j),$$

where

$$E(\omega):=\frac{1}{2}\int_D\int_D G(x,y)\omega(x)\omega(y)dxdy,\quad \psi(x)=(-\Delta)^{-1}\omega(x):=\int_D G(x,y)\omega(y)dy.$$

Let us explain the definition of $\mathcal{E}$ briefly. The first term $E(\omega)$ represents the self-interacting energy of the background vorticity ω, the second term represents the mutual interaction energy between the background vorticity and the l point vortices, the third term represents the total interaction energy between any two different point vortices, and the fourth term represents the interaction energy between the l point vortices and the boundary of D. As we have mentioned in Section 1, we have dropped the infinite self-interacting energy for each point vortex.

Now we consider the maximization of $\mathcal{E}$ on $N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$.

Lemma 3.2. *For fixed λ, there exists $(\omega^\lambda, x_{p+1}^\lambda, \cdots, x_k^\lambda) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$, such that*

$$\mathcal{E}(\omega^\lambda, x_{p+1}^\lambda, \cdots, x_k^\lambda) = \sup_{(\omega, x_{p+1}, \cdots, x_k) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}} \mathcal{E}(\omega, x_{p+1}, \cdots, x_k). \tag{3.1}$$

Proof. First, for any $(\omega, x_{p+1}, \cdots, x_k) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$, since $G \in L^1(D \times D)$, $dist(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$ for any $i \neq j, p+1 \leq i, j \leq k$, and $dist(B_{\delta_0}(\bar{x}_i), \partial D) > 0, p+1 \leq i \leq k$, we have

$$\begin{aligned}
&\mathcal{E}(\omega, x_{p+1}, \cdots, x_k) \\
=&\frac{1}{2}\int_D\int_D G(x,y)\omega(x)\omega(y)dxdy + \sum_{j=p+1}^{k} \kappa_j \psi(x_j) \\
&+ \frac{1}{2}\sum_{\substack{i\neq j \\ p+1\leq i,j\leq k}} \kappa_i\kappa_j G(x_i, x_j) - \sum_{j=p+1}^{k} \kappa_j^2 H(x_j) \\
\leq&\frac{\lambda^2}{2}\int_D\int_D |G(x,y)|dxdy + \sum_{j=p+1}^{k} |\kappa_j||\psi|_{L^\infty(D)} + \sum_{j=p+1}^{k} \kappa_j^2 |H|_{L^\infty(B_\delta(\bar{x}_j))}.
\end{aligned} \tag{3.2}$$

Thus

$$M := \sup_{(\omega, x_{p+1}, \cdots, x_k) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}} \mathcal{E}(\omega, x_{p+1}, \cdots, x_k) < +\infty.$$

Now we choose $(\omega^n, x_{p+1}^n, \cdots, x_k^n) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$ such that

$$\lim_{n\to+\infty} \mathcal{E}(\omega^n, x_{p+1}^n, \cdots, x_k^n) = M.$$

Since N_p^λ is weakly star closed in $L^\infty(D)$ (for a detailed proof of this fact, see Theorem 2.1 in [16]) and $\overline{B_\delta(\bar{x}_j)}$ is closed in the Euclidean topology for $j = p+1, \cdots, k$, there exists $(\omega^\lambda, x_{p+1}^\lambda, \cdots, x_k^\lambda) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$ such that (up to a subsequence)

$$\omega^n \to \omega^\lambda, \quad \text{weakly star in } L^\infty(D),$$

$$x_j^n \to x_j^\lambda, \quad j = p+1, \cdots, k.$$

Then obviously

$$\mathcal{E}(\omega^\lambda, x_{p+1}^\lambda, \cdots, x_k^\lambda) = \lim_{n\to+\infty} \mathcal{E}(\omega^n, x_{p+1}^n, \cdots, x_k^n) = M,$$

which completes the proof. $\square$

Remark 3.3. It is easy to see that

$$E(\omega)+\sum_{j=p+1}^{k}\kappa_j\psi(x_j^\lambda)\le E(\omega^\lambda)+\sum_{j=p+1}^{k}\kappa_j\psi^\lambda(x_j^\lambda) \tag{3.3}$$

for any $\omega\in N_p^\lambda$, and

$$\begin{aligned}&\sum_{j=p+1}^{k}\kappa_j\psi^\lambda(x_j)+\frac{1}{2}\sum_{\substack{i\neq j\\ p+1\le i,j\le k}}\kappa_i\kappa_jG(x_i,x_j)-\sum_{j=p+1}^{k}\kappa_j^2H(x_j)\\ \le&\sum_{j=p+1}^{k}\kappa_j\psi^\lambda(x_j^\lambda)+\frac{1}{2}\sum_{\substack{i\neq j\\ p+1\le i,j\le k}}\kappa_i\kappa_jG(x_i^\lambda,x_j^\lambda)-\sum_{j=p+1}^{k}\kappa_j^2H(x_j^\lambda)\end{aligned} \tag{3.4}$$

for any $(x_{p+1},\cdots,x_k)\in\overline{B_\delta(\bar{x}_{p+1})}\times\cdots\times\overline{B_\delta(\bar{x}_k)}$, where $\psi^\lambda=(-\Delta)^{-1}\omega^\lambda$ and $\psi=(-\Delta)^{-1}\omega$.

