

ON A CLASS OF HESSIAN TYPE EQUATIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we consider a class of Hessian type equations which include the $(n - 1)$ Monge-Ampère equation on Riemannian manifolds. The *a priori* C^2 estimates and the existence of solutions are established.

Keywords: Hessian type equations; $(n - 1)$ Monge-Ampère equation; *a priori* C^2 estimate.

1. INTRODUCTION

In this paper, we consider the C^2 estimates for the Hessian equations of the form

$$(1.1) \quad \sigma_k(\lambda(\Delta u g - \nabla^2 u + \chi)) = f(x, u, \nabla u)$$

on a compact Riemannian manifold (M, g) of dimension $n \geq 2$ with smooth boundary $\partial M \neq \emptyset$, where χ is a $(0, 2)$ -tensor field, $\nabla^2 u$ is the Hessian of u , $\lambda(h)$ denotes the eigenvalues of a $(0, 2)$ -tensor field h with respect to the metric g , f is a smooth positive function which may depend on $(x, \nabla u) \in T^*M$ and u , and

$$\sigma_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k}, \quad k = 1, \dots, n,$$

are the elementary symmetric functions.

Equation (1.1) falls in the frame of equations considered in [12] or [14] (see (2.5)) where f is assumed to be convex with respect to ∇u . Using the idea of [6] where the star-shaped hypersurface with a class of prescribed curvatures were considered, we are able to establish the C^2 estimates without this condition.

When $k = n$, the equation (1.1) becomes

$$(1.2) \quad \det(\Delta u g - \nabla^2 u + \chi) = f(x, u, \nabla u)$$

which is called the $(n - 1)$ Monge-Ampère equation. Our interest to study equations (1.1) and (1.2) is partly from their complex analogues. The complex $(n - 1)$ Monge-Ampère equation is closely related to the Gauduchon conjecture ([9, §IV.5]) which was solved by Székelyhidi-Tosatti-Weinkove [31]. For more results about the complex $(n - 1)$

Monge-Ampère equation, the reader is referred to [7, 8, 27, 30, 32, 33] and references therein.

When $k = n = 2$, equation (1.1) is the classic Monge-Ampère equation

$$\det(\nabla^2 u + \chi) = f(x, u, \nabla u)$$

which plays a crucial role in many geometric problems such as the Minkowski problem ([4], [24], [25], [26]) and the Weyl problem ([16], [19], [24]).

The Gårding cone Γ_k is defined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0 \text{ for } j = 1, \dots, k\}, k = 1, \dots, n.$$

We state some well known results about σ_k in Γ_k :

$$(1.3) \quad \frac{\partial \sigma_k}{\partial \lambda_i} = \sigma_{k-1;i}(\lambda) > 0, \text{ for all } \lambda \in \Gamma_k \text{ and } 1 \leq i \leq n,$$

where $\sigma_{k-1;i}(\lambda)$ is the symmetric function with $\lambda_i = 0$,

$$(1.4) \quad \sigma_k^{1/k} \text{ is concave in } \Gamma_k.$$

Definition 1.1. *A function $u \in C^2(M)$ is called admissible if $\lambda(\Delta u g - \nabla^2 u + \chi)(x) \in \Gamma_k$ for all $x \in M$.*

We note that (1.1) is elliptic with respect to admissible solutions. In this paper, we suppose there exists an admissible subsolution $\underline{u} \in C^2(\overline{M})$ satisfying

$$(1.5) \quad \sigma_k(\Delta \underline{u} g - \nabla^2 \underline{u} + \chi) \geq f(x, \underline{u}, \nabla \underline{u}) \text{ in } M.$$

Our main results are the following second order estimates.

Theorem 1.2. *Suppose there exists an admissible function $\underline{u} \in C^2(\overline{M})$. Let $u \in C^4(M) \cap C^2(\overline{M})$ be an admissible solution to (1.1) on \overline{M} . Then there exists a positive constant C depending on $|u|_{C^1(\overline{M})}$, $|\underline{u}|_{C^2(\overline{M})}$, $|f|_{C^2}$ and $\inf f$ such that*

$$(1.6) \quad \sup_{\overline{M}} |\nabla^2 u| \leq C(1 + \sup_{\partial M} |\nabla^2 u|).$$

