Journal of Scientific Computing _####################_
https://doi.org/10.1007/s10915-021-01446-7

Superconvergence of the Local Discontinuous Galerkin Method for One Dimensional Nonlinear Convection-Diffusion Equations

Xiaobin Liu[1] · Dazhi Zhang[1] · Xiong Meng[2] · Boying Wu[2]

Received: 2 September 2020 / Revised: 12 January 2021 / Accepted: 26 February 2021

Abstract

In this paper, we study superconvergence properties of the local discontinuous Galerkin (LDG) methods for solving nonlinear convection-diffusion equations in one space dimension. The main technicality is an elaborate estimate to terms involving projection errors. By introducing a new projection and constructing some correction functions, we prove the $(2k+1)$th order superconvergence for the cell averages and the numerical flux in the discrete L^2 norm with polynomials of degree $k \geq 1$, no matter whether the flow direction $f'(u)$ changes or not. Superconvergence of order $k+2$ $(k+1)$ is obtained for the LDG error (its derivative) at interior right (left) Radau points, and the convergence order for the error derivative at Radau points can be improved to $k+2$ when the direction of the flow doesn't change. Finally, a supercloseness result of order $k+2$ towards a special Gauss–Radau projection of the exact solution is shown. The superconvergence analysis can be extended to the generalized numerical fluxes and the mixed boundary conditions. All theoretical findings are confirmed by numerical experiments.

Keywords Local discontinuous Galerkin method · Nonlinear convection-diffusion equation · Superconvergence · Correction function · Projection

---

This work is supported by National Natural Science Foundation of China (Grant No. 11971132, 11971131, U1637208), the Fundamental Research Funds for the Central Universities (Grant No. HIT. NSRIF. 2020081) and the National Key Research and Guangdong Basic and Applied Basic Research Foundation (2020B1515310006).

✉ Xiong Meng
xiongmeng@hit.edu.cn

Xiaobin Liu
liuxiaobin@hit.edu.cn

Dazhi Zhang
zhangdazhi@hit.edu.cn

Boying Wu
mathwby@hit.edu.cn

1 School of Mathematics, Harbin Institute of Technology, Harbin 150001, Heilongjiang, China

2 School of Mathematics and Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, Heilongjiang, China

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Mathematics Subject Classification 65M12 · 65M60

1 Introduction

In this paper, we investigate superconvergence of the local discontinuous Galerkin (LDG) method for one-dimensional nonlinear convection-diffusion equations

$$u_t + f(u)_x - bu_{xx} = g(x,t), \quad (x,t) \in [0, 2\pi] \times (0, T], \tag{1.1a}$$

$$u(x, 0) = u_0(x), \quad x \in [0, 2\pi], \tag{1.1b}$$

where $b > 0$ is a constant, u_0 is smooth, and g is a smooth function. We assume that the nonlinear flux function $f(u)$ is sufficiently smooth with respect to the variable u, and the exact solution is assumed to be smooth on $[0, 2\pi] \times [0, T]$ for a fixed T. The main task in deriving superconvergence is a delicate treatment for terms involving projection errors. By defining a new projection and constructing some correction functions, superconvergence properties for Radau points, cell averages and supercloseness are shown. Both the periodic boundary condition and mixed boundary conditions are considered.

As a class of efficient methods for solving partial differential equations (PDEs) involving high order spatial derivatives, the LDG method was proposed by [17] for convection-diffusion equations. Typically, an auxiliary variable will be introduced so that the standard discontinuous Galerkin (DG) methods can be applied to the resulting first-order system. Due to its numerical stability and local solvability of auxiliary variables, the LDG method has been widely used for solving a series of high order equations; see, e.g., [9,15].

In addition to optimal error estimates of LDG methods, the research of superconvergence has been a hot topic in recent years. Superconvergence results for Radau points have been obtained by using the Fourier approach [18] and the finite element technique [8,10,14] for different types of PDEs. Suboptimal supercloseness results of order $k+3/2$ (with k being the polynomial degree) is proved for linear convection-diffusion equations in [12], which was latter improved to be sharp of order $k+2$ in [23]. There is another kind of superconvergence, which is measured in the L^2 norm for post-processed errors. For linear hyperbolic equations, [16] proved that the post-processed solution through a convolution with some kernel functions is of order $2k+1$ superconvergent to the exact solution. Based on the duality argument and divided different estimates, the post-processing technique is extended to linear convection-diffusion equations in [19] and nonlinear symmetric systems of hyperbolic conservation laws in [20].

Recently, a systematic way via constructing special interpolation functions was successfully applied to the DG and LDG methods for linear hyperbolic and parabolic equations in [5,6]. Moreover, for nonlinear scalar conservation laws, by suitably choosing a local projection and analyzing correction functions, [4] proved that the order between the DG solution and the particular projection can achieves $(k+2)$th order when the direction of the flow doesn't change, and the order is less than $k+2$ when $f'(u)$ changes its sign. Also, the DG flux function $f(u_h)$ is proved to be superconvergent to a particular flux function of the exact solution.

In current paper, we aim at analyzing the superconvergence properties of LDG methods for nonlinear convection-diffusion equations. Different from using a weighed projection and a special operator for constructing correction function when $f'(u)$ is fixed or introducing a special projection consisting of four local projections when $f'(u)$ does change its sign in [4], we propose a new approach based on the balance of leading errors between the nonlinear

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

convection term and the diffusion term. To this end, we construct a new combined projection $\Pi(u, q) = (P_h^- u, \mathbb{P}_h^+ q)$ depending on the "reference" numerical flux $\tilde{f}$. To be more specific, the standard local Gauss–Radau projection $P_h^- u$ is used to eliminate the boundary term and integral term resulting from the prime variable u for the diffusion term, while $\mathbb{P}_h^+$ plays the role in dealing with difficulties coming from the auxiliary variable q and the nonlinear convection term $f(u)$. When the direction of the flow doesn't change, the projection $\mathbb{P}_h^+$ is $(k+2)$th order superclose to the local Gauss–Radau projection P_h^+. Further, some special interpolation functions consisting of the difference between the newly designed projections and correction functions are constructed. The interpolation function is thus superclose to the LDG solution, and superconvergence results can be obtained, which provides a solid foundation for illustrating the inherent interactive mechanism of the leading errors between the nonlinear convection term and the diffusion term.

An overview of this paper is as follows. In Sect 2, we present the semi-discrete LDG method for nonlinear convection-diffusion problems. In Sect. 3, we introduce a new projection and construct special correction functions, and the corresponding properties are analyzed. Section 4 is devoted to the superconvergence analysis, in which we show superconvergence for cell averages, Radau points as well as supercloseness. Extensions of the results to generalized alternating fluxes, mixed boundary conditions and the auxiliary variable are given in Sect. 5. In Sect. 6, numerical experiments are displayed that demonstrate the sharpness of our theoretical results. We end in Sect. 7 with concluding remarks and some possible future work.

2 The LDG Scheme

The usual notations of the LDG method are adopted here. For any positive integer r, we denote $\mathbb{Z}_r = \{0, 1, \dots, r\}$ and $\mathbb{Z}_r^+ = \mathbb{Z}_r \backslash \{0\}$. The computational domain $\Omega = [0, 2\pi]$ is divided into N elements with $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi$, The cell center and cell length are denoted by $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, respectively. The following polynomial space is chosen as the finite element space

$$V_h^k = \{v \in L^2(\Omega) : v|_{I_j} \in P^k(I_j), \quad j \in \mathbb{Z}_N^+\}$$

with $P^k(I_j)$ the set of polynomials of degree up to k defined on I_j. Since functions in V_h^k may be discontinuous across element boundaries, we use

$$\{\!\{v\}\!\}_{j+\frac{1}{2}} = \frac{1}{2}\left(v_{j+\frac{1}{2}}^+ + v_{j+\frac{1}{2}}^-\right), \quad [\![v]\!]_{j+\frac{1}{2}} = v_{j+\frac{1}{2}}^+ - v_{j+\frac{1}{2}}^-$$

to denote the mean and jump of the function v at each element boundary point $x_{j+\frac{1}{2}}$, where $v_{j+\frac{1}{2}}^+$ and $v_{j+\frac{1}{2}}^-$ are the traces from the right and left cells.

Throughout this paper, we use $W^{\ell,p}(D)$ to denote the standard Sobolev space on D equipped with the norm $\|\cdot\|_{W^{\ell,p}(D)} = \|\cdot\|_{\ell,p,D}$ with $\ell \geq 0$, $p = 2$ and $p = \infty$. For $p = 2$, $\|\cdot\|_{W^{\ell,2}(D)} = \|\cdot\|_{\ell,D}$, and the subscript D will be omitted when $D = \Omega$ with an unmarked norm $\|\cdot\|$ denoting the standard L^2 norm on Ω. For $v \in H^1(\Omega)$, the L^2 norm at cell boundaries is defined as follows:

$$\|v\|_{\Gamma_h} = \left(\sum_{j=1}^{N}[(v_{j-\frac{1}{2}}^+)^2 + (v_{j+\frac{1}{2}}^-)^2]\right)^{\frac{1}{2}}.$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

As usual, by introducing an auxiliary variable $q = \sqrt{b}u_x$, the problem (1.1) can be written as a first order system

$$u_t + f(u)_x - \sqrt{b}q_x = g(x,t), \qquad q - \sqrt{b}u_x = 0.$$

Then the semi-discrete LDG scheme is formulated as follows: find $u_h, q_h \in V_h^k$ such that for $\forall v, \varphi \in V_h^k$

$$\int_{I_j} (u_h)_t v \,\mathrm{d}x + \hat{f}(u_h)v^-|_{j+\frac{1}{2}} - \hat{f}(u_h)v^+|_{j-\frac{1}{2}} - \int_{I_j} f(u_h)v_x \,\mathrm{d}x$$
$$-\sqrt{b}\Big(\hat{q}_h v^-|_{j+\frac{1}{2}} - \hat{q}_h v^+|_{j-\frac{1}{2}} - \int_{I_j} q_h v_x \,\mathrm{d}x\Big) = \int_{I_j} g(x,t)v \,\mathrm{d}x, \tag{2.1a}$$

$$\int_{I_j} q_h\varphi \,\mathrm{d}x - \sqrt{b}\Big(\hat{u}_h\varphi^-|_{j+\frac{1}{2}} - \hat{u}_h\varphi^+|_{j-\frac{1}{2}} - \int_{I_j} u_h\varphi_x \,\mathrm{d}x\Big) = 0, \tag{2.1b}$$

where $\hat{f}(u_h)$ is the Godunov flux, i.e.,

$$\hat{f}(u_h) \triangleq \hat{f}(u_h^-, u_h^+) = \begin{cases} \min_{u_h^- \le \omega \le u_h^+} f(\omega), & \text{if } u_h^- < u_h^+, \\ \max_{u_h^+ \le \omega \le u_h^-} f(\omega), & \text{if } u_h^- \ge u_h^+, \end{cases}$$

and $\hat{u}_h, \hat{q}_h$ are a pair of alternating fluxes. For example, one can use the following alternating fluxes

$$\hat{u}_h = u_h^-, \qquad \hat{q}_h = q_h^+, \tag{2.2a}$$

or

$$\hat{u}_h = u_h^+, \qquad \hat{q}_h = q_h^-. \tag{2.2b}$$

Motivated by [4], to deal with the nonlinear term, a "reference" numerical flux is introduced, which plays an important role in the design of new projections in Sect. 3.1 below. That is,

$$\tilde{f}(u_h) = \begin{cases} f(u_h^-), & \text{if } f'(u)|_{j+\frac{1}{2}} \ge 0, \\ f(u_h^+), & \text{if } f'(u)|_{j+\frac{1}{2}} < 0, \end{cases} \tag{2.3}$$

where u_h and u are the numerical solution and the exact solution of (1.1), respectively.