Since $\omega^\lambda\in N_p^\lambda$, we can write $\omega^\lambda=\sum_{i=1}^{p}\omega_i^\lambda$, where $\int_D\omega_i^\lambda(x)dx=\kappa_i, supp(\omega_i^\lambda)\subset B_\delta(\bar{x}_i)$ and $0\le sgn(\kappa_i)\omega_i^\lambda\le\lambda$. The profile of each ω_i^λ is as follows.

Lemma 3.4. *For $i=1,\cdots,p$, ω_i^λ has the form*

$$\omega_i^\lambda=sgn(\kappa_i)\lambda I_{A_i^\lambda},$$

where

$$A_i^\lambda:=\left\{x\in D\;\middle|\;sgn(\kappa_i)\left(\psi^\lambda(x)+\sum_{j=p+1}^{k}\kappa_jG(x_j^\lambda,x)\right)>c_i^\lambda\right\}\cap B_\delta(\bar{x}_i)$$

for some $c_i^\lambda\in\mathbb{R}$ depending on λ and δ.

Proof. To make it clear, we divide the proof into two cases.

Case 1: $\kappa_i>0$. For $s>0$ we define a family of test functions $\omega_s^\lambda=\omega^\lambda+s(z_0-z_1)$, where

$$\begin{cases}z_0,z_1\in L^\infty(D), z_0,z_1\ge0, \int_D z_0(x)dx=\int_D z_1(x)dx,\\ supp(z_0), supp(z_1)\subset B_\delta(\bar{x}_i),\\ z_0=0\quad\text{in } B_\delta(\bar{x}_i)\setminus\{\omega_i^\lambda\le\lambda-\mu\},\\ z_1=0\quad\text{in } B_\delta(\bar{x}_i)\setminus\{\omega_i^\lambda\ge\mu\},\end{cases} \tag{3.5}$$

where $\mu\in(0,\lambda)$. It is not hard to check that for fixed z_0,z_1 and μ, if s is sufficiently small (depending on z_0,z_1,μ), then $\omega_s^\lambda\in N_p^\lambda$. By (3.3) we have

$$\frac{d}{ds}\left(E(\omega_s^\lambda)+\sum_{j=p+1}^{k}\kappa_j\psi_s^\lambda(x_j^\lambda)\right)\Bigg|_{s=0^+}\le0,$$

where $\psi_s^\lambda = (-\Delta)^{-1}\omega_s^\lambda$. That is,

$$\int_D \left(\psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x)\right) z_0(x)dx \le \int_D \left(\psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x)\right) z_1(x)dx.$$

Since z_0, z_1, μ are chosen arbitrarily as above we have

$$\sup_{\{\omega_i^\lambda<\lambda\}\cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) \le \inf_{\{\omega_i^\lambda>0\}\cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right).$$

But $\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)$ is a continuous function on $\overline{B_\delta(\bar{x}_i)}$ (notice that $x_j^\lambda \notin \overline{B_\delta(\bar{x}_i)}$ for $p+1 \le j \le k$), we obtain

$$\sup_{\{\omega_i^\lambda<\lambda\}\cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) = \inf_{\{\omega_i^\lambda>0\}\cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right).$$

Now we define

$$\begin{aligned} c_i^\lambda &:= \sup_{\{\omega_i^\lambda<\lambda\}\cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) \\ &= \inf_{\{\omega_i^\lambda>0\}\cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right). \end{aligned} \tag{3.6}$$

It is easy to see that

$$\omega_i^\lambda \equiv \lambda \quad \text{a.e. on } \left\{ x \in D \;\middle|\; \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) > c_i^\lambda \right\} \cap B_\delta(\bar{x}_i), \tag{3.7}$$

$$\omega_i^\lambda \equiv 0 \quad \text{a.e. on } \left\{ x \in D \;\middle|\; \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) < c_i^\lambda \right\} \cap B_\delta(\bar{x}_i). \tag{3.8}$$

On the level set $\left\{ x \in D \;\middle|\; \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) = c_i^\lambda \right\} \cap B_\delta(\bar{x}_i)$, we have

$$\nabla(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)) = 0, \quad \text{a.e.}, \tag{3.9}$$

which gives

$$\omega_i^\lambda = -\Delta(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)) = 0, \quad \text{a.e.}. \tag{3.10}$$

(3.7),(3.8) and (3.10) together give

$$\omega_i^\lambda = \lambda I_{\{\psi^\lambda+\sum_{j=p+1}^k \kappa_j G(x_j^\lambda,\cdot)>c_i^\lambda\}\cap B_\delta(\bar{x}_i)}, \tag{3.11}$$

which completes the proof of Case 1.