In this paper we consider the Dirichlet boundary condition

$$(1.7) \quad u = \varphi \text{ on } \partial M.$$

Theorem 1.3. *Suppose there exists an admissible function $\underline{u} \in C^2(\overline{M})$ with $\underline{u} = \varphi$ on ∂M . Let $u \in C^3(M) \cap C^2(\overline{M})$ be an admissible solution to (1.1) and (1.7) with $u \geq \underline{u}$ on \overline{M} . Then, there exists a positive constant C depending on $|u|_{C^1(\overline{M})}$, $|\underline{u}|_{C^2(\overline{M})}$, $|\varphi|_{C^4(\partial M)}$, $|f|_{C^1}$ and $\inf f$ such that*

$$(1.8) \quad \sup_{\partial M} |\nabla^2 u| \leq C.$$

Since the function f may depend on the gradient of u , to obtain the gradient estimates, we need the following growth condition

$$(1.9) \quad \begin{aligned} p \cdot \nabla_p f^{1/k}(x, z, p) &\leq \bar{f}(x, z)(1 + |p|^{\gamma_1}) \\ p \cdot \nabla_x f^{1/k}(x, z, p) + |p|^2 f_z^{1/k}(x, z, p) &\geq -\bar{f}(x, z)(1 + |p|^{\gamma_2}) \end{aligned}$$

when $|p|$ is sufficiently large, where $\gamma_1 < 2$, $\gamma_2 < 4$ are positive constants and \bar{f} is a positive continuous function of $(x, z) \in \bar{M} \times \mathbb{R}$. Based on the *a priori* estimates, in particular, Theorem 1.2 and Theorem 1.3, we can prove the existence and regularity result as follows.

Theorem 1.4. *Suppose there exists an admissible function $\underline{u} \in C^2(\bar{M})$ satisfying (1.5) and $\underline{u} = \varphi$ on ∂M . Assume that $f > 0$ satisfies (1.9) and*

$$(1.10) \quad \sup_{(x,z,p) \in T^*M \times \mathbb{R}} \frac{-f_z(x, z, p)}{f(x, z, p)} < \infty.$$

Then there exists an admissible solution $u \in C^\infty(\bar{M})$ of (1.1) and (1.7). Moreover, the solution is unique if $f_z \geq 0$.

Remark 1.5. *The function \underline{u} does not need to be a subsolution to derive the second order estimates in Theorem 1.2 and Theorem 1.3. The conditions that \underline{u} is a subsolution (satisfying (1.5)) and (1.10) are used to derive the existence of solutions to (1.1) and (1.7) by the continuity method and degree theory as in [12].*

The general Hessian type equations as well as their Dirichlet problems and other boundary value problem were studied by many people. We refer the reader to [3, 5, 11, 12, 13, 14, 15, 20, 23, 34, 35, 36, 37, 21, 22] and the reference therein for previous studies.

The equation (1.1) is related to the k -Hessian equation

$$(1.11) \quad \sigma_k(\lambda(\nabla^2 u + \chi)) = f(x, u, \nabla u).$$

It is an interesting question to establish the second order estimates for the equation (1.11) without the condition that f is convex in ∇u . This is true for the special cases $k = 2$, $k = n - 1$ and $k = n - 2$ according to [17], [28] and [29] respectively. It is of interest to consider the general k .

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2. PRELIMINARIES

In this section, we provide some preliminaries which may be used in the following sections. Throughout the paper let ∇ denote the Levi-Civita connection of (M, g) . The curvature tensor is defined by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z.$$

Under a local frame e_1, \dots, e_n on M we write $g_{ij} = g(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, while the Christoffel symbols Γ_{ij}^k and curvature coefficients are given respectively by $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ and

$$R_{ijkl} = g(R(e_k, e_l)e_j, e_i), \quad R_{jkl}^i = g^{im} R_{mjkl}.$$

We shall write $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j - \Gamma_{ij}^k \nabla_k$, etc.

For any $v \in C^4(\overline{M})$, we usually identify ∇v with its gradient, and use $\nabla^2 v$ to denote its Hessian which is locally given by $\nabla_{ij} v = \nabla_i(\nabla_j v) - \Gamma_{ij}^k \nabla_k v$. We recall that $\nabla_{ij} v = \nabla_{ji} v$ and

$$(2.1) \quad \nabla_{ijk} v - \nabla_{jik} v = R_{kij}^l \nabla_l v,$$

$$(2.2) \quad \begin{aligned} \nabla_{ijkl} v - \nabla_{klij} v &= R_{ljk}^m \nabla_{im} v + \nabla_i R_{ljk}^m \nabla_m v + R_{lik}^m \nabla_{jm} v \\ &\quad + R_{jik}^m \nabla_{lm} v + R_{jil}^m \nabla_{km} v + \nabla_k R_{jil}^m \nabla_m v. \end{aligned}$$