The LDG scheme (2.1) will be simplified if one adopts the DG spatial discretization operator given by

$$\mathcal{H}\big(w, v; \hat{w}\big) = \sum_{j=1}^{N} \mathcal{H}_j\big(w, v; \hat{w}\big)$$

with

$$\mathcal{H}_j\big(w, v; \hat{w}\big) = -(w, v_x)_j + \hat{w}v^-|_{j+\frac{1}{2}} - \hat{w}v^+|_{j-\frac{1}{2}}, \tag{2.4}$$

where $(\cdot,\cdot)_j$ denotes the L^2 inner product on I_j. By Galerkin orthogonality, one has the following cell error equation

$$\big((e_u)_t, v\big)_j + (e_q, \varphi)_j + \mathcal{H}_j\big(f(u) - f(u_h), v;\, f(u) - \hat{f}(u_h)\big)$$
$$- \sqrt{b}\mathcal{H}_j\big(e_q, v; \hat{e}_q\big) - \sqrt{b}\mathcal{H}_j\big(e_u, \varphi; \hat{e}_u\big) = 0, \tag{2.5}$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

which holds for $\forall v, \varphi \in V_h^k$ and $j \in \mathbb{Z}_N^+$. Here $e_u = u - u_h$, $e_q = q - q_h$. In what follows, let us recall the local Gauss–Radau projections of a function $\phi \in H^1(\Omega)$ into the finite element space V_h^k, denoted by P_h^- or P_h^+, which are defined as the unique function in V_h^k such that

$$(P_h^- \phi, v_h)_j = (\phi, v_h)_j, \quad \forall v_h \in P^{k-1}(I_j), \quad P_h^- \phi(x^-_{j+\frac{1}{2}}) = \phi(x^-_{j+\frac{1}{2}}), \tag{2.6a}$$

$$(P_h^+ \phi, v_h)_j = (\phi, v_h)_j, \quad \forall v_h \in P^{k-1}(I_j), \quad P_h^+ \phi(x^+_{j-\frac{1}{2}}) = \phi(x^+_{j-\frac{1}{2}}). \tag{2.6b}$$

To facilitate analysis, we use the following Legendre expansion in each element I_j, $j \in \mathbb{Z}_N^+$. That is, for $\phi \in H^1(I_j)$

$$\phi(x,t) = \sum_{m=0}^{\infty} \phi_{j,m}(t) L_{j,m}(x), \qquad \phi_{j,m}(t) = \frac{2m+1}{h_j}(\phi, L_{j,m})_j,$$

where $L_{j,m}$ denotes the rescaled Legendre polynomial of degree m on I_j, namely $L_{j,m}(x) = L_m(\frac{2(x-x_j)}{h_j})$. By the definition of P_h^-, P_h^+ in combination with the orthogonality property of Legendre polynomials, one has

$$(\phi - P_h^- \phi)(x,t) = \overrightarrow{\phi}_{j,k} L_{j,k} + \sum_{m=k+1}^{\infty} \phi_{j,m} L_{j,m}, \tag{2.7a}$$

$$(\phi - P_h^+ \phi)(x,t) = \overleftarrow{\phi}_{j,k} L_{j,k} + \sum_{m=k+1}^{\infty} \phi_{j,m} L_{j,m}, \tag{2.7b}$$

in which $\overrightarrow{\phi}_{j,k}$, $\overleftarrow{\phi}_{j,k}$ can be determined by the boundary collocation conditions in (2.6). It reads,

$$\overrightarrow{\phi}_{j,k} = \sum_{m=0}^{k} \phi_{j,m} - \phi(x^-_{j+\frac{1}{2}}), \quad \overleftarrow{\phi}_{j,k} = \sum_{m=0}^{k} (-1)^{k-m} \phi_{j,m} - (-1)^k \phi(x^+_{j-\frac{1}{2}}). \tag{2.7c}$$

After a simple application of the Bramble–Hilbert lemma [1, Lemma 2.2.2] and scaling arguments, we obtain

$$|\overrightarrow{\phi}_{j,k}| \le C h^{k+1} \|\phi\|_{k+1,\infty}, \qquad |\overleftarrow{\phi}_{j,k}| \le C h^{k+1} \|\phi\|_{k+1,\infty},$$

where C is a constant independent of ϕ and the mesh size h.

For the correction function construction procedure, the following integral operator D_x^{-1} is essential, which aims at eliminating the leading term of the error equation via integration by parts, and thus superconvergence results can be obtained; see, e.g., [6]. That is,

$$D_x^{-1}\phi(x) = \frac{1}{\bar{h}_j} \int_{x_{j-\frac{1}{2}}}^{x} \phi(\tau)\,\mathrm{d}\tau, \quad \tau \in I_j,$$

where $\bar{h}_j = h_j/2$. Obviously,

$$\phi(x) = \bar{h}_j \big(D_x^{-1}\phi(x)\big)_x. \tag{2.8}$$

Moreover, by the properties of Legendre polynomials, for $m \in \mathbb{Z}_k$ with $L_{j,-1} = 0$, we have

$$D_x^{-1} L_{j,m}(x) = \frac{1}{2m+1}(L_{j,m+1} - L_{j,m-1})(x). \tag{2.9}$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Finally, we list some inverse properties of the element space V_h^k that will be used in our analysis. For any $v_h \in V_h^k$, there exists a positive constant C independent of v_h and h such that

$$\text{(i)}\|\partial_x v_h\| \le Ch^{-1}\|v_h\|; \quad \text{(ii)}\|v_h\|_{\Gamma_h} \le Ch^{-1/2}\|v_h\|; \quad \text{(iii)}\|v_h\|_\infty \le Ch^{-1/2}\|v_h\|.$$

3 A New Projection and Correction Functions

To derive superconvergence results, interpolation functions consisting of special projections and correction functions need to be carefully designed, which are mainly used to obtain a superconvergent bound for the contribution of projection errors; see Sect. 3.3 below. Since the Gauss–Radau projections P_h^- or P_h^+ are not sufficient to deal with the nonlinear term that changes its flow direction, we shall first introduce a new projection, which is a modification of P_h^+. In what follows, we mainly concentrate on the fluxes (2.2a), and the case with (2.2b) will be discussed in Remark 3.2.

3.1 A New Projection

Motivated by [11, Sect. 4.2], we define the following modified projection

$$\Pi(u,q) = (P_h^- u, \mathbb{P}_h^+ q),$$

where $P_h^- u \in V_h^k$ has been given in (2.6a), and $\mathbb{P}_h^+ q \in V_h^k$ depends on both u and q such that

$$\int_{I_j} (q - \mathbb{P}_h^+ q) v_h \,\mathrm{d}x - \frac{1}{\sqrt{b}} \int_{I_j} f'(u)(u - P_h^- u) v_h \,\mathrm{d}x = 0, \quad \forall v_h \in P^{k-1}(I_j), \tag{3.1a}$$

$$\mathbb{P}_h^+ q(x_{j-\frac{1}{2}}^+) = q(x_{j-\frac{1}{2}}^+) - \frac{1}{\sqrt{b}} f'(u_{j-\frac{1}{2}})(\widetilde{u - P_h^- u})_{j-\frac{1}{2}}, \quad \forall j \in \mathbb{Z}_N^+, \tag{3.1b}$$

where $\widetilde{w}$ has been defined in (2.3). It is easy to see that $\mathbb{P}_h^+ q = P_h^+ q$ when $f'(u) = 0$, and $(q - \mathbb{P}_h^+ q)_{j-\frac{1}{2}}^+ = 0$ when $f'(u) \ge 0$. Thus, $\mathbb{P}_h^+$ can be viewed as an extension of the local Gauss–Radau projection P_h^+. Moreover, for $v \in V_h^k$ and $j \in \mathbb{Z}_N^+$, $\Pi(u,q) = (P_h^- u, \mathbb{P}_h^+ q)$ satisfies the following identity

$$\mathcal{H}_j\big(f'(u)(u - P_h^- u), v; f'(u)(\widetilde{u - P_h^- u})\big) - \sqrt{b}\mathcal{H}_j\big(q - \mathbb{P}_h^+ q, v; (q - \mathbb{P}_h^+ q)^+\big) = 0. \tag{3.2}$$

The properties of the projection $\mathbb{P}_h^+$ in the following lemma are essential to the proof of superconvergence; see Lemma 3.3 below.

Lemma 3.1 *Suppose $|\partial_x^k f'(u)| \le C$, then the projection $\mathbb{P}_h^+$ in (3.1) is well defined. Moreover, if $q - \mathbb{P}_h^+ q$ has the following expression in each element I_j*

$$q - \mathbb{P}_h^+ q\big|_{I_j} = \sum_{m=0}^{k} \bar{q}_{j,m} L_{j,m} + \sum_{m=k+1}^{\infty} q_{j,m} L_{j,m}, \tag{3.3}$$

then there holds the following results:

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

(1) The coefficients $\bar{q}_{j,m}$ satisfies

$$|\bar{q}_{j,m}| \le Ch^{2k+1-m}\|u\|_{k+2,\infty}, \quad m \in \mathbb{Z}_k. \tag{3.4}$$

(2) The cell average of the projection error $q - \mathbb{P}_h^+ q$ in each element I_j is superconvergent with an order of $2k+1$, i.e.,

$$\left|\frac{1}{h_j}\int_{I_j}(q - \mathbb{P}_h^+ q)(x)\,\mathrm{d}x\right| \le Ch^{2k+1}\|u\|_{k+2,\infty}.$$

Especially, when $f'(u) \ge 0$, namely $(\widetilde{u - P_h^- u})_{j-\frac{1}{2}} = (u - P_h^- u)^-_{j-\frac{1}{2}} = 0$, we have the following superclosenes results.