Case 2: $\kappa_i<0$. The argument is similar. For $s>0$, define $\omega_s^\lambda=\omega^\lambda+s(z_1-z_0)$, where

$$\begin{cases} z_0, z_1 \in L^\infty(D), z_0, z_1 \geq 0, \int_D z_0(x)dx = \int_D z_1(x)dx, \\ supp(z_0), supp(z_1) \subset B_\delta(\bar{x}_i), \\ z_0 = 0 \quad \text{in } B_\delta(\bar{x}_i)\setminus\{\omega_i^\lambda \geq \mu-\lambda\}, \\ z_1 = 0 \quad \text{in } B_\delta(\bar{x}_i)\setminus\{\omega_i^\lambda \leq -\mu\}, \end{cases} \tag{3.12}$$

where $\mu\in(0,\lambda)$. Then

$$\frac{d}{ds}\left(E(\omega_s^\lambda)+\sum_{j=p+1}^k \kappa_j\psi_s^\lambda(x_j^\lambda)\right)\Bigg|_{s=0^+}\leq 0.$$

Repeating the argument in Case 1, we obtain

$$\sup_{\{\omega_i^\lambda<0\}\cap B_\delta(\bar{x}_i)}\left(\psi^\lambda+\sum_{j=p+1}^k \kappa_j G(x_j^\lambda,\cdot)\right)=\inf_{\{\omega_i^\lambda>-\lambda\}\cap B_\delta(\bar{x}_i)}\left(\psi^\lambda+\sum_{j=p+1}^k \kappa_j G(x_j^\lambda,\cdot)\right),$$

then we can define

$$c_i^\lambda := -\sup_{\{\omega_i^\lambda<0\}\cap B_\delta(\bar{x}_i)}\left(\psi^\lambda+\sum_{j=p+1}^k \kappa_j G(x_j^\lambda,\cdot)\right)=-\inf_{\{\omega_i^\lambda>-\lambda\}\cap B_\delta(\bar{x}_i)}\left(\psi^\lambda+\sum_{j=p+1}^k \kappa_j G(x_j^\lambda,\cdot)\right). \tag{3.13}$$

Similarly we have

$$\omega_i^\lambda = -\lambda I_{\{\psi^\lambda+\sum_{j=p+1}^k \kappa_j G(x_j^\lambda,\cdot)<-c_i^\lambda\}\cap B_\delta(\bar{x}_i)}.$$

□

Lemma 3.5. *For the c_i^λ given in Lemma 3.4, we have the following estimate*

$$c_i^\lambda > -\frac{|\kappa_i|}{2\pi}\ln\delta - C,$$

where $C>0$ is independent of λ and δ.

Proof. We only prove the case $\kappa_i > 0$. The other case is similar. By the definition of c_i^λ (recall (3.6)), we have

$$
\begin{aligned}
c_i^\lambda &= \inf_{\{\omega_i^\lambda>0\}\cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) \\
&\geq \inf_{B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) \quad \text{(by maximum principle)} \\
&\geq \inf_{\partial B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) \\
&\geq \inf_{\partial B_\delta(\bar{x}_i)} \psi^\lambda - C
\end{aligned} \tag{3.14}
$$

for some $C > 0$ not depending on λ and δ. But for any $x \in \partial B_\delta(\bar{x}_i)$,

$$
\begin{aligned}
\psi^\lambda(x) &= -\frac{1}{2\pi} \int_D \ln|x-y| \omega_i^\lambda(y) dy - \int_D h(x,y) \omega_i^\lambda(y) dy + \int_D G(x,y) \sum_{j=1, j\neq i}^{p} \omega_j^\lambda(y) dy \\
&\geq -\frac{\kappa_i}{2\pi} \ln|2\delta| - C
\end{aligned} \tag{3.15}
$$

for some $C > 0$ not depending on λ and δ. Here we use the fact that $dist(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$ for any $i \neq j, 1 \leq i,j \leq p$, and $dist(B_{\delta_0}(\bar{x}_i), \partial D) > 0, 1 \leq i \leq p$.

Combining (3.14) with (3.15) we get the desired result. □

Lemma 3.6. *For δ sufficiently small, not depending on λ, the following assertion holds true*

$$
sgn(\kappa_i) \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) - c_i^\lambda \leq 0 \quad on\ \partial B_{\delta_0}(\bar{x}_i) \tag{3.16}
$$

for each $1 \leq i \leq p$.