For $\lambda \in \mathbb{R}^n$ let

$$\mu_i = \sum_{j \neq i} \lambda_j, \quad i = 1, \dots, n.$$

Define the function $h(\lambda)$ on the cone

$$\Gamma := \{\lambda \in \mathbb{R}^n : (\mu_1, \dots, \mu_n) \in \Gamma_k\}$$

by

$$(2.3) \quad h(\lambda) := \sigma_k^{1/k}(\mu_1, \dots, \mu_n).$$

Let

$$(2.4) \quad \chi_1 := \frac{\text{tr}_g(\chi)}{n-1} g - \chi.$$

We find the equation (1.1) can be rewritten by

$$(2.5) \quad F(\nabla^2 u + \chi_1) := h(\lambda(\nabla^2 u + \chi_1)) = \tilde{f}(x, u, \nabla u) := f^{1/k}(x, u, \nabla u)$$

and a function $u \in C^2(M)$ is admissible if and only if $\lambda(\nabla^2 u + \chi_1) \in \Gamma$ in M . It is easy to derive from (1.3) and (1.4) that

$$(2.6) \quad h_i(\lambda) = \frac{\partial h(\lambda)}{\partial \lambda_i} > 0, \quad \text{in } \Gamma, i = 1, \dots, n$$

and

$$(2.7) \quad h(\lambda) \text{ is concave in } \Gamma.$$

Furthermore, $h > 0$ in Γ and $h = 0$ on $\partial\Gamma$.

Lemma 2.1. *There exists a positive constant ν_0 depending only on n and k such that*

$$(2.8) \quad h_j(\lambda) \geq \nu_0 \left(1 + \sum_i h_i(\lambda)\right), \text{ if } \lambda_j < 0, \forall \lambda \in \Gamma.$$

Proof. Suppose

$$\tilde{h}_i(\lambda) = \frac{\partial \sigma_k^{1/k}}{\partial \mu_i}, i = 1, \dots, n.$$

It is easy to find

$$h_i(\lambda) = \sum_{l \neq i} \tilde{h}_l(\lambda), i = 1, \dots, n$$

and by Maclaurin's inequality,

$$\sum_{i=1}^n \tilde{h}_i(\lambda) \geq C_{n,k}$$

for some positive constant $C_{n,k}$ depending only on n and k . If $\lambda_j < 0$, there exists an index $l \neq j$ such that $\lambda_l > 0$ since $\Gamma \subset \Gamma_1$. Thus, $\mu_j > \mu_l$ and since $\mu \in \Gamma_k$, we have

$$\sigma_{k-1;j}(\mu) \leq \sigma_{k-1;l}(\mu).$$

Hence $\tilde{h}_j(\lambda) = \frac{1}{k} \sigma_k(\mu)^{1/k-1} \sigma_{k-1;\mu_j}(\mu) \leq \frac{1}{k} \sigma_k(\mu)^{1/k-1} \sigma_{k-1;\mu_l}(\mu) = \tilde{h}_l(\lambda)$.

It follows that

$$\begin{aligned} 2h_j(\lambda) &= 2 \sum_{s \neq j} \tilde{h}_s(\lambda) \geq \sum_{s \neq j} \tilde{h}_s(\lambda) + \tilde{h}_l(\lambda) \\ &\geq \sum_{s \neq j} \tilde{h}_s(\lambda) + \tilde{h}_j(\lambda) = \sum_{s=1}^n \tilde{h}_s(\lambda). \end{aligned}$$

Therefore,

$$h_j(\lambda) \geq \frac{1}{2} \sum_{s=1}^n \tilde{h}_s(\lambda) \geq \frac{1}{4(n-1)} \sum_{i=1}^n h_i(\lambda) + \frac{C_{n,k}}{4}$$

and (2.8) is proved. \square

It is also easy to see

$$(2.9) \quad \sum h_i(\lambda) \lambda_i = \sum \tilde{h}_i(\lambda) \mu_i = h(\lambda) > 0 \text{ in } \Gamma.$$

Lemma 2.2. *For any constant $A > 0$ and any compact set K in Γ there is a number $R = R(A, K)$ such that*

$$(2.10) \quad h(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) \geq A, \text{ for all } \lambda \in K.$$

Proof. First there exists a compact set $K' \subset \Gamma_k$ such that

$$\mu = \left(\sum_{j \neq 1} \lambda_j, \dots, \sum_{j \neq n} \lambda_j \right) \in K' \text{ for all } \lambda \in K.$$

Next, for $R > 0$ we have

$$\begin{aligned} h^k(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) &= \sigma_k(\mu_1 + R, \dots, \mu_{n-1} + R, \mu_n) \\ &\geq \sigma_k(\mu_1 + R, \mu_2, \dots, \mu_n) \\ &= (\mu_1 + R)\sigma_{k-1}(\mu_2, \dots, \mu_n) + \sigma_k(\mu_2, \dots, \mu_n) \\ &= \sigma_k(\mu) + R\sigma_{k-1}(\mu_2, \dots, \mu_n) \geq Ra_0 \end{aligned}$$

for some positive constant a_0 depending on K' . (2.10) follows when R is sufficiently large. \square

Next, we have (cf. e.g. [37] and [18]),

$$\prod_{i=1}^n \sigma_{k-1;i}(\mu) \geq \frac{k^n}{n^n} (C_n^k)^{n/k} [\sigma_k(\mu)]^{n(k-1)/k} \text{ for any } \mu \in \Gamma_k.$$

Therefore, for any positive constants f_1, f_2 with $0 < \tilde{f}_1 < \tilde{f}_2 < \infty$ there exists a positive constant c_0 depending only on n, k, \tilde{f}_1 and \tilde{f}_2 such that

$$(2.11) \quad \prod_{i=1}^n h_i(\lambda) \geq \prod_{i=1}^n \tilde{h}_i(\lambda) \geq c_0$$

for any $\lambda \in \Gamma_{\tilde{f}_1, \tilde{f}_2} := \{\lambda \in \Gamma : \tilde{f}_1 \leq h(\lambda) \leq \tilde{f}_2\}$.