(3) $\mathbb{P}_h^+ q$ is superclose to the Gauss–Radau projection $P_h^+ q$, i.e.,

$$\|\mathbb{P}_h^+ q - P_h^+ q\|_\infty \le Ch^{k+2}\|u\|_{k+2,\infty}. \tag{3.5}$$

(4) The function value approximation of $\mathbb{P}_h^+ q$ is superconvergent at left Radau points $\ell_{j,m}$, $m \in \mathbb{Z}_k^+$ (zeros of left Radau polynomial $L_{j,m+1} + L_{j,m}$), and the derivative value approximation is superconvergent at the interior right Radau points $r_{j,m}$, $m \in \mathbb{Z}_k^+$ (zeros of right Radau polynomial $L_{j,m+1} - L_{j,m}$, except the point $x = x_{j+\frac{1}{2}}$), namely

$$\left|(q - \mathbb{P}_h^+ q)(\ell_{j,m})\right| \le Ch^{k+2}\|q\|_{k+2,\infty}, \tag{3.6a}$$

$$\left|\partial_x(q - \mathbb{P}_h^+ q)(r_{j,m})\right| \le Ch^{k+1}\|q\|_{k+2,\infty}. \tag{3.6b}$$

The constant C is independent of h.

Proof (1) Since $\mathbb{P}_h^+ q \in V_h^k$, we express in each element I_j

$$\mathbb{P}_h^+ q|_{I_j} = \sum_{m=0}^{k} b_{j,m}L_{j,m}, \tag{3.7}$$

where $b_{j,m}$ are coefficients to be determined later. Using the orthogonality of Legendre polynomials, (3.1a), and (2.6a), we obtain, for $\forall m \in \mathbb{Z}_{k-1}$

$$\begin{aligned}\bar{q}_{j,m} &= \frac{2m+1}{h_j}\int_{I_j}(q - \mathbb{P}_h^+ q)L_{j,m}\,\mathrm{d}x\\ &= \frac{2m+1}{\sqrt{b}h_j}\int_{I_j} f'(u)(u - P_h^- u)L_{j,m}\,\mathrm{d}x\\ &= \frac{2m+1}{\sqrt{b}h_j}\int_{I_j}\big(f'(u) - I_{k-1-m}f'(u)\big)(u - P_h^- u)L_{j,m}\,\mathrm{d}x.\end{aligned}$$

Here and below, $I_m w \in P^m(I_j)$ represents an interpolation of w. By the Bramble–Hilbert lemma,

$$\|f'(u) - I_{k-1-m}f'(u)\|_{\infty,I_j} \le Ch^{k-m}\|\partial_x^{k-m}f'(u)\|_\infty \le Ch^{k-m},$$

which yields

$$|\bar{q}_{j,m}| \le Ch^{k-m}\|u - P_h^- u\|_{\infty,I_j} \le Ch^{2k+1-m}\|u\|_{k+1,\infty}, \ m \in \mathbb{Z}_{k-1}. \tag{3.8}$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Next, let us consider the estimate to $\bar{q}_{j,k}$. On the one hand, by (3.7), using the definition of $\mathbb{P}_h^+$ in (3.1b) and the fact that $(-1)^{2m} = 1$ ($\forall m \in \mathbb{Z}_{k-1}$), we get

$$b_{j,k} = (-1)^k q(x^+_{j-\frac{1}{2}}) - \sum_{m=0}^{k-1} (-1)^{k-m} b_{j,m} - \Phi,$$

where

$$\Phi = (-1)^k \frac{1}{\sqrt{b}} f'(u) (\widetilde{u - P_h^- u})_{j-\frac{1}{2}}.$$

On the other hand, it follows from $q_{j,k} = -(-1)^k(-1)^{k+1} q_{j,k}$ and the expression for $\overleftarrow{q}_{j,k}$ in (2.7c) that

$$\begin{aligned}
\bar{q}_{j,k} = q_{j,k} - b_{j,k} &= q_{j,k} + (-1)^{k+1} q(x^+_{j-\frac{1}{2}}) + \sum_{m=0}^{k-1} (-1)^{k-m} b_{j,m} + \Phi \\
&= (-1)^{k+1} \Big(q(x^+_{j-\frac{1}{2}}) - \sum_{m=0}^{k} (-1)^m q_{j,m} + \sum_{m=0}^{k-1} (-1)^m \bar{q}_{j,m} \Big) + \Phi \\
&= \overleftarrow{q}_{j,k} + \sum_{m=0}^{k-1} (-1)^{k-m+1} \bar{q}_{j,m} + \Phi, \qquad (3.9)
\end{aligned}$$

where we have also used the relation $b_{j,m} = q_{j,m} - \bar{q}_{j,m},\ m \in \mathbb{Z}_{k-1}$. Moreover,

$$|\Phi| \le \|u - P_h^- u\|_\infty \le C h^{k+1} \|u\|_{k+1,\infty}.$$

Consequently,

$$|\bar{q}_{j,k}| \le |\overleftarrow{q}_{j,k}| + \sum_{m=0}^{k-1} |\bar{q}_{j,m}| + |\Phi| \le C h^{k+1} \|u\|_{k+2,\infty}.$$

This completes the proof of (3.4).

(2) Using (3.3) and (3.4) in combination with the orthogonality property of Legendre polynomials, we have

$$\left| \frac{1}{h_j} \int_{I_j} (q - \mathbb{P}_h^+ q)\,\mathrm{d}x \right| = \frac{1}{h_j} \left| \int_{I_j} \bar{q}_{j,0}\,\mathrm{d}x \right| \le |\bar{q}_{j,0}| \le C h^{2k+1} \|u\|_{k+2,\infty}.$$

(3) When $f'(u) \ge 0$, namely $(\widetilde{u - P_h^- u})_{j-\frac{1}{2}} = 0$, then $\Phi = 0$, and we can express $P_h^+ q$ in terms of the orthogonal basis $L_{j,m}$ ($m \in \mathbb{Z}_k$) as

$$P_h^+ q = \sum_{m=0}^{k} q_{j,m} L_{j,m} - \overleftarrow{q}_{j,k} L_{j,k}$$

with $\overleftarrow{q}_{j,k}$ defined in (2.7c). This, together with (3.7) and (3.9), leads to

$$\begin{aligned}
P_h^+ q - \mathbb{P}_h^+ q &= \sum_{m=0}^{k-1} (q_{j,m} - b_{j,m}) L_{j,m} + \big(\bar{q}_{j,k} - \overleftarrow{q}_{j,k}\big) L_{j,k}, \\
&= \sum_{m=0}^{k-1} \bar{q}_{j,m} L_{j,m} + \sum_{m=0}^{k-1} (-1)^{k-m+1} \bar{q}_{j,m} L_{j,k},
\end{aligned}$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

where we have used the fact that $q_{j,m} - b_{j,m} = \bar{q}_{j,m}$, $m \in \mathbb{Z}_k$. Then, by (3.4)

$$\|P_h^+ q - \mathbb{P}_h^+ q\|_{\infty,I_j} \le \sum_{m=0}^{k-1} |\bar{q}_{j,m}| \le Ch^{k+2}\|u\|_{k+2,\infty}, \quad \forall j \in \mathbb{Z}_N.$$

This finishes the proof of (3.5).

(4) The inverse inequality $\|\partial_x(P_h^+ q - \mathbb{P}_h^+ q)\|_{\infty,I_j} \le Ch^{k+1}\|u\|_{k+2,\infty}$ together with the superconvergence results for P_h^+ in [7], namely

$$\left|(q - P_h^+ q)(\ell_{j,m})\right| \le Ch^{k+2}\|q\|_{k+2,\infty}, \quad \left|\partial_x(q - P_h^+ q)(r_{j,m})\right| \le Ch^{k+1}\|q\|_{k+2,\infty}$$

gives us the desired results (3.6). This completes the proof of Lemma 3.1. □

3.2 Correction Functions

In order to construct the interpolation functions $(u_I^\ell,\ q_I^\ell)$, let us begin by defining a series of functions $w_{u,i},\ w_{q,i} \in V_h^k,\ i \in \mathbb{Z}_k^+$ as follows

$$(\sqrt{b}w_{u,i} - \bar{h}_j D_x^{-1} w_{q,i-1}, v)_j = 0, \qquad \left(w_{u,i}^-\right)_{j+\frac{1}{2}} = 0, \tag{3.10a}$$

$$(\sqrt{b}w_{q,i} - f'(u)w_{u,i} - \bar{h}_j D_x^{-1}\partial_t w_{u,i-1}, v)_j = 0, \quad (w_{q,i}^+)_{j-\frac{1}{2}} = \frac{1}{\sqrt{b}}\left(f'(u)\widetilde{w}_{u,i}\right)_{j-\frac{1}{2}}, \tag{3.10b}$$

where $v \in P^{k-1}(I_j)$, and

$$w_{u,0} = u - P_h^- u, \qquad w_{q,0} = q - \mathbb{P}_h^+ q.$$

Further, for any positive integer $\ell \in \mathbb{Z}_k^+$, we define in each element I_j the correction functions

$$W_u^\ell = \sum_{i=1}^{\ell} w_{u,i}, \qquad W_q^\ell = \sum_{i=1}^{\ell} w_{q,i}, \tag{3.11}$$

and the special interpolation functions are

$$u_I^\ell = P_h^- u - W_u^\ell, \qquad q_I^\ell = \mathbb{P}_h^+ q - W_q^\ell. \tag{3.12}$$

The components $w_{u,i}, w_{q,i}$ in correction functions have the following property.