Proof. Let $1 \leq i \leq p$ be fixed. By Lemma 3.5 it suffices to show that

$$
\sup_{x \in \partial B_{\delta_0}(\bar{x}_i)} \left| \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) \right| \leq C
$$

for some $C > 0$ not depending on λ and δ. In fact, since for any $j \neq i, j = p+1, \cdots, k$, $dist(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$, we have

$$
\sup_{x \in \partial B_{\delta_0}(\bar{x}_i)} \left| \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) \right| \leq C.
$$

It remains to show that $\sup_{x\in\partial B_{\delta_0}(\bar{x}_i)}|\psi^\lambda(x)|\leq C$. For any $x\in\partial B_{\delta_0}(\bar{x}_i)$, we estimate $\psi^\lambda(x)$ as follows

$$
\begin{aligned}
|\psi^\lambda(x)| &\leq \left|\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i^\lambda(y)dy\right| + \left|\int_D h(x,y)\omega_i^\lambda(y)dy\right| + \left|\int_D G(x,y)\sum_{j\neq i,j=1}^{p}\omega_j^\lambda(y)dy\right| \\
&\leq -\frac{|\kappa_i|}{2\pi}\ln(\delta_0-\delta)+C \quad (\text{recall that } \delta<\frac{\delta_0}{2}) \\
&\leq -\frac{|\kappa_i|}{2\pi}\ln(\frac{\delta_0}{2})+C.
\end{aligned}
$$

Thus the proof is completed.

□

From now on we fix δ such that (3.16) holds true.

Now we turn to analyze the asymptotic behavior of the maximizer $(\omega^\lambda, x_{p+1}^\lambda,\cdot\cdot\cdot,x_k^\lambda)$ as $\lambda\to+\infty$. We will show that as $\lambda\to+\infty$, the support of ω_i^λ shrinks to $\bar{x}_i$ for $i=1,\cdot\cdot\cdot,p$ and $x_j^\lambda\to\bar{x}_j$ for $j=p+1,\cdot\cdot\cdot,k$. To achieve this goal, we estimate the energy of the background vorticity first.

For simplicity, hereafter we will use C to denote various positive numbers not depending on λ, and $o(1)$ to denote various quantities that go to zero as $\lambda\to+\infty$.

Lemma 3.7. $E(\omega^\lambda)\geq -\frac{1}{4\pi}\sum_{i=1}^p\kappa_i^2\ln\varepsilon_i - C$*, where* ε_i *is the positive number such that* $\lambda\pi\varepsilon_i^2=|\kappa_i|$.

Proof. Choose the test function $\omega=\sum_{i=1}^p\omega_i$, where $\omega_i=sgn(\kappa_i)\lambda I_{B_{\varepsilon_i}(\bar{x}_i)}$. It is obvious that $\omega\in N_p^\lambda$, so by (3.3)

$$
E(\omega)+\sum_{j=p+1}^{k}\kappa_j\psi(x_j^\lambda)\leq E(\omega^\lambda)+\sum_{j=p+1}^{k}\kappa_j\psi^\lambda(x_j^\lambda),
$$

where $\psi=(-\Delta)^{-1}\omega$. It is easy to check that

$$
\left|\sum_{j=p+1}^{k}\kappa_j\psi(x_j^\lambda)\right|\leq C, \quad \left|\sum_{j=p+1}^{k}\kappa_j\psi^\lambda(x_j^\lambda)\right|\leq C,
$$

so

$$
E(\omega^\lambda)\geq E(\omega)-C. \tag{3.17}
$$

On the other hand,

$$\begin{aligned} E(\omega) &= \frac{1}{2}\int_D\int_D G(x,y)\omega(x)\omega(y)dxdy \\ &= \sum_{i=1}^{p} E(\omega_i) + \sum_{1\le i<j\le p}\int_D\int_D G(x,y)\omega_i(x)\omega_j(y)dxdy \\ &\ge \sum_{i=1}^{p} E(\omega_i) - C. \end{aligned} \tag{3.18}$$

Each $E(\omega_i)$ can be calculated directly as follows

$$\begin{aligned} E(\omega_i) =&\frac{1}{2}\int_D\int_D G(x,y)\omega_i(x)\omega_i(y)dxdy \\ =&\frac{1}{2}\int_{B_{\varepsilon_i}(\bar{x}_i)}\int_{B_{\varepsilon_i}(\bar{x}_i)} -\frac{1}{2\pi}\ln|x-y|\omega_i(x)\omega_i(y)dxdy \\ &-\frac{1}{2}\int_{B_{\varepsilon_i}(\bar{x}_i)}\int_{B_{\varepsilon_i}(\bar{x}_i)} h(x,y)\omega_i(x)\omega_i(y)dxdy \\ \ge& -\frac{\kappa_i^2}{4\pi}\ln(2\varepsilon_i) - \sum_{i=1}^{p}\kappa_i^2 H(\bar{x}_i) + o(1). \end{aligned} \tag{3.19}$$

Now (3.17),(3.18) and (3.19) together give the desired result. □

Now we define

$$T^\lambda := \sum_{i=1}^{p}\frac{sgn(\kappa_i)}{2}\int_D \omega_i^\lambda(x)\left(sgn(\kappa_i)(\psi^\lambda(x) + \sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda, x)) - c_i^\lambda\right)dx, \tag{3.20}$$

which represents the total kinetic energy of the fluid on the vorticity set $\cup_{i=1}^{p} A_i^\lambda$.

Lemma 3.8. $T^\lambda \le C$.

Proof. For simplicity we denote

$$\zeta_i^\lambda := sgn(\kappa_i)\left(\psi^\lambda + \sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda, \cdot)\right) - c_i^\lambda.$$

It suffices to prove that

$$sgn(\kappa_i)\int_D \omega_i^\lambda \zeta_i^\lambda dx \le C$$

for each $1 \le i \le p$.