In the following sections, u will denote an admissible solution of (1.1) and (1.7) with $u \geq \underline{u}$ in M . For simplicity we denote $U := \nabla^2 u + \chi_1$ and, under a local frame e_1, \dots, e_n ,

$$U_{ij} := U(e_i, e_j) = \nabla_{ij} u + (\chi_1)_{ij}.$$

Denote $G(X) = \sigma_k^{1/k}(\lambda(X))$ for a $(0, 2)$ -tensor field X on \bar{M} . Let $\eta = \Delta u g - \nabla^2 u + \chi$ and locally $\eta_{ij} = (g^{pq} \nabla_{pq} u) \delta_{ij} - g^{il} \nabla_{lj} u + g^{il} \chi_{lj}$. Throughout this paper we denote

$$G^{ij} = \frac{\partial G}{\partial \eta_{ij}} \text{ and } F^{ij} = \frac{\partial F}{\partial U_{ij}}.$$

Then we have

$$(2.12) \quad F^{ij} = \sum_l G^{ll} g^{ij} - g^{il} G^{lj}.$$

3. A REMARK FOR THE *a priori* C^1 ESTIMATES

We first note that $\Gamma_k \subset \Gamma_1$. Let $u \in C^2(M) \cap C^0(\overline{M})$ be an admissible solution of (1.1) and (1.7) with $u \geq \underline{u}$, we have

$$\Delta u + \text{tr}_g(\chi) > 0 \text{ in } M.$$

Let ϕ be the solution of

$$\Delta \phi + \text{tr}_g(\chi) = 0 \text{ in } M$$

with $\phi = \varphi$ on ∂M . By the maximum principle, we have

$$\underline{u} \leq u \leq \phi.$$

Therefore, we obtain

$$(3.1) \quad \sup_{\overline{M}} |u| + \sup_{\partial M} |\nabla u| \leq C,$$

where the positive constant C depends only on $|\underline{u}|_{C^1(\overline{M})}$ and $|\phi|_{C^1(\overline{M})}$.

Theorem 3.1. *Let $u \in C^3(M) \cap C^1(\overline{M})$ be an admissible solution of (1.1). Suppose f satisfies (1.9). Then we have*

$$(3.2) \quad \sup_{\overline{M}} |\nabla u| \leq C(1 + \sup_{\partial M} |\nabla u|).$$

Since h satisfies (2.6)-(2.9), Theorem 3.1 follows by Theorem 5.1 in [12] or Theorem 4.3 in [15]. Combining (3.1) and (3.2), the C^1 estimates are proved.

4. SECOND ORDER ESTIMATES

In this section, we deal with the second order estimates for the Hessian type equation (1.1). The idea is mainly from [6]. First since $\lambda(\Delta \underline{u}g - \nabla^2 \underline{u} + \chi) \in \Gamma_k$, we can find a positive constant ϵ_0 such that $\lambda(\Delta \underline{u}g - \nabla^2 \underline{u} + \chi - \epsilon_0 g)(x) \in \Gamma_k$ for all $x \in \overline{M}$. Note that if $|u|_{C^1(\overline{M})}$ is under control there exist uniform constants $\tilde{f}_2 \geq \tilde{f}_1 > 0$ such that

$$(4.1) \quad \tilde{f}_1 \leq \tilde{f}(x, u, \nabla u) \leq \tilde{f}_2 \text{ on } \overline{M}.$$

Hence by the concavity of $\sigma_k^{1/k}$ in Γ_k , we have

$$\begin{aligned} & F^{ij} \nabla_{ij}(\underline{u} - u) - \epsilon_0 \sum G^{ii} \\ &= G^{ij} [(\Delta \underline{u} - \Delta u) \delta_{ij} - g^{il} (\nabla_{lj} \underline{u} - \nabla_{lj} u) - \epsilon_0 \delta_{ij}] \\ &\geq G(\Delta \underline{u}g - \nabla^2 \underline{u} + \chi - \epsilon_0 g) - G(\Delta u g - \nabla^2 u + \chi) \\ &= G(\Delta \underline{u}g - \nabla^2 \underline{u} + \chi - \epsilon_0 g) - \tilde{f}(x, u, \nabla u) \geq -C \end{aligned}$$

on \overline{M} . It follows that

$$(4.2) \quad F^{ij} \nabla_{ij}(\underline{u} - u) \geq \epsilon_0 \sum G^{ii} - C \quad \text{on } \overline{M}.$$