Lemma 3.2 *The functions $w_{u,i}, w_{q,i}$, $i \in \mathbb{Z}_k^+$ defined in (3.10) are uniquely determined. Moreover, suppose that $f'(u)$ is a sufficiently smooth function satisfying*

$$|\partial_x^{k+1} f'(u)| \le C, \qquad |\partial_t^{k+1} f'(u)| \le C,$$

and the functions $w_{u,i}$ and $w_{q,i}$ in each element I_j are expressed by

$$w_{u,i}|_{I_j} = \sum_{m=0}^{k} \beta_{i,m} L_{j,m}, \qquad w_{q,i}|_{I_j} = \sum_{m=0}^{k} \gamma_{i,m} L_{j,m}.$$

Then, the coefficients $\beta_{i,m}$ and $\gamma_{i,m}$ satisfy

$$|\partial_t^n \beta_{i,m}| \le Ch^{\max\{k+1+i,\ 2k+1-m\}}\|\partial_t^n u\|_{k+i+1,\infty}, \quad n = 0, 1, \tag{3.13a}$$

$$|\partial_t^n \gamma_{i,m}| \le Ch^{\max\{k+1+i,\ 2k+1-m\}}\|\partial_t^n u\|_{k+i+2,\infty}, \quad n = 0, 1. \tag{3.13b}$$

SPI Journal: 10915 Article No.: 1446 □ TYPESET □ DISK □ LE □ CP Disp.:2021/3/8 Pages: 30 Layout: Small

As a consequence,

$$\|\partial_t w_{u,i}\|_\infty + \|w_{q,i}\|_\infty \le Ch^{k+i+1}\|u\|_{k+i+3,\infty}, \tag{3.14a}$$
$$\|W_u^\ell\|_\infty + \|W_q^\ell\|_\infty \le Ch^{k+2}\|u\|_{k+\ell+3,\infty}. \tag{3.14b}$$

Proof We prove this lemma by induction, consisting of the following two steps. Since the case with $n = 1$ is quite similar to that with $n = 0$, we mainly consider the case with $n = 0$.

Step 1: When $i = 1$, taking $v = L_{j,m}, m \in \mathbb{Z}_{k-1}$ in (3.10a) and using the orthogonality property of Legendre polynomials, we get

$$(\sqrt{b}w_{u,1} - \bar{h}_j D_x^{-1} w_{q,0}, v)_j = \left(\sqrt{b}\sum_{m=0}^{k}\beta_{1,m}L_{j,m} - \bar{h}_j\sum_{m=0}^{k}\bar{q}_{j,m}D_x^{-1}L_{j,m}, v\right)_j = 0,$$

where $\bar{q}_{j,m}$ are the coefficients defined in (3.3). Using the relation (2.9) and the orthogonality property of Legendre polynomials again, we arrive at

$$\sqrt{b}\beta_{1,0} = -\frac{\bar{q}_{j,1}}{3}\bar{h}_j,$$
$$\sqrt{b}\beta_{1,m} = \frac{\bar{q}_{j,m-1}}{2m-1}\bar{h}_j - \frac{\bar{q}_{j,m+1}}{2m+3}\bar{h}_j, \quad m \in \mathbb{Z}_{k-1}^+.$$

Using (3.8), we have

$$|\beta_{1,0}| \le h|\bar{q}_{j,1}| \le Ch^{2k+1}\|u\|_{k+1,\infty},$$
$$|\beta_{1,m}| \le h(|\bar{q}_{j,m-1}| + |\bar{q}_{j,m+1}|) \le Ch^{2k+1-m}\|u\|_{k+1,\infty}, \ m \in \mathbb{Z}_{k-1}^+.$$

Using the fact that $\left(w_{u,1}^-\right)_{j+\frac{1}{2}} = 0$, we obtain

$$|\beta_{1,k}| = |\sum_{m=0}^{k-1}\beta_{1,m}| \le \sum_{m=0}^{k-1}|\beta_{1,m}| \le Ch^{k+2}\|u\|_{k+2,\infty}.$$

Analogously, taking $v = L_{j,m}, m \in \mathbb{Z}_{k-1}$ in (3.10b) and using (2.7a) as well as (2.9), we get

$$\sqrt{b}\gamma_{1,m} = \frac{2m+1}{h_j}\left(f'(u)w_{u,1}, L_{j,m}\right)_j, \quad m \in \mathbb{Z}_{k-2},$$
$$\sqrt{b}\gamma_{1,k-1} = \frac{2k-1}{h_j}\left(f'(u)w_{u,1}, L_{j,k-1}\right)_j - \frac{\bar{h}_j\partial_t\overrightarrow{u}_{j,k}}{2k+1}.$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Specifically, when $m \in \mathbb{Z}_{k-2}$, we have

$$
\begin{aligned}
\left|\sqrt{b}\gamma_{1,m}\right| &= \Big|\frac{2m+1}{h_j}\sum_{\nu=0,\nu\neq m}^{k}\big(f'(u)\beta_{1,\nu}L_{j,\nu},L_{j,m}\big)_j + \frac{2m+1}{h_j}\big(f'(u)\beta_{1,m}L_{j,m},L_{j,m}\big)_j\Big| \\
&= \Big|\frac{2m+1}{h_j}\sum_{\nu=0,\nu\neq m}^{k}\Big(\big(f'(u)-I_{|\nu-m|-1}f'(u)\big)\beta_{1,\nu}L_{j,\nu},L_{j,m}\Big)_j \\
&\quad + \frac{2m+1}{h_j}\big(f'(u)\beta_{1,m}L_{j,m},L_{j,m}\big)_j\Big| \\
&\leq \Big|\frac{2m+1}{h_j}\sum_{\nu=0,\nu\neq m}^{k}\Big(\big(f'(u)-I_{|\nu-m|-1}f'(u)\big)\beta_{1,\nu}L_{j,\nu},L_{j,m}\Big)_j\Big| + C|\beta_{1,m}| \\
&\leq \sum_{\nu=0,\nu\neq m}^{k} Ch^{|\nu-m|}\left|\beta_{1,\nu}\right| + C|\beta_{1,m}|,
\end{aligned}
$$

where in the second step we have used the orthogonality property of Legendre polynomials, in the third step we have employed Hölder's inequality and in the last step we have used the following interpolation error estimate

$$\|f'(u)-I_{|\nu-m|-1}f'(u)\|_\infty \leq Ch^{|\nu-m|}\|\partial_x^{|\nu-m|}f'(u)\|_\infty \leq Ch^{|\nu-m|}.$$

It is easy to see that no matter $\nu > m$ or $\nu < m$, the following formula is valid

$$\left|\gamma_{1,m}\right| \leq Ch^{2k+1-m}\|u\|_{k+2,\infty}, \quad m \in \mathbb{Z}_{k-2}.$$

By the same arguments, we can obtain

$$\left|\gamma_{1,k-1}\right| \leq \sum_{\nu=0,\nu\neq k-1}^{k} Ch^{|\nu-(k-1)|}\left|\beta_{1,\nu}\right| + C|\beta_{1,k-1}| + h_j\left|\partial_t \overrightarrow{u}_{j,k}\right| \leq Ch^{k+2}\|u\|_{k+3,\infty},$$

since $\|\partial_t u\|_{k+1,\infty} = \|f'(u)\partial_x u - \partial_x^2 u\|_{k+1,\infty} \leq C\|u\|_{k+3,\infty}$. It remains to bound $\gamma_{1,k}$. We use $(w_{q,1}^+)_{j-\frac{1}{2}} = \frac{1}{\sqrt{b}}\big(f'(u)\widetilde{w}_{u,1}\big)_{j-\frac{1}{2}}$ in (3.10b) to obtain

$$|\gamma_{1,k}| \leq \sum_{m=0}^{k-1}|\gamma_{1,m}| + \sum_{m=0}^{k}C|\beta_{1,m}| \leq Ch^{k+2}\|u\|_{k+3,\infty}.$$

Therefore, (3.13) is valid for $i = 1$.

Step 2: Suppose that (3.13) holds for i, $1 \leq i \leq k-1$, and we need to prove that it is also valid for $i+1$.

We choose $v = L_{j,m}$, $m \in \mathbb{Z}_{k-1}$ in (3.10a) to obtain

$$
\begin{aligned}
\sqrt{b}\beta_{i+1,0} &= -\frac{\bar{h}_j}{3}\gamma_{i,1}, \\
\sqrt{b}\beta_{i+1,m} &= \frac{\bar{h}_j\gamma_{i,m-1}}{2m-1} - \frac{\bar{h}_j\gamma_{i,m+1}}{2m+3}, \quad m \in \mathbb{Z}_{k-1}^+.
\end{aligned}
$$

It is easy to deduce that

$$
\begin{aligned}
&|\beta_{i+1,0}| \leq h|\gamma_{i,1}| \leq Ch^{2k+1}\|u\|_{k+i+2,\infty}, \\
&|\beta_{i+1,m}| \leq h\big(|\gamma_{i,m-1}| + |\gamma_{i,m+1}|\big) \leq Ch^{\max\{k+2+i,2k+1-m\}}\|u\|_{k+i+2,\infty}, \ m \in \mathbb{Z}_{k-1}^+.
\end{aligned}
$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Using the fact that $(w^-_{u,i+1})_{j+\frac{1}{2}} = 0$, we get

$$|\beta_{i+1,k}| = |\sum_{m=0}^{k-1} \beta_{i+1,m}| \le \sum_{m=0}^{k-1} |\beta_{i+1,m}| \le Ch^{k+2+i} \|u\|_{k+i+2,\infty}.$$

Analogously, taking $v = L_{j,m}$, $m \in \mathbb{Z}_{k-1}$ in (3.10b), we have

$$\sqrt{b}\gamma_{i+1,0} = \frac{1}{h_j}\big(f'(u)w_{u,i+1}, L_{j,0}\big)_j - \frac{\bar{h}_j}{3}\partial_t \beta_{i,1},$$

$$\sqrt{b}\gamma_{i+1,m} = \frac{2m+1}{h_j}\big(f'(u)w_{u,i+1}, L_{j,m}\big)_j + \frac{\bar{h}_j \partial_t \beta_{i,m-1}}{2m-1} - \frac{\bar{h}_j \partial_t \beta_{i,m+1}}{2m+3}, \quad m \in \mathbb{Z}^+_{k-1}.$$