On the one hand, by Hölder's inequality and Sobolev embedding $W^{1,1}(B_\delta(\bar{x}_i)) \hookrightarrow L^2(B_\delta(\bar{x}_i))$, we have

$$\begin{aligned}
&sgn(\kappa_i)\int_D \omega_i^\lambda \zeta_i^\lambda dx = \lambda \int_{A_i^\lambda} \zeta_i^\lambda dx \\
\leq & \lambda |A_i^\lambda|^{\frac{1}{2}} \left(\int_{A_i^\lambda} |\zeta_i^\lambda|^2 dx\right)^{\frac{1}{2}} \quad (\zeta_i^\lambda \geq 0 \text{ on } A_i^\lambda) \\
\leq & \lambda |A_i^\lambda|^{\frac{1}{2}} \left(\int_{B_\delta(\bar{x}_i)} |(\zeta_i^\lambda)^+|^2 dx\right)^{\frac{1}{2}} \\
\leq & C\lambda |A_i^\lambda|^{\frac{1}{2}} \left(\int_{B_\delta(\bar{x}_i)} (\zeta_i^\lambda)^+ dx + \int_{B_\delta(\bar{x}_i)} |\nabla(\zeta_i^\lambda)^+| dx\right) \\
\leq & C\lambda |A_i^\lambda|^{\frac{1}{2}} \left(\int_{A_i^\lambda} \zeta_i^\lambda dx + \int_{A_i^\lambda} |\nabla \zeta_i^\lambda| dx\right) \\
\leq & C\lambda |A_i^\lambda|^{\frac{1}{2}} |A_i^\lambda|^{\frac{1}{2}} \left(\int_{A_i^\lambda} |\nabla \zeta_i^\lambda|^2 dx\right)^{\frac{1}{2}} + C|A_i^\lambda|^{\frac{1}{2}} sgn(\kappa_i) \int_D \omega_i^\lambda \zeta_i^\lambda dx \\
\leq & C \left(\int_{A_i^\lambda} |\nabla \zeta_i^\lambda|^2 dx\right)^{\frac{1}{2}} + o(1) sgn(\kappa_i) \int_D \omega_i^\lambda \zeta_i^\lambda dx,
\end{aligned} \tag{3.21}$$

which implies

$$sgn(\kappa_i)\int_D \omega_i^\lambda \zeta_i^\lambda dx \leq C \left(\int_{A_i^\lambda} |\nabla \zeta_i^\lambda|^2 dx\right)^{\frac{1}{2}}. \tag{3.22}$$

On the other hand, since $\zeta_i^\lambda \leq 0$ on $\partial B_{\delta_0}(\bar{x}_i)$ (recall (3.16)), integration by parts gives

$$\begin{aligned}
sgn(\kappa_i)\int_D \omega_i^\lambda \zeta_i^\lambda dx &= sgn(\kappa_i)\int_{B_{\delta_0}(\bar{x}_i)} \omega_i^\lambda (\zeta_i^\lambda)^+ dx \\
&= \int_{B_{\delta_0}(\bar{x}_i)} |\nabla(\zeta_i^\lambda)^+|^2 dx \\
&\geq \int_{A_i^\lambda} |\nabla \zeta_i^\lambda|^2 dx.
\end{aligned} \tag{3.23}$$

Combining (3.22) with (3.23) we complete the proof. □

Lemma 3.9. $\sum_{i=1}^p c_i^\lambda |\kappa_i| \geq -\frac{1}{2\pi}\sum_{i=1}^p \kappa_i^2 \ln \varepsilon_i - C$.

Proof. By the definition of T^λ, the following identity holds true

$$T^\lambda = E(\omega^\lambda) + \frac{1}{2}\sum_{i=1}^p \sum_{j=p+1}^k \int_D \omega_i^\lambda(x)\kappa_j G(x_j^\lambda, x) dx - \frac{1}{2}\sum_{i=1}^p c_i^\lambda |\kappa_i|.$$

Since $dist(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$ for any $i = 1, \cdots, p$ and $j = p+1, \cdots, k$, it is easy to see that

$$\left| \frac{1}{2} \sum_{i=1}^{p} \sum_{j=p+1}^{k} \int_D \omega_i^\lambda(x) \kappa_j G(x_j^\lambda, x) dx \right| \le C,$$

then the desired result follows from Lemma 3.7 and Lemma 3.8. □

Now we are ready to estimate the size of $supp(\omega_i^\lambda)$.