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Consider the quantity

$$W = \max_{x \in \overline{M}} \max_{\xi \in T_x M, |\xi|=1} U_{\xi\xi} e^\phi,$$

where $U_{\xi\xi} = U(\xi, \xi)$,

$$\phi = \frac{\delta |\nabla u|^2}{2} + b(\underline{u} - u)$$

and δ (sufficiently small), b (sufficiently large) are positive constants to be determined. Suppose W is achieved at an interior point $x_0 \in M$ and $\xi_0 \in T_{x_0} M$. Choose a smooth orthonormal local frame e_1, \dots, e_n about x_0 such that at x_0 , $e_1 = \xi_0$, $\nabla_{e_i} e_j = 0$ and $\{U_{ij}\}$ is diagonal. So $\{\eta_{ij}(x_0)\}$, $\{G^{ij}(x_0)\}$ and $\{F^{ij}(x_0)\}$ are all diagonal. We may assume

$$U_{11}(x_0) \geq \dots \geq U_{nn}(x_0)$$

so that

$$\eta_{11}(x_0) \leq \dots \leq \eta_{nn}(x_0)$$

since $\eta_{jj}(x_0) = \sum_{i \neq j} U_{ii}(x_0)$ for $1 \leq j \leq n$. We have, at x_0 where the function $\log U_{11} + \phi$ attains its maximum,

$$(4.3) \quad \frac{\nabla_i U_{11}}{U_{11}} + \delta \nabla_i u \nabla_{ii} u + b \nabla_i(\underline{u} - u) = 0,$$

for each $i = 1, \dots, n$ and

$$(4.4) \quad F^{ii} \left\{ \frac{\nabla_{ii} U_{11}}{U_{11}} - \left(\frac{\nabla_i U_{11}}{U_{11}} \right)^2 + \delta (\nabla_{ii} u)^2 + b \nabla_{ii}(\underline{u} - u) + \delta \nabla_l u \nabla_{iil} u \right\} \leq 0.$$

Differentiating the equation (1.1) twice we get, at x_0 ,

$$(4.5) \quad F^{ii} \nabla_i U_{ii} = \nabla'_i \tilde{f} + \tilde{f}_u \nabla_l u + \tilde{f}_{p_l} \nabla_l u,$$

where $\nabla'_i \tilde{f}$ denotes the partial covariant derivative of \tilde{f} when viewed as depending on $x \in M$ only for $l = 1, \dots, n$. Combining with (2.1) and (4.3), we obtain

$$(4.6) \quad \begin{aligned} F^{ii} \nabla_{11} U_{ii} + G^{ij, st} \nabla_1 \eta_{ij} \nabla_1 \eta_{st} &= \nabla_{11} \tilde{f} \\ &\geq -C U_{11}^2 + \tilde{f}_{p_l} \nabla_{11} u \geq -C U_{11}^2 - C b U_{11} \end{aligned}$$

provided U_{11} is sufficiently large. Now we consider two cases.

Case 1. $|U_{ii}| \leq \epsilon U_{11}$ for all $i > 1$, where the positive constant ϵ will be determined later.

Thus, we find

$$|\eta_{11}| \leq (n-1)\epsilon U_{11}$$

and

$$(1 - (n-2)\epsilon)U_{11} \leq \eta_{22} \leq \dots \leq \eta_{nn} \leq (1 + (n-2)\epsilon)U_{11}.$$

It follows that

$$\begin{aligned} \sigma_{k-1}(\eta) &\geq [(1 - (n-2)\epsilon)^{k-1} - (n-1)\epsilon(1 + (n-2)\epsilon)^{k-2}]U_{11}^{k-1} \\ &\geq \frac{1}{2}U_{11}^{k-1} \end{aligned}$$

if ϵ is sufficiently small. Therefore, there exists a positive constant c_0 depending only on n, k and \tilde{f}_1 such that

$$(4.7) \quad \sum G^{ii} = \frac{n-k+1}{k} f^{1/k-1} \sigma_{k-1}(\eta) \geq c_0 U_{11}^{k-1}.$$