In particular, when $m \in \mathbb{Z}_{k-1}$, we have

$$\begin{aligned}
&\Big|\frac{2m+1}{h_j}\big(f'(u)w_{u,i+1}, L_{j,m}\big)_j\Big| \\
&= \Big|\frac{2m+1}{h_j} \sum_{\nu=0,\nu\neq m}^{k} \big(f'(u)\beta_{i+1,\nu}L_{j,\nu}, L_{j,m}\big)_j + \frac{2m+1}{h_j}\big(f'(u)\beta_{i+1,m}L_{j,m}, L_{j,m}\big)_j\Big| \\
&\le \Big|\frac{2m+1}{h_j} \sum_{\nu=0,\nu\neq m}^{k} \Big(\big(f'(u) - I_{|\nu-m|-1}f'(u)\big)\beta_{i+1,\nu}L_{j,\nu}, L_{j,m}\Big)_j\Big| + C|\beta_{i+1,m}| \\
&\le \sum_{\nu=0,\nu\neq m}^{k} Ch^{|\nu-m|}\big|\beta_{i+1,\nu}\big| + C|\beta_{i+1,m}| \\
&\le Ch^{\max\{k+2+i,2k+1-m\}} \|u\|_{k+i+2,\infty}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\gamma_{i+1,0}| &\le Ch^{2k+1}\|u\|_{k+i+2,\infty} + h|\partial_t \beta_{i,1}| \le Ch^{2k+1}\|u\|_{k+i+3,\infty}, \\
|\gamma_{i+1,m}| &\le Ch^{\max\{k+2+i,2k+1-m\}}\|u\|_{k+i+2,\infty} + h\big(|\partial_t\beta_{i,m-1}| + |\partial_t\beta_{i,m+1}|\big) \\
&\le Ch^{\max\{k+2+i,2k+1-m\}}\|u\|_{k+i+3,\infty}, \qquad m \in \mathbb{Z}^+_{k-1}.
\end{aligned}$$

In addition, it follows from (3.10b) that

$$\big|\gamma_{i+1,k}\big| \le \sum_{m=0}^{k-1}\big|\gamma_{i+1,m}\big| + \sum_{m=0}^{k} C|\beta_{i+1,m}| \le Ch^{k+2+i}\|u\|_{k+i+3,\infty}.$$

Therefore (3.13) holds for $i+1$ and this finishes the proof of Lemma 3.2. □

3.3 The Superconvergent Bound for the Projection Errors

To clearly see how to cancel terms involving projection errors with the goal of obtaining superconvergence, we split the error e_u, e_q into two parts:

$$\begin{aligned}
e_u &= u - u_h = u - u_I^\ell + u_I^\ell - u_h \triangleq \eta_u + \xi_u, \\
e_q &= q - q_h = q - q_I^\ell + q_I^\ell - q_h \triangleq \eta_q + \xi_q.
\end{aligned}$$

Here u_I^ℓ and q_I^ℓ are the two special interpolation functions of u and q given in (3.12).

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Lemma 3.3 *Suppose that $u \in W^{k+\ell+3,\infty}$, $\ell \in \mathbb{Z}_k^+$ is the exact solution of (1.1), and u_I^ℓ, q_I^ℓ are the interpolation functions defined by (3.12), then for $\forall v, \varphi \in V_h^k$, we have*

$$\left|((\eta_u)_t, v)_j + \mathcal{H}_j\big(f'(u)\eta_u, v; f'(u)\widetilde{\eta}_u\big) - \sqrt{b}\mathcal{H}_j\big(\eta_q, v; \eta_q^+\big)\right| \le Ch^{k+\ell+1}\|u\|_{k+\ell+3,\infty}\|v\|_{1,I_j},$$
$$\left|(\eta_q, \varphi)_j - \sqrt{b}\mathcal{H}_j\big(\eta_u, \varphi; \eta_u^-\big)\right| \le Ch^{k+\ell+1}\|u\|_{k+\ell+2,\infty}\|\varphi\|_{1,I_j}.$$

Proof Since $\eta_u = u - P_h^- u + W_u^\ell$ and $\eta_q = q - \mathbb{P}_h^+ q + W_q^\ell$, using the identity (3.2), we get

$$\begin{aligned} S &\triangleq ((\eta_u)_t, v)_j + \mathcal{H}_j\big(f'(u)\eta_u, v; f'(u)\widetilde{\eta}_u\big) - \sqrt{b}\mathcal{H}_j\big(\eta_q, v; \eta_q^+\big) \\ &= ((\eta_u)_t, v)_j + \mathcal{H}_j\big(f'(u)W_u^\ell, v; f'(u)\widetilde{W}_u^\ell\big) - \sqrt{b}\mathcal{H}_j\big(W_q^\ell, v; (W_q^\ell)^+\big), \end{aligned}$$

which, by the definition of the DG spatial operator in (2.4) and the boundary collocation for correction functions in (3.10b), is

$$S = ((w_{u,0} + W_u^\ell)_t, v)_j + (\sqrt{b}W_q^\ell - f'(u)W_u^\ell, v_x)_j.$$

Let us now work on $(\partial_t w_{u,i}, v)_j$ for $i \in \mathbb{Z}_{\ell-1}$, which consists of the first term in S except $(\partial_t w_{u,\ell}, v)_j$. It follows from (2.8) and integration by parts that

$$\begin{aligned} (\partial_t w_{u,i}, v)_j &= \big(\bar{h}_j (D_x^{-1}\partial_t w_{u,i})_x, v\big)_j \\ &= -(\bar{h}_j D_x^{-1}\partial_t w_{u,i}, v_x)_j + \bar{h}_j D_x^{-1}\partial_t w_{u,i}(x_{j+\frac{1}{2}}^-)v(x_{j+\frac{1}{2}}^-) \\ &\quad - \bar{h}_j D_x^{-1}\partial_t w_{u,i}(x_{j-\frac{1}{2}}^+)v(x_{j-\frac{1}{2}}^+). \end{aligned} \tag{3.15}$$

Consequently, substituting the relation (3.10b) regarding the integral terms and the following boundary values implied by the definition of the integral operator D_x^{-1} into (3.15)

$$\begin{aligned} &\bar{h}_j D_x^{-1}\partial_t w_{u,i}(x_{j+\frac{1}{2}}^-) = \int_{I_j}\sum_{m=0}^{k}\partial_t\beta_{i,m}L_{j,m}\,\mathrm{d}x = h_j\partial_t\beta_{i,0}, \\ &\bar{h}_j D_x^{-1}\partial_t w_{u,i}(x_{j-\frac{1}{2}}^+) = 0, \end{aligned}$$

we obtain

$$S = (\partial_t w_{u,\ell}, v)_j + \sum_{i=0}^{\ell-1} h_j\partial_t\beta_{i,0}v(x_{j+\frac{1}{2}}^-) \le Ch^{k+\ell+1}\|u\|_{k+\ell+3,\infty}\|v\|_{1,I_j},$$

where $\beta_{0,0} = 0$ due to (2.7a), and we have also used $\|\partial_t w_{u,\ell}\|_\infty \le Ch^{k+\ell+1}\|u\|_{k+\ell+3,\infty}$ in (3.14a), the inverse property $\|v\|_{\infty,j} \le Ch_j^{-1}\|v\|_{1,I_j}$ and the fact that $|\partial_t\beta_{i,0}| \le Ch^{2k+1}\|u\|_{k+\ell+3,\infty}$ for $i \in \mathbb{Z}_{\ell-1}^+$ in (3.13a).

Analogously, there holds

$$\begin{aligned} |(\eta_q, \varphi)_j - \sqrt{b}\mathcal{H}_j\big(\eta_u, \varphi; \eta_u^-\big)| &= \left|\sum_{i=0}^{\ell-1} h_j\gamma_{i,0}\varphi(x_{j+\frac{1}{2}}^-) + (w_{q,\ell}, \varphi)_j\right| \\ &\le Ch^{k+\ell+1}\|u\|_{k+\ell+2,\infty}\|\varphi\|_{1,I_j}, \end{aligned}$$

where $\gamma_{0,0} = \bar{q}_{j,0} \le Ch^{2k+1}\|u\|_{k+2,\infty}$ owing to (3.4). This completes the proof of Lemma 3.3. □

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Remark 3.1 In contrast to the linear parabolic equations [6], $w_{u,i}$ and $w_{q,i}$ ($i \in \mathbb{Z}_{k-1}$) defined for nonlinear convection-diffusion equations in this paper are no longer orthogonal to $L_{j,0}(x)$. Thus, the boundary terms containing $\beta_{i,0}$ or $\gamma_{i,0}$ will be generated. Fortunately, as shown in the proof of Lemma 3.3, these boundary terms are of high order and will not affect our superconvergence results.

Remark 3.2 If we choose the numerical fluxes (2.2b), we can define the following modified projection

$$\mathring{\Pi}(u,q) = (P_h^+ u, \mathring{\mathbb{P}}_h^- q),$$

in which $P_h^+ u \in V_h^k$ has been given in (2.6b), and $\mathring{\mathbb{P}}_h^- q \in V_h^k$ depends on both u and q such that

$$\int_{I_j} (q - \mathring{\mathbb{P}}_h^- q) v_h \, \mathrm{d}x - \frac{1}{\sqrt{b}} \int_{I_j} f'(u)(u - P_h^+ u) v_h \, \mathrm{d}x = 0, \quad \forall v_h \in P^{k-1}(I_j),$$

$$\mathring{\mathbb{P}}_h^- q(x_{j+\frac{1}{2}}^-) = q(x_{j+\frac{1}{2}}^-) - \frac{1}{\sqrt{b}} \widetilde{f'(u)(u - P_h^+ u)}_{j+\frac{1}{2}}, \qquad \forall j \in \mathbb{Z}_N^+.$$

We can see that $\mathring{\mathbb{P}}_h^-$ is a generalized version of the local Gauss–Radau projection P_h^-. For this case, the correction functions $w_{u,i}$, $w_{q,i}$ are

$$(\sqrt{b} w_{u,i} - \bar{h}_j D_x^{-1} w_{q,i-1}, v)_j = 0, \qquad (w_{u,i}^+)_{j-\frac{1}{2}} = 0,$$

$$(\sqrt{b} w_{q,i} - f'(u) w_{u,i} - \bar{h}_j D_x^{-1} \partial_t w_{u,i-1}, v)_j = 0, \quad (w_{q,i}^-)_{j+\frac{1}{2}} = \frac{1}{\sqrt{b}} \big(f'(u) \widetilde{w}_{u,i}\big)_{j+\frac{1}{2}},$$

where $v \in P^{k-1}(I_j)$, and

$$w_{u,0} = u - P_h^+ u, \qquad w_{q,0} = q - \mathring{\mathbb{P}}_h^- q.$$

By similar arguments as those used for fluxes (2.2a), we conclude that the results in Lemma 3.1–Lemma 3.3 are still valid for the fluxes (2.2b).

4 Superconvergence

In this section, we will prove the superconvergence properties for the LDG solution regarding cell averages and Radau points. To this end, let us first show a supercloseness result for $\|u_I^\ell - u_h\|$.