Lemma 3.10. *There exists $R_0 > 0$ not depending on λ such that $diam(supp(\omega_i^\lambda)) \le R_0 \varepsilon_i$, $i = 1, \cdots, p$.*

Proof. For any $x_i \in supp(\omega_i^\lambda)$, $i = 1, \cdots, p$, we have

$$sgn(\kappa_i) \left(\psi^\lambda(x_i) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x_i) \right) \ge c_i^\lambda.$$

It is easy to see that

$$\left| \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x_i) \right| \le C,$$

so we have

$$sgn(\kappa_i) \psi^\lambda(x_i) \ge c_i^\lambda - C,$$

which gives

$$\int_D -\frac{1}{2\pi} \ln |x_i - y| |\omega_i^\lambda(y)| dy \ge c_i^\lambda - C. \tag{3.24}$$

Combining (3.24) with Lemma 3.9 we obtain

$$\sum_{i=1}^{p} |\kappa_i| \int_D -\frac{1}{2\pi} \ln |x_i - y| |\omega_i^\lambda(y)| dy \ge -\frac{1}{2\pi} \sum_{i=1}^{p} \kappa_i^2 \ln \varepsilon_i - C,$$

or equivalently

$$\sum_{i=1}^{p} \frac{|\kappa_i|}{2\pi} \int_D \ln \frac{\varepsilon_i}{|x_i - y|} |\omega_i^\lambda(y)| dy \ge -C.$$

For any $R > 1$ to be determined, we have

$$\sum_{i=1}^{p} \frac{|\kappa_i|}{2\pi} \int_{D \backslash B_{R\varepsilon_i}(x_i)} \ln \frac{\varepsilon_i}{|x_i - y|} |\omega_i^\lambda(y)| dy + \sum_{i=1}^{p} \frac{|\kappa_i|}{2\pi} \int_{B_{R\varepsilon_i}(x_i)} \ln \frac{\varepsilon_i}{|x_i - y|} |\omega_i^\lambda(y)| dy \ge -C. \tag{3.25}$$

The second integral in (3.25) is bounded (in fact, it can be calculated explicitly), that is,

$$\left| \sum_{i=1}^{p} \frac{|\kappa_i|}{2\pi} \int_{B_{R\varepsilon_i}(x_i)} \ln \frac{\varepsilon_i}{|x_i - y|} |\omega_i^\lambda(y)| dy \right| \le C,$$

so we get

$$\sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_{D\backslash B_{R\varepsilon_i}(x_i)}\ln\frac{\varepsilon_i}{|x_i-y|}|\omega_i^\lambda(y)|dy\geq -C, \tag{3.26}$$

which implies

$$\sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_{D\backslash B_{R\varepsilon_i}(x_i)}|\omega_i^\lambda(y)|dy\leq\frac{C}{\ln R}, \tag{3.27}$$

consequently for each $1\leq i\leq p$ we obtain

$$\frac{|\kappa_i|}{2\pi}\int_{D\backslash B_{R\varepsilon_i}(x_i)}|\omega_i^\lambda(y)|dy\leq\frac{C}{\ln R}. \tag{3.28}$$

Since $\int_D|\omega_i^\lambda(y)|dy=|\kappa_i|$, we can choose R large enough such that

$$\int_{B_{R\varepsilon_i}(x_i)}|\omega_i^\lambda(y)|dy>\frac{|\kappa_i|}{2},\quad i=1,\cdots,p. \tag{3.29}$$

Now we claim that

$$diam(supp(\omega_i^\lambda))\leq 2R\varepsilon_i.$$

In fact, supposing that $diam(supp(\omega_i^\lambda))>2R\varepsilon_i$, we can choose $x_i,y_i\in supp(\omega_i^\lambda)$ such that $|x_i-y_i|>2R\varepsilon_i$, then by (3.29) (recall that in (3.29) $x_i\in supp(\omega_i^\lambda)$ is arbitrary)

$$\int_D|\omega_i^\lambda(y)|dy\geq\int_{B_{R\varepsilon_i}(x_i)}|\omega_i^\lambda(y)|dy+\int_{B_{R\varepsilon_i}(y_i)}|\omega_i^\lambda(y)|dy>|\kappa_i|,$$

which is a contradiction.

Finally by choosing $R_0=2R$ we complete the proof.

□

By now we have constructed $\omega_i^\lambda\in N_p^\lambda, i=1,\cdots,p$, and $x_j^\lambda\in\overline{B_\delta(\bar{x}_j)}, j=p+1\cdots,k$. Moreover, we have shown that the diameter of $supp(\omega_i^\lambda)$ vanishes as $\lambda\to+\infty$. To analyze their limiting positions, we define the center of ω_i^λ as follows

$$z_i^\lambda:=\frac{1}{\kappa_i}\int_D x\omega_i^\lambda dx,\quad i=1,\cdots,p.$$

It is easy to see that $z_i^\lambda\in\overline{B_\delta(\bar{x}_i)}$.

Lemma 3.11. *$z_i^\lambda\to\bar{x}_i$ for $1\leq i\leq p$ and $x_j^\lambda\to\bar{x}_j$ for $p+1\leq j\leq k$ as $\lambda\to+\infty$.*

Proof. Up to a subsequence we assume that $z_i^\lambda\to z_i\in\overline{B_\delta(\bar{x}_i)}, 1\leq i\leq p$, and $x_j^\lambda\to z_j\in\overline{B_\delta(\bar{x}_j)}, p+1\leq j\leq k$. It suffices to show that $(z_1,\cdots,z_k)=(\bar{x}_1,\cdots,\bar{x}_k)$.