As in [13], using an inequality due to Andrews [1] and Gerhardt [10] and by (2.1) and (4.3), we have

$$\begin{aligned} -G^{ij, st} \nabla_1 \eta_{ij} \nabla_1 \eta_{st} &\geq \sum_{i \neq j} \frac{G^{ii} - G^{jj}}{\eta_{jj} - \eta_{ii}} (\nabla_1 \eta_{ij})^2 \\ &\geq 2 \sum_{i \geq 2} \frac{F^{ii} - F^{11}}{U_{11} - U_{ii}} (\nabla_i U_{11})^2 \\ (4.8) \quad &\geq \frac{2(1-\theta)}{1+\theta} \sum_{i \geq 2} \frac{F^{ii} (\nabla_i U_{11})^2}{U_{11}} - \frac{C}{\theta U_{11}} \sum F^{ii} - U_{11} F^{11} |\nabla \phi|^2 \\ &\geq \sum_{1 \leq i \leq n} \frac{F^{ii} (\nabla_i U_{11})^2}{U_{11}} - \frac{C}{\theta U_{11}} \sum F^{ii} \\ &\quad - C U_{11} (b^2 + \delta^2 U_{11}^2) F^{11} \end{aligned}$$

by fixing θ small enough so that

$$\frac{2(1-\theta)}{1+\theta} \geq 1.$$

Now we can derive from (4.2) and (4.4)-(4.8) that, at x_0 ,

$$\begin{aligned} 0 &\geq -C \sum F^{ii} - C U_{11} - \frac{C}{\theta U_{11}^2} \sum F^{ii} + \delta F^{ii} U_{ii}^2 \\ &\quad - C(b^2 + \delta^2 U_{11}^2) F^{11} - Cb + b\epsilon_0 \sum G^{ii} - C\delta U_{11} \\ (4.9) \quad &\geq (\delta - C\delta^2) F^{ii} U_{ii}^2 - Cb^2 F^{11} + \frac{b\epsilon_0 c_0}{2} U_{11}^{k-1} \\ &\quad - C U_{11} + \frac{b\epsilon_0}{2} \sum G^{ii} - C(n-1) \sum G^{ii} \end{aligned}$$

provided $U_{11} \geq b$. Suppose b is large enough such that in (4.9),

$$\frac{b\epsilon_0 c_0}{2} U_{11}^{k-1} - C U_{11} + \frac{b\epsilon_0}{2} \sum G^{ii} - C(n-1) \sum G^{ii} \geq 0.$$

Therefore, we have

$$(4.10) \quad 0 \geq (\delta - C\delta^2) F^{ii} U_{ii}^2 - Cb^2 F^{11}.$$

Case 2. $U_{22} > \epsilon U_{11}$ or $U_{nn} < -\epsilon U_{11}$.

By the definition of F^{ij} , we have, at x_0 ,

$$F^{jj} = \sum_{i \neq j} G^{ii} \geq G^{11} \geq \frac{1}{n} \sum G^{ii}, \text{ for all } j = 2, \dots, n.$$

It follows that, in this case,

$$(4.11) \quad F^{ii} U_{ii}^2 \geq \frac{\epsilon U_{11}^2}{n} \sum G^{ii}.$$

Next, by MacLaurin's inequality, $\sum G^{ii} \geq c_1 > 0$ for some positive constant c_1 depending only on n and k . We have, by (4.2)-(4.6), (4.11) and the concavity of G ,

$$(4.12) \quad \begin{aligned} 0 &\geq -C U_{11} - Cb - F^{ii} (\nabla_i \phi)^2 + \delta F^{ii} U_{ii}^2 - C \sum F^{ii} \\ &\geq \frac{(\delta - C\delta^2)\epsilon}{n} U_{11}^2 \left(\frac{1}{2} \sum G^{ii} + \frac{c_1}{2} \right) \\ &\quad - C U_{11} - Cb^2 \sum G^{ii} \end{aligned}$$

provided $U_{11} \geq b$.

Now in view of (4.10) and (4.12) in both cases, we can choose $0 < \delta \ll 1 \ll b$ to obtain

$$U_{11}(x_0) \leq \frac{Cb}{\epsilon(\delta - C\delta^2)}$$

and Theorem 1.2 is proved. \square

5. BOUNDARY ESTIMATES FOR SECOND ORDER DERIVATIVES

The key step to prove Theorem 1.3 is constructions of suitable barrier functions. For a point x_0 on ∂M , we choose smooth orthonormal local frames e_1, \dots, e_n around x_0 such that when restricted to ∂M , e_n is the interior normal to ∂M .

The estimate for $|\nabla_{\alpha\beta} u(x_0)|$ follows naturally from the formula

$$(5.1) \quad \nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n(u - \underline{u}) II(e_\alpha, e_\beta), \quad \forall 1 \leq \alpha, \beta < n \text{ on } \partial M$$

where II denotes the second fundamental form of ∂M .