4.1 Supercloseness

To deal with the nonlinearity of the flux function $f(u)$, we should make an a priori assumption that for small enough h there holds

$$\|P_h^- u - u_h\| \le h^2. \tag{4.1}$$

Note that this a priori assumption doesn't make sense when $k = 0$. Therefore, all the following theorems only hold for $k \ge 1$.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Theorem 4.1 *Let $u \in W^{k+\ell+3,\infty}$, $\ell \in \mathbb{Z}_k^+ (k \geq 1)$ be the exact solution of the problem (1.1), and u_h, q_h are the numerical solutions of LDG scheme (2.1) satisfying (4.1). For periodic boundary conditions, if the initial discretization is chosen such that $u_h(\cdot, 0) = u_I^\ell(\cdot, 0)$, then*

$$\|u_I^\ell - u_h\| + \Big(\int_0^t \|q_I^\ell - q_h\|^2 d\tau\Big)^{\frac{1}{2}} \leq Ch^{k+\ell+1}, \tag{4.2}$$

where C depends on t and $\|u\|_{k+\ell+3,\infty}$.

Proof Choosing $v = \xi_u, \varphi = \xi_q$ in the cell error equation (2.5), and summing up them over all cells, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 + \|\xi_q\|^2 &= \big(-(\eta_u)_t, \xi_u\big) - (\eta_q, \xi_q) - \mathcal{H}\big(f(u) - f(u_h), \xi_u; f(u) - \tilde{f}(u_h)\big) \\
&\quad + \sum_{j=1}^{N} \big(\tilde{f}(u_h) - \hat{f}(u_h)\big)[\![\xi_u]\!]\Big|_{j+\frac{1}{2}} + \sqrt{b}\mathcal{H}\big(\eta_q, \xi_u; \eta_q^+\big) \\
&\quad + \sqrt{b}\mathcal{H}\big(\eta_u, \xi_q; \eta_u^-\big) \\
&\quad + \sqrt{b}\mathcal{H}\big(\xi_q, \xi_u; \xi_q^+\big) + \sqrt{b}\mathcal{H}\big(\xi_u, \xi_q; \xi_u^-\big),
\end{aligned} \tag{4.3}$$

where, for the nonlinear boundary terms, we have added and subtracted the "reference" function $\tilde{f}(u_h)$ defined by (2.3) in $f(u) - \hat{f}(u_h)$. By using the second order Taylor expansion with respect to the variable u, we write out the nonlinear terms as follows

$$f(u) - f(u_h) = f'(u)\xi_u + f'(u)\eta_u - \frac{1}{2}\bar{f}_u''(\xi_u + \eta_u)^2, \tag{4.4a}$$

$$f(u) - \tilde{f}(u_h) = f'(u)\tilde{\xi}_u + f'(u)\tilde{\eta}_u - \frac{1}{2}\bar{\bar{f}}_u''(\tilde{\xi}_u + \tilde{\eta}_u)^2, \tag{4.4b}$$

where $\bar{f}_u''$ and $\bar{\bar{f}}_u''$ are the mean values, which can be given in the integral form of the remainder. Substituting (4.4) into (4.3), and using Lemma 3.3 in combination with the following skew-symmetry property

$$\mathcal{H}(\xi_q, \xi_u; \xi_q^+) + \mathcal{H}(\xi_u, \xi_q; \xi_u^-) = 0,$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 + \|\xi_q\|^2 \leq Ch^{k+\ell+1}(\|\xi_u\| + \|\xi_q\|) + \Lambda + \Theta + \Psi, \tag{4.5}$$

where

$$\begin{aligned}
\Lambda &= -\mathcal{H}\big(f'(u)\xi_u, \xi_u; f'(u)\tilde{\xi}_u\big), \\
\Theta &= \frac{1}{2}\mathcal{H}\big(\bar{f}''(u)e_u^2, \xi_u; \bar{f}''(u)\tilde{e}_u^2\big), \\
\Psi &= \sum_{j=1}^{N} \big(\tilde{f}(u_h) - \hat{f}(u_h)\big)[\![\xi_u]\!]\Big|_{j+\frac{1}{2}}
\end{aligned}$$

will be estimated separately.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

A simple integration by parts gives us the estimate for Λ; it reads,

$$\begin{aligned}
\Lambda &= -\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}\partial_x f'(u)\xi_u^2\,\mathrm{d}x + \sum_{j=1}^{N}\left(f'(u)(\tilde{\xi}_u - \{\!\{\xi_u\}\!\})[\![\xi_u]\!]\right)_{j+\frac{1}{2}} \\
&\le C\|\xi_u\|^2 - \frac{1}{2}\sum_{j=1}^{N}\left(|f'(u)|[\![\xi_u]\!]^2\right)_{j+\frac{1}{2}} \\
&\le C\|\xi_u\|^2. \qquad (4.6a)
\end{aligned}$$

To deal with the high order term Θ, let us first show a "rough" bound of order $k+1$ for $\|\xi\|_\infty$. It is easy to show that

$$\begin{aligned}
|\Theta| &\le Ch^{-1}\|e_u\|_\infty\|e_u\|\|\xi_u\| + C\|e_u\|_\infty\left(\|\xi_u\|_{\Gamma_h} + \|\eta_u\|_{\Gamma_h}\right)\|\xi_u\|_{\Gamma_h} \\
&\le Ch^k\|e_u\|_\infty\|\xi_u\| + Ch^{-1}\|e_u\|_\infty\|\xi_u\|^2 \qquad (4.6b) \\
&\le (Ch^{-1}\|e_u\|_\infty + Ch^{-3}\|e_u\|_\infty^2)\|\xi_u\|^2 + Ch^{2k+3}, \qquad (4.6c)
\end{aligned}$$

where in the last step we have rewritten $h^k\|e_u\|_\infty\|\xi_u\| = h^{-\frac{3}{2}}\|e\|_\infty\|\xi_u\|h^{k+\frac{3}{2}}$ followed by the application of Young's inequality. For Ψ, using the Taylor expansion of f, the Cauchy–Schwarz inequality and the inverse inequality, we have the following estimate; see [4, Theorem 4.3]

$$|\Psi| \le Ch^{-2}\|e_u\|_\infty^2\|\eta_u\|_\infty^2 + C(1 + h^{-1}\|e_u\|_\infty)\|\xi_u\|^2. \qquad (4.6d)$$

By the a priori error assumption (4.1), we have

$$\|e_u\|_\infty \le \|u - P_h^- u\|_\infty + \|P_h^- u - u_h\|_\infty \le Ch^{\frac{3}{2}}. \qquad (4.6e)$$

Inserting the estimates (4.6a), (4.6c)–(4.6e) into (4.5), using Young's inequality and the Gronwall inequality, one has

$$\|\xi_u\| \le Ch^{k+\frac{3}{2}}.$$

Remark that the above estimate for $\|\xi_u\|$ is sufficient to verify the a priori error assumption (4.1) with $k \ge 1$; see, e.g., [22,24]. Then, we arrive at the following error estimate of order $k+1$ for $\|\xi_u\|_\infty$ and thus $\|e_u\|_\infty$.

$$\|e_u\|_\infty \le \|\eta_u\|_\infty + h^{-\frac{1}{2}}\|\xi_u\| \le Ch^{k+1}. \qquad (4.7)$$

We are now ready to prove the supercloseness result in (4.2). Substituting (4.7) into (4.6b), (4.6d) and (4.5), and taking into account (4.6a), we obtain, after using Young's inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 + \|\xi_q\|^2 \le Ch^{k+\ell+1}(\|\xi_u\| + \|\xi_q\|) + \|\xi_u\|^2 + Ch^{4k+2}.$$

Choosing $u_h(\cdot,0) = u_I^\ell(\cdot,0)$ and using Gronwall inequality, we have

$$\|\xi_u\| + \left(\int_0^t \|\xi_q\|^2\mathrm{d}\tau\right)^{\frac{1}{2}} \le Ch^{k+\ell+1}.$$

This finishes the proof of Theorem 4.1. □

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

4.2 Superconvergence

To derive superconvergence properties for the derivative approximation at Radau points, the following Lemma is needed.

Lemma 4.1 [2] *Let $q_h, u_h \in V_h^k$ satisfy*

$$(q_h, \varphi)_j = \mathcal{H}_j\big(u_h, \varphi; \hat{u}_h\big), \quad \forall \varphi \in V_h^k.$$

Then there holds for $\hat{u}_h = u_h^-$

$$\partial_x u_h(\ell_{j,m}) = q_h(\ell_{j,m}), \quad (j, m) \in \mathbb{Z}_N \times \mathbb{Z}_k,$$

and for $\hat{u}_h = u_h^+$

$$\partial_x u_h(r_{j,m}) = q_h(r_{j,m}), \quad (j, m) \in \mathbb{Z}_N \times \mathbb{Z}_k.$$

Due to the supercloseness result between u_I^ℓ and u_h in Theorem 4.1, taking $\ell \le k$ in the correction functions, we have the following superconvergence results for the LDG solution u_h.

Theorem 4.2 *Assume that $u \in W^{2k+3,\infty}(\Omega)$, $k \ge 1$ is the solution of* (1.1)*, and u_h, q_h are the numerical solutions of the LDG scheme* (1.1) *when the alternating fluxes* (2.2a) *are used with the initial solution $u_h(\cdot, 0) = u_I^k(\cdot, 0)$. Then for periodic boundary conditions, we have the following superconvergence results*

(1) Superconvergence of the numerical fluxes

$$\|e_{un}\| = \Big(\frac{1}{N}\sum_{j=1}^{N}\big|(u - \hat{u}_h)(x_{j+\frac{1}{2}}, t)\big|^2\Big)^{\frac{1}{2}} \le Ch^{2k+1}.$$

(2) Superconvergence for the cell averages

$$\|e_u\|_c = \Big(\frac{1}{N}\sum_{j=1}^{N}\Big|\frac{1}{h_j}\int_{I_j}(u - u_h)(x, t)\,\mathrm{d}x\Big|^2\Big)^{\frac{1}{2}} \le Ch^{2k+1}.$$

(3) When $\ell \ge 2$, the function value approximation of the LDG solution is $(k+2)$th order superconvergent at right Radau points $r_{j,m}$, and the derivative value approximation is $(k+1)$th order superconvergence at interior left Radau points (except the point $x = x_{j-\frac{1}{2}}$), i.e.,

$$\|e_{ur}\| = \max_{j \in \mathbb{Z}_N} |(u - u_h)(r_{j,m})| \le Ch^{k+2},$$

$$\|e_{u\ell}\| = \max_{j \in \mathbb{Z}_N} |\partial_x(u - u_h)(\ell_{j,m})| \le Ch^{k+1}.$$

It is worth pointing out that, when the direction of the flow doesn't change, the order of $\|e_{u\ell}\|$ can also be $k+2$.