Define $\omega = \sum_{i=1}^{p} \omega_i$, where $\omega_i = sgn(\kappa_i)\lambda I_{B_{\varepsilon_i}(\bar{x}_i)}$. It is easy to see that $\omega \in N_p^\lambda$, so we have

$$\mathcal{E}(\omega, \bar{x}_{p+1}, \cdots, \bar{x}_k) \le \mathcal{E}(\omega^\lambda, x_{p+1}^\lambda, \cdots, x_k^\lambda),$$

that is,

$$\begin{aligned} &E(\omega) + \sum_{j=p+1}^{k} \kappa_j \psi(\bar{x}_j) + \frac{1}{2} \sum_{\substack{i\neq j \\ p+1\le i,j\le k}} \kappa_i\kappa_j G(\bar{x}_i, \bar{x}_j) - \sum_{j=p+1}^{k} \kappa_j^2 H(\bar{x}_j) \\ \le &E(\omega^\lambda) + \sum_{j=p+1}^{k} \kappa_j \psi^\lambda(x_j^\lambda) + \frac{1}{2} \sum_{\substack{i\neq j \\ p+1\le i,j\le k}} \kappa_i\kappa_j G(x_i^\lambda, x_j^\lambda) - \sum_{j=p+1}^{k} \kappa_j^2 H(x_j^\lambda), \end{aligned} \tag{3.30}$$

where $\psi^\lambda = (-\Delta)^{-1}\omega^\lambda$ and $\psi = (-\Delta)^{-1}\omega$. It is easy to check that

$$\begin{aligned} E(\omega) = &\frac{1}{2} \sum_{i=1}^{p} \int_D \int_D -\frac{1}{2\pi} \ln|x-y| \omega_i(x)\omega_i(y) dxdy - \sum_{i=1}^{p} \kappa_i^2 H(\bar{x}_i) \\ &+ \frac{1}{2} \sum_{\substack{i\neq j \\ 1\le i,j\le p}} \kappa_i\kappa_j G(\bar{x}_i, \bar{x}_j) + o(1), \end{aligned} \tag{3.31}$$

and

$$\sum_{j=p+1}^{k} \kappa_j \psi(\bar{x}_j) = \sum_{i=1}^{p} \sum_{j=p+1}^{k} \kappa_i\kappa_j G(\bar{x}_i, \bar{x}_j) + o(1). \tag{3.32}$$

Similarly

$$\begin{aligned} E(\omega^\lambda) = &\frac{1}{2} \sum_{i=1}^{p} \int_D \int_D -\frac{1}{2\pi} \ln|x-y| \omega_i^\lambda(x)\omega_i^\lambda(y) dxdy - \sum_{i=1}^{p} \kappa_i^2 H(z_i) \\ &+ \frac{1}{2} \sum_{\substack{i\neq j \\ 1\le i,j\le p}} \kappa_i\kappa_j G(z_i, z_j) + o(1), \end{aligned} \tag{3.33}$$

and

$$\sum_{j=p+1}^{k} \kappa_j \psi^\lambda(x_j^\lambda) = \sum_{i=1}^{p} \sum_{j=p+1}^{k} \kappa_i\kappa_j G(z_i, z_j) + o(1). \tag{3.34}$$

Hence from all the above we obtain

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{p}\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i(x)\omega_i(y)dxdy-\sum_{i=1}^{p}\kappa_i^2H(\bar{x}_i)\\
&+\frac{1}{2}\sum_{\substack{i\neq j\\1\leq i,j\leq p}}\kappa_i\kappa_jG(\bar{x}_i,\bar{x}_j)+\sum_{i=1}^{p}\sum_{j=p+1}^{k}\kappa_i\kappa_jG(\bar{x}_i,\bar{x}_j)\\
&+\frac{1}{2}\sum_{\substack{i\neq j\\p+1\leq i,j\leq k}}\kappa_i\kappa_jG(\bar{x}_i,\bar{x}_j)-\sum_{j=p+1}^{k}\kappa_j^2H(\bar{x}_j)\\
\leq&\frac{1}{2}\sum_{i=1}^{p}\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i^\lambda(x)\omega_i^\lambda(y)dxdy-\sum_{i=1}^{p}\kappa_i^2H(z_i)\\
&+\frac{1}{2}\sum_{\substack{i\neq j\\1\leq i,j\leq p}}\kappa_i\kappa_jG(z_i,z_j)+\sum_{i=1}^{p}\sum_{j=p+1}^{k}\kappa_i\kappa_jG(z_i,z_j)\\
&+\frac{1}{2}\sum_{\substack{i\neq j\\p+1\leq i,j\leq k}}\kappa_i\kappa_jG(z_i,z_j)-\sum_{j=p+1}^{k}\kappa_j^2H(z_j)+o(1).
\end{aligned}
\tag{3.35}
$$