For $x \in \overline{M}$ let $\rho(x)$ and $d(x)$ denote the distances from x to x_0 and ∂M , respectively,

$$\rho(x) := \text{dist}_M(x, x_0), \quad d(x) := \text{dist}_M(x, \partial M)$$

and $M_\delta = \{x \in M : \rho(x) < \delta\}$. As in [13], we shall construct barriers of the form

$$(5.2) \quad \Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{\beta < n} |\nabla_\beta(u - \varphi)|^2,$$

where $v = (u - \underline{u}) + td - \frac{Nd^2}{2}$ and t, N, A_1, A_2, A_3 are positive constants to be chosen. Define the linear operator \mathcal{L} locally by

$$\mathcal{L}w := F^{ij} \nabla_{ij} w - f_{p_i}^{1/k}(x, u, \nabla u) \nabla_i w, \quad w \in C^2(M).$$

It follows that Lemma 6.2 of [12] also holds for equation (2.5) and (1.7). Thus, there exists positive constants $t, \delta \ll 1 \ll N$ and θ_0 such that

$$(5.3) \quad \mathcal{L}v \leq -\theta_0 \left(1 + \sum F^{ii}\right) \text{ and } v \geq 0 \text{ in } M_\delta.$$

Lemma 5.1. *Let $h \in C(\overline{M}_\delta)$ satisfy $h \leq C\rho^2$ on $\overline{M}_\delta \cap \partial M$ and $h \leq C$ on \overline{M}_δ . Then for any positive constant K there exist uniform positive constants $A_1 \gg A_2 \gg A_3 \gg 1$ such that $\Psi \geq h$ on ∂M_δ and*

$$(5.4) \quad \mathcal{L}\Psi \leq -K \left(1 + \sum f_i |\lambda_i| + \sum F^{ii}\right) \text{ in } M_\delta,$$

where f_1, \dots, f_n and $\lambda_1, \dots, \lambda_n$ are eigenvalues of $\{F^{ij}\}$ and $\{U_{ij}\}$ respectively.

We refer the reader to Lemma 5.2 in [14] for the proof of Lemma 5.1 and a bound of $|\nabla_{\alpha n} u(x_0)|$ ($1 \leq \alpha \leq n-1$) can be obtained as in [14]. A bound of $|\nabla_{nn} u(x_0)|$ can be obtained as in [12] since the equation (2.5) satisfies (2.10) so that (2.5) is a special case of equations considered in [12]. In this paper, we provide a different proof which depends on the special structure of (2.5). It suffices to prove

$$(5.5) \quad \nabla_{nn} u(x_0) \leq C$$

since the trace of U with respect to the metric g , $\text{tr}_g(U) > 0$.

We consider two cases $k < n$ and $k = n$. First we may assume $\{U_{\alpha\beta}(x_0)\}_{1 \leq \alpha, \beta \leq n-1}$ is diagonal by rotating e_1, \dots, e_{n-1} . Let $\mu_\alpha = U_{nn}(x_0) + \sum_{\beta \neq \alpha} U_{\beta\beta}(x_0)$ for $1 \leq \alpha \leq n-1$, $\mu' = (\mu_1, \dots, \mu_{n-1})$ and $\mu_n = \sum_{\alpha < n} U_{\alpha\alpha}(x_0)$. When $k < n$, we find, at x_0 , the equation (2.5) can be written by

$$(5.6) \quad \mu_n \sigma_{k-1}(\mu') + \sigma_k(\mu') - \sum_{\alpha < n} U_{\alpha n}^2 \sigma_{k-2; \alpha}(\mu') = f.$$

We see if $U_{nn}(x_0)$ is sufficiently large, so are μ_α for $1 \leq \alpha \leq n-1$. It follows that $U_{nn}(x_0)$ being sufficiently large will contradict the equation (5.6). Therefore, (5.5) is proved.

For the case $k = n$, we have

Lemma 5.2. *Suppose $k = n$. Let \tilde{g} , $\tilde{\nabla}$ and ν be the induced metric, connection on ∂M and the interior unit normal vector field to ∂M respectively. Let $\lambda'(\tilde{\nabla}\varphi - \nabla_\nu u\Pi + \tilde{\chi}_1)$ denote the eigenvalues of $\tilde{\nabla}\varphi - \nabla_\nu u\Pi + \tilde{\chi}_1$ with respect to the metric \tilde{g} on ∂M . Then there exists a uniform constant $c_0 > 0$ such that*

$$(5.7) \quad \text{tr}_{\tilde{g}}(\tilde{\nabla}\varphi - \nabla_\nu u\Pi + \tilde{\chi}_1) \geq c_0 \text{ on } \partial M,$$

where $\text{tr}_{\tilde{g}}(X')$ denotes the trace of X' with respect to \tilde{g} for a $(0, 2)$ -tensor field X' on ∂M .