(4) The numerical solution u_h is superconvergent with order $k+2$ towards the Gauss–Radau projection $P_h^- u$ of the exact solution, namely,

$$\|u_h - P_h^- u\| \le Ch^{k+2}.$$

The constant C is independent of h.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Proof (1) It follows from the exact collocation of P_h^- in (2.6a) as well as $(w_{u,i}^-)_{j+\frac{1}{2}} = 0$ in (3.10a), the inverse inequality and the supercloseness result in Theorem 4.1 that

$$\begin{aligned} \|e_{un}\| &= \Big(\frac{1}{N}\sum_{j=1}^{N}\big|\widehat{(u_I^k - u_h)}(x_{j+\frac{1}{2}}, t)\big|^2\Big)^{\frac{1}{2}} \\ &\le \Big(\frac{1}{N}\sum_{j=1}^{N} C h_j^{-1}\|u_I^k - u_h\|_{I_j}\Big)^{\frac{1}{2}} \\ &\le C\|u_I^k - u_h\| \le Ch^{2k+1}. \end{aligned} \tag{4.8}$$

(2) By the orthogonality property of P_h^- in (2.6a) and the definition of u_I^k, we obtain

$$(e_u, 1)_j = (u_I^k - u_h, 1)_j + (W_u^k, 1)_j.$$

Then, by a direct calculation and taking into account (3.13a) with $m = 0$ due to the orthogonality property of Legendre polynomials, we get

$$\|e_u\|_c \le \|u_I^k - u_h\| + Ch^{2k+1}\|u\|_{2k+3,\infty} \le Ch^{2k+1}.$$

(3) If we take $\ell \ge 2$ in Theorem 4.1, and use the inverse inequality, we obtain

$$\|\xi_u\|_\infty \le Ch^{k+\frac{5}{2}}.$$

By the triangle inequality,

$$\begin{aligned} |(u - u_h)(r_{j,m})| &\le |(u - P_h^- u)(r_{j,m})| + \|\xi_u\|_\infty + \|W_u^\ell\|_\infty \\ &\le Ch^{k+2}, \end{aligned}$$

where we have also used $|(u - P_h^- u)(r_{j,m})| \le Ch^{k+2}$ due to the standard approximation theory. The result of the other equation for the derivative approximations can be obtained by the same arguments.

Moreover, if the direction of the flow doesn't change, combining Lemma 3.1, Lemma 3.2 and Lemma 4.1 we have

$$|\partial_x(u - u_h)(\ell_{j,m})| = |(q - q_h)(\ell_{j,m})| \le Ch^{k+2}.$$

(4) Using the triangle inequality,

$$\|u_h - P_h^- u\| \le \|P_h^- u - W_u^k - u_h\| + \|W_u^k\| \le Ch^{k+2}.$$

This finishes the proof of Theorem 4.2. □

Remark 4.1 From the construction of the special projection in (3.1), we can see that the conclusion is only valid for $b = \mathcal{O}(1)$. For convection dominated problems with small diffusion coefficient $b \ll 1$, the exact solution often exists a boundary layer near the outflow boundary. When the direction of the flow doesn't change, we can observe superconvergence property similar to the nonlinear hyperbolic equations [4] out of the local subdomain with pollution width of $\mathcal{O}(h\ln N)$.

Remark 4.2 For the strongly anisotropic problems when b is very large, the theoretical results are still valid, since $\Phi = (-1)^k \frac{1}{\sqrt{b}} f'(u)\widetilde{(u - P_h^- u)}_{j-\frac{1}{2}}$ has an additional order $\frac{1}{\sqrt{b}}$. However, this case requires a smaller time step when explicit time discretization methods are used.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Remark 4.3 For high dimensions, we need to introduce more auxiliary variables, such as $p = u_x, q = u_y$ for the two-dimensional case. Unfortunately, it is difficult to construct an interpolation function to deal with $p = u_x$ and $q = u_y$ simultaneously. The main technical difficulty is that the conditions they need to satisfy are interactive restricted in the process of constructing interpolation functions.

4.3 The Initial Discretization

In this section, we consider how to discretize the initial datum. Initial value discretization is very important for the study of superconvergence, which can be obtained using the same technique as that in [2]. Specifically, for periodic boundary conditions,

1. according to the definition of projection $P_h^-, \mathbb{P}_h^+$, calculate the $w_{u,0}, w_{q,0}$;
2. calculate $w_{u,i}, w_{q,i}$ by the equations (3.10);
3. calculate $W_u^\ell = \sum_{i=1}^{\ell} w_{u,i}, u_I^\ell = P_h^- u - W_u^\ell$;
4. let $u_h(\cdot, 0) = u_I^\ell(\cdot, 0)$.

5 Extensions

5.1 Generalized Alternating Numerical Fluxes

In this section, we extend the superconvergence results to generalized alternating numerical fluxes. To be more specific, the numerical fluxes can be in the following form

$$\hat{v}_{j+\frac{1}{2}} = v^{(\theta)}_{j+\frac{1}{2}} = \theta v^-_{j+\frac{1}{2}} + \tilde{\theta} v^+_{j+\frac{1}{2}}, \quad \tilde{\theta} = 1 - \theta.$$

When the numerical fluxes $(u_h^{(\theta)}, q_h^{(\tilde{\theta})})$ are used, we introduce a modified projection $\widetilde{\Pi}(u, q) = (P_\theta u, \mathbb{P}_{\tilde{\theta}} q)$ satisfying

$$\int_{I_j} (P_\theta u - u) v_h \, \mathrm{d}x = 0, \qquad \forall v_h \in P^{k-1}(I_j),$$
$$\left(P_\theta u\right)^{(\theta)}_{j+\frac{1}{2}} = u^{(\theta)}_{j+\frac{1}{2}}, \qquad \forall j \in \mathbb{Z}_N^+,$$

and $\mathbb{P}_{\tilde{\theta}} q \in V_h^k$ depends on both u and q such that

$$\int_{I_j} (q - \mathbb{P}_{\tilde{\theta}} q) v_h \, \mathrm{d}x - \frac{1}{\sqrt{b}} \int_{I_j} f'(u)(u - P_\theta u) v_h \, \mathrm{d}x = 0, \qquad \forall v_h \in P^{k-1}(I_j),$$
$$(\mathbb{P}_{\tilde{\theta}} q)^{(\tilde{\theta})}_{j+\frac{1}{2}} = q^{(\tilde{\theta})}_{j+\frac{1}{2}} - \frac{1}{\sqrt{b}} f'(u_{j+\frac{1}{2}}) (\widetilde{u - P_\theta u})_{j+\frac{1}{2}}, \qquad \forall j \in \mathbb{Z}_N^+.$$

Similar to (3.2), the boundary terms of the projection errors for both convection and diffusion parts can be eliminated. For more properties of global projections; see, e.g., [11,21].

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Analogously, we define a series of functions $w_{u,i}, w_{q,i}, i \in \mathbb{Z}_k^+$ as follows

$$(\sqrt{b}w_{u,i} - \bar{h}_j D_x^{-1} w_{q,i-1}, v_h)_j = 0, \qquad (w_{u,i}^{(\theta)})_{j+\frac{1}{2}} = 0, \tag{5.1a}$$

$$(\sqrt{b}w_{q,i} - w_{u,i} - \bar{h}_j D_x^{-1} \partial_t w_{u,i-1}, v_h)_j = 0, \quad (w_{q,i}^{(\tilde{\theta})})_{j+\frac{1}{2}} = \frac{1}{\sqrt{b}} f'(u_{j+\frac{1}{2}})(\widetilde{w}_{u,i})_{j+\frac{1}{2}}, \tag{5.1b}$$

where $v_h \in P^{k-1}(I_j)$, and

$$w_{u,0} = u - P_\theta u, \qquad w_{q,0} = q - \mathbb{P}_{\tilde{\theta}} q.$$

Following the same argument as that in Sect. 3, we can obtain superconvergence results similar to Lemmas 3.1–3.3. The main difference is that we need to solve linear coupled systems involving the coefficients $\bar{q}_{j,k}, \beta_{i,k}$, and $\gamma_{i,k}$ for $j \in \mathbb{Z}_N^+$.

Next, let us present some preliminary results related to the superconvergence results based on generalized alternating numerical fluxes. The generalized Radau polynomials are defined as in [3]

$$R_{k+1}^\theta = \begin{cases} L_{k+1} - (2\theta - 1)L_k, & \text{when } k \text{ is even,} \\ (2\theta - 1)L_{k+1} - L_k, & \text{when } k \text{ is odd.} \end{cases}$$

For any positive $\theta \neq \frac{1}{2}$, $j \in \mathbb{Z}_N^+$, if the following local projection $P_h u \in V_h^k$ in [3] is introduced,

$$\int_{I_j} (P_h u - u) v \, \mathrm{d}x = 0, \quad \forall v \in P^{k-1}(I_j),$$

$$\theta P_h u(x_{j+\frac{1}{2}}^-) + (1-\theta) P_h u(x_{j-\frac{1}{2}}^+) = \theta u(x_{j+\frac{1}{2}}^-) + (1-\theta) u(x_{j-\frac{1}{2}}^+).$$

Then the following superconvergence results hold.

Lemma 5.1 [3] *Suppose $u \in W^{k+2,\infty}(\Omega)$ and $P_h u$ is the local projection of u defined above with $\theta \neq \frac{1}{2}$, then*

$$|(u - P_h u)(\mathcal{R}_{j,m})| \leq Ch^{k+2},$$

$$|\partial_x (u - P_h u)(\mathcal{R}_{j,m}^\star)| \leq Ch^{k+1},$$

$$\| P_h u - P_\theta u \|_\infty \leq Ch^{k+2}.$$

Here $\mathcal{R}_{j,m}, \mathcal{R}_{j,m}^\star$ are the roots of $R_{j,m+1}^\theta$ and $\partial_x R_{j,m+1}^\theta$, and C is independent of h.

Following the same argument as what we did in Sect. 4, we obtain the superconvergence results based on generalized alternating numerical fluxes, whose detailed proofs are omitted to save space.