On the other hand, by Riesz's rearrangement inequality we have for each $1\leq i\leq p$

$$
\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i^\lambda(x)\omega_i^\lambda(y)dxdy\leq\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i(x)\omega_i(y)dxdy. \tag{3.36}
$$

Therefore by (3.35) and (3.36) we get

$$
\frac{1}{2}\sum_{\substack{i\neq j\\1\leq i,j\leq k}}\kappa_i\kappa_jG(\bar{x}_i,\bar{x}_j)-\sum_{j=1}^{k}\kappa_j^2H(\bar{x}_j)\leq\frac{1}{2}\sum_{\substack{i\neq j\\1\leq i,j\leq k}}\kappa_i\kappa_jG(z_i,z_j)-\sum_{j=1}^{k}\kappa_j^2H(z_j), \tag{3.37}
$$

or equivalently

$$
\mathcal{K}_k(z_1,\cdots,z_k)\leq\mathcal{K}_k(\bar{x}_1,\cdots,\bar{x}_k). \tag{3.38}
$$

Since $(\bar{x}_1,\cdots,\bar{x}_k)$ is the unique minimum point for $\mathcal{K}_k$ on $\overline{B_\delta(\bar{x}_1)}\times\cdots\times\overline{B_\delta(\bar{x}_k)}$, we obtain

$$
(z_1,\cdots,z_k)=(\bar{x}_1,\cdots,\bar{x}_k),
$$

which completes the proof. □

Remark 3.12. It is easy to see that $x_j^\lambda\in B_\delta(\bar{x}_j)$ for $j=p+1,\cdots,k$, and $dist(supp(\omega_i^\lambda),\partial B_\delta(\bar{x}_i))>0$ for $i=1,\cdots,p$, provided that λ is sufficiently large.

4. Proof of Theorem 2.4

Having made all the preparations in the preceding sections, we are now ready to give the proof of Theorem 2.4.

Proof of Theorem 2.4. We only need to prove that $(\omega^\lambda, x^\lambda_{p+1}, \cdots, x^\lambda_k)$ satisfies (2.16) if λ is sufficiently large, since other assertions have been verified in Section 3.

First, by Lemma 3.11, $x^\lambda_j \in B_\delta(\bar{x}_j)$ for $j = p+1, \cdots, k$, thus by (3.4) we have

$$\nabla \left(\psi^\lambda(x) + \sum_{j=p+1, j\neq i}^{k} \kappa_j G(x^\lambda_j, x) - \kappa_i H(x) \right) \Bigg|_{x=x^\lambda_i} = 0, \;\; j = p+1, \cdots, k.$$

Now for any given $\phi \in C^\infty_c(D)$, define a family of transformations $\Phi_t(x), t \in \mathbb{R}$, from D to D by the following ordinary differential equation:

$$\begin{cases} \frac{d\Phi_t(x)}{dt} = J\nabla\phi(\Phi_t(x)), \;\; t \in \mathbb{R}, \\ \Phi_0(x) = x. \end{cases} \tag{4.1}$$

Since $J\nabla\phi$ is a smooth vector field with compact support in D, (4.1) is solvable for all $t \in \mathbb{R}$. It is also easy to see that $J\nabla\phi$ is divergence-free, so by Liouville theorem (see [24], Appendix 1.1) $\Phi_t(x)$ is area-preserving, or equivalently, for any measurable set $A \subset D$

$$|\Phi_t(A)| = |A|, \quad \forall\, t \in \mathbb{R}. \tag{4.2}$$

Now we define a family of test functions

$$\omega^\lambda_t(x) := \omega^\lambda(\Phi_{-t}(x)). \tag{4.3}$$

Since Φ_t is area-preserving and $dist(supp(\omega^\lambda_i), \partial B_\delta(\bar{x}_i)) > 0$ for each $1 \leq i \leq p$, we have $\omega^\lambda_t \in N^\lambda_p$ as long as $|t|$ is sufficiently small. Then by (3.3) we have

$$\frac{d}{dt}\left(E(\omega^\lambda_t) + \sum_{j=p+1}^{k} \kappa_j \psi^\lambda_t(x^\lambda_j) \right) \Bigg|_{t=0} = 0,$$

where $\psi^\lambda_t = (-\Delta)^{-1}\omega^\lambda_t$. It is easy to check that (see [28] for example)

$$E(\omega^\lambda_t) = E(\omega^\lambda) + t\int_D \omega^\lambda(x)\partial(\psi^\lambda(x), \phi(x))dx + o(t),$$

and

$$\psi^\lambda_t(x^\lambda_j) = \psi^\lambda(x^\lambda_j) + t\int_D \omega^\lambda(x)\partial(G(x^\lambda_j, x), \phi(x))dx + o(t),$$

where $o(t)/t \to 0$ as $t \to 0$. Therefore we get

$$\int_D \omega^\lambda(x)\partial\left(\psi^\lambda(x)+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,x),\phi(x)\right)dx=0,$$

which completes the proof of Theorem 2.4.

□

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