Proof. First, for any $\xi, \eta \in T_x \partial M$ at $x \in \partial M$,

$$\nabla_{\xi\eta} u = \tilde{\nabla}\varphi - \nabla_\nu u\Pi(\xi, \eta).$$

Suppose the minimum of $\psi(x) := \text{tr}_{\tilde{g}}(\tilde{\nabla}\varphi - \nabla_\nu u\Pi + \tilde{\chi}_1)(x)$ is achieved at a point $y_0 \in \partial\Omega$. It suffices to show $\psi(y_0) \geq c_0 > 0$. Choose a local orthonormal frame e_1, \dots, e_n about y_0 as before such that $\{U_{\alpha\beta}(y_0)\}_{1 \leq \alpha, \beta \leq n-1}$ is diagonal. It follows that

$$\psi(x) = \sum_{\alpha < n} U_{\alpha\alpha}(x) \text{ for } x \in \partial M \text{ near } y_0.$$

Set

$$(5.8) \quad B_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle, \quad 1 \leq \alpha, \beta \leq n-1.$$

Thus, $B_{\alpha\beta} = \Pi(e_\alpha, e_\beta)$ on ∂M near y_0 . We have

$$(5.9) \quad \sum_{\alpha < n} U_{\alpha\alpha}(x) = \psi(x) \geq \psi(y_0) \text{ for all } x \in \partial M \text{ near } y_0.$$

By (5.1) and (5.9) we have for $x \in \partial M$ near y_0 ,

$$(5.10) \quad \nabla_n(u - \underline{u})(x) \sum_{\alpha < n} B_{\alpha\alpha}(x) \leq \sum_{\alpha < n} \underline{U}_{\alpha\alpha}(x) - \psi(y_0),$$

where $\underline{U} := \nabla^2 \underline{u} + \chi_1$. Since $\lambda(\underline{U}) \in \Gamma$, there exists a uniform constant $\tilde{c} > 0$ such that $\sum_{\alpha < n} \underline{U}_{\alpha\alpha}(y_0) \geq \tilde{c}$. We may assume $\psi(y_0) < \frac{\tilde{c}}{2}$ for otherwise we are done. By (5.1), we have

$$(5.11) \quad \nabla_n(u - \underline{u})(y_0) \sum_{\alpha < n} B_{\alpha\alpha}(y_0) \geq \sum_{\alpha < n} \underline{U}_{\alpha\alpha}(y_0) - \psi(y_0) \geq \frac{\tilde{c}}{2}.$$

It follows that

$$\sum_{\alpha < n} B_{\alpha\alpha} \geq c_1$$

in $\overline{M}_{\delta'}$ for some uniform positive constant c_1 and δ' . Thus, by (5.10),

$$(5.12) \quad \nabla_n(u - \underline{u}) \leq \Phi \text{ on } \partial M \cap \overline{M}_{\delta'},$$

where

$$\Phi(x) := \frac{\sum_{\alpha < n} U_{\alpha\alpha}(x) - \psi(y_0)}{\sum_{\alpha < n} B_{\alpha\alpha}(x)}$$

is smooth in $M_{\delta'}$. By [13], we see

$$|\mathcal{L}\nabla_n(u - \underline{u})| \leq C \left(1 + \sum f_i |\lambda_i| + \sum F^{ii} \right) \text{ in } M_{\delta'}.$$

Thus, by Lemma 5.2 we can choose Ψ such that

$$\begin{aligned} \mathcal{L}(\nabla_n(u - \underline{u}) - \Phi - \Psi) &\geq 0 \text{ in } M_{\delta_0} \\ \nabla_n(u - \underline{u}) - \Phi - \Psi &\leq 0 \text{ on } \partial M_{\delta_0} \end{aligned}$$

for some positive constant $\delta_0 \leq \delta'$. By the maximum principle, we get a bound

$$\nabla_{nn}u(y_0) \leq C.$$

Note that when $k = n$, at y_0 , the equation (2.5) is

$$(5.13) \quad \mu_n \sigma_{n-1}(\mu') - \sum_{\alpha < n} U_{\alpha n}^2 \sigma_{n-2;\alpha}(\mu') = f,$$

where $\mu_i (1 \leq i \leq n)$ is defined as in (5.6) with x_0 replaced by y_0 . It follows that

$$\frac{f}{\mu_n} = \sigma_{n-1}(\mu') + \frac{\sum_{\alpha < n} U_{\alpha n}^2 \sigma_{n-2;\alpha}(\mu')}{\mu_n} \leq \sigma_{n-1}(\mu') \leq C$$

since $\lambda(U)(y_0)$ lies in an *a priori* bounded subset of Γ . Therefore,

$$\mu_n = \sum_{\alpha < n} U_{\alpha\alpha}(y_0) = \psi(y_0) \geq \frac{f}{C} \geq \frac{\tilde{f}_1^k}{C} \geq c_0$$

for some $c_0 > 0$ by (4.1). \square

(5.5) follows from (5.7) and (5.13) (with y_0 replaced by x_0). Theorem 1.3 is proved.

After establishing the *a priori* C^2 estimates, the equation (2.5) becomes uniformly elliptic with respect to admissible solutions by (4.1). The $C^{2,\alpha}$ estimates can be proved using Evans-Krylov theory since (2.5) is concave for admissible solutions. The higher order estimates are derived from the classic Schauder estimates. Theorem 1.4 can be proved by standard arguments using the continuity method and degree theory for which the reader is referred to Section 8 in [12] and [2] where Monge-Ampère equations are treated for more details.

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