Theorem 5.1 *Assume that $u \in W^{2k+3,\infty}(\Omega)$, $k \geq 1$ is the solution of (1.1), and u_h, q_h are the numerical solutions of LDG scheme (1.1) when the numerical fluxes $(u_h^{(\theta)}, q_h^{(\tilde{\theta})})$ are used with the initial solution $u_h(\cdot, 0) = u_I^k(\cdot, 0)$. Then for periodic boundary conditions, we have the following superconvergence results*

1. Superconvergence of the numerical flux

$$\|e_{un}\| = \Big(\frac{1}{N} \sum_{j=1}^{N} \big| (u - u_h)^{(\theta)} (x_{j+\frac{1}{2}}, t) \big|^2 \Big)^{\frac{1}{2}} \leq Ch^{2k+1}.$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

2. Superconvergence for the cell averages

$$\|e_u\|_c = \Big(\frac{1}{N}\sum_{j=1}^{N}\Big|\frac{1}{h_j}\int_{I_j}(u-u_h)(x,t)\,\mathrm{d}x\Big|^2\Big)^{\frac{1}{2}} \le Ch^{2k+1}.$$

3. When $\ell \ge 2$, the function value approximation of the LDG solution is $(k+2)$th order superconvergent at interior generalized Radau points $\mathcal{R}_{j,m}$, and the derivative value approximation is $(k+1)$th order superconvergence at interior generalized derivative Radau points $\mathcal{R}^\star_{j,m}$, i.e.,

$$\|e_{ur}\| = \max_{j\in\mathbb{Z}_N} |(u-u_h)(\mathcal{R}_{j,m})| \le Ch^{k+2},$$

$$\|e^\star_{ur}\| = \max_{j\in\mathbb{Z}_N} |\partial_x(u-u_h)(\mathcal{R}^\star_{j,m})| \le Ch^{k+1}.$$

4. The numerical solution u_h is superconvergent with order $k+2$ towards the global projection $P_\theta u$ of the exact solution, namely,

$$\|u_h - P_\theta u\| \le Ch^{k+2}.$$

The constant C is independent of h.

5.2 Mixed Boundary Conditions

Consider the following mixed boundary conditions

$$u(0,t) = g_1(t), \quad u_x(2\pi,t) = g_2(t). \tag{5.2}$$

For simplicity, we choose the numerical fluxes as

$$\big(\hat f(u_h),\ \hat u_h,\ \hat q_h\big)_{j+\frac12} = \begin{cases} \big(f(g_1),\ g_1,\ q_h^+\big)_{\frac12}, & j=0,\\ \big(\text{Godunov flux},\ u_h^-,\ q_h^+\big)_{j+\frac12}, & j=1,\dots,N-1,\\ \big(f(u_h^-),\ u_h^-,\ g_2\big)_{N+\frac12}, & j=N. \end{cases} \tag{5.3}$$

The projection $\mathbb{P}_h^+$ defined in (3.1) is modified to $\tilde{\mathbb{P}}_h^+$ determined by

$$\int_{I_j}(q-\tilde{\mathbb{P}}_h^+ q)v_h\,\mathrm{d}x - \frac{1}{\sqrt b}\int_{I_j} f'(u)(u-P_h^- u)v_h\,\mathrm{d}x = 0, \qquad \forall v_h \in P^{k-1}(I_j),$$

$$\tilde{\mathbb{P}}_h^+ q(x^+_{j-\frac12}) = q(x^+_{j-\frac12}) - \frac{1}{\sqrt b}\widetilde{f'(u)(u-P_h^- u)}_{j-\frac12}, \qquad \forall j \in \mathbb{Z}_N^+\backslash\{1\},$$

$$\tilde{\mathbb{P}}_h^+ q(x^+_{\frac12}) = q(x^+_{\frac12}),$$

where $\widetilde{w}$ has been defined in (2.3). Then we construct the following correction functions

$$(w_{u,i} - \bar h_j D_x^{-1} w_{q,i-1}, z)_j = 0, \qquad (w^-_{u,i})_{j+\frac12} = 0, \qquad \forall j \in \mathbb{Z}_N^+,$$

$$(w_{q,i} - w_{u,i} - \bar h_j D_x^{-1}\partial_t w_{u,i-1}, z)_j = 0, \qquad (w^+_{q,i})_{j-\frac12} = \frac{1}{\sqrt b} f'(u)\widetilde{w}_{u,i}\big|_{j-\frac12}, \qquad \forall j \in \mathbb{Z}_N^+\backslash\{1\},$$

$$(w^+_{q,i})_{\frac12} = 0$$

for $\forall z \in P^{k-1}(I_j)$. The superconvergence results can thus be obtained if we follow the same arguments as those in Sects. 3 and 4.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Remark 5.1 The novelty in designing a new projection is that the diffusion term is used to balance the convection term. Therefore, when constructing $\Pi(u, q)$ in Sect. 3.1, the projection P_h^- dealing with convection term should be designed first, and then the projection $\mathbb{P}_h^+$. For the case of Dirichlet boundary conditions

$$u(0, t) = g_3(t), \qquad u(2\pi, t) = g_4(t),$$

it is difficult to modify the projection $\mathbb{P}_h^+$ to eliminate the boundary term introduced by the auxiliary variable q. However, the superconvergence phenomenon can still be observed when we follow [13] and define the numerical fluxes as follows

$$\left(\hat{f}(u_h),\ \hat{u}_h,\ \hat{q}_h\right)_{j+\frac{1}{2}} = \begin{cases} \left(f(g_3),\ g_3,\ q_h^+\right)_{\frac{1}{2}}, & j = 0, \\ \left(\text{Godunov flux},\ u_h^-,\ q_h^+\right)_{j+\frac{1}{2}}, & j = 1, \ldots, N-1, \\ \left(f(u_h^-) - \kappa(g_4 - u_h^-),\ g_4,\ q_h^-\right)_{N+\frac{1}{2}}, & j = N, \end{cases} \tag{5.4}$$

where $\kappa = \mathcal{O}(h^{-1})$ is a positive constant. See Table 8 for numerical results.

5.3 Superconvergence for the Auxiliary Variable

For the numerical flux in (2.2a), the superconvergence properties still hold for the auxiliary variable q, if the direction of the flow doesn't change.

Theorem 5.2 *Assume that $u \in W^{2k+3,\infty}(\Omega)$, $k \geq 1$ is the solution of* (1.1)*, and u_h, q_h are the numerical solutions of LDG scheme* (1.1) *when the alternating fluxes* (2.2a) *are used with the initial solution $u_h(\cdot, 0) = u_I^k(\cdot, 0)$. Then for periodic boundary conditions, we have the following superconvergence results for the auxiliary variable q_h.*

1. Superconvergence of the numerical flux

$$\left(\int_0^t \|e_{qn}\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \leq Ch^{2k+1}.$$

2. Superconvergence for the cell averages

$$\left(\int_0^t \|e_q\|_c^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \leq Ch^{2k+1}.$$

3. When $\ell \geq 2$, the function value approximation of the LDG solution is $(k+2)$th order superconvergent at left Radau points $\ell_{j,m}$, and the derivative value approximation is $(k+1)$th order superconvergence at the interior right Radau points, except the point $x = x_{j+\frac{1}{2}}$, i.e.,

$$\left(\int_0^t \|e_{q\ell}\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \leq Ch^{k+2}, \qquad \left(\int_0^t \|e_{qr}\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \leq Ch^{k+1}.$$

4. The numerical solution q_h is superconvergent with order $k+2$ towards the Gauss–Radau projection $P_h^+ q$ of the exact solution, namely,

$$\left(\int_0^t \|q_h - P_h^+ q\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \leq Ch^{k+2}.$$

The norms aforementioned can be defined as the same way as in Theorem 4.2 and C is independent of h.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Table 1 CFL constants for different numerical examples

	Table 2	Table 3	Table 4	Table 5	Table 6	Table 7	Table 8	Table 9
CFL_1	0.02	0.02	0.02	0.02	0.005	0.0005	0.02	0.1
CFL_2	0.01	0.005	0.01	0.005	0.003	0.0001	0.01	0.1
CFL_3	0.002	0.002	0.005	0.002	0.001	–	–	–

Remark 5.2 In the superconvergence analysis, the correction functions (W_u^ℓ, W_q^ℓ) we designed should satisfy the following properties

$$\left(W_u^\ell\right)^-_{j+\frac{1}{2}} = 0, \quad \left(W_q^\ell\right)^+_{j-\frac{1}{2}} = 0, \quad j \in \mathbb{Z}_N,$$

which are needed to derive superconvergence of the numerical flux in (4.8). Therefore, it is easy to see that superconvergence of the auxiliary variable q_h is no longer valid when the flow direction changes in (3.10b), or the generalized alternating numerical fluxes (5.1b) are used. The superconvergence can be observed numerically in the $L^\infty([0, T]; L^2(\Omega))$ norm.

6 Numerical Experiments

In this section, we provide numerical examples to verify our theoretical results. For time discretization, we use the third order explicit total variation diminishing Runge–Kutta method and take $\Delta t = CFL_k * h^2$ for $P^k (1 \le k \le 3)$ polynomials. In all examples, uniform meshes are considered and the parameters CFL_k are listed in Table 1.

Example 6.1 We first consider the following problem with the direction of the flow not change

$$\begin{aligned} &u_t + (e^u)_x - bu_{xx} = g(x, t), && (x, t) \in [0, 2\pi] \times (0, T], \\ &u(x, 0) = \sin(5x), && x \in [0, 2\pi] \end{aligned}$$

with the periodic boundary condition. $g(x, t)$ is suitably chosen such that the exact solution is

$$u(x, t) = e^{-bt} \sin(5x + t).$$

Table 2 lists results for u and q when $b = 1.2$, $T = 1$, from which we observe $(2k + 1)$th order superconvergence for the numerical trace as well as cell averages. In addition, for the prime variable u_h, superconvergence of the function value approximation and the derivative approximation at Radau points both achieve $(k + 2)$th order. In Table 3, we present the L^2 errors of ξ_u, ξ_q, $P_h^+ q - q_h$ and $P_h^- u - u_h$, which demonstrates that the LDG solution u_h (q_h) is superconvergent with order $k+2$ towards the Gauss–Radau projection $P_h^- u$ $(P_h^+ q)$. Moreover, by correcting the local projection, the order of L^2 error between numerical solution and interpolation function can reach $2k + 1$.

Example 6.2 In this example, we consider the following problem with the direction of the flow changes

$$\begin{aligned} &u_t + (u^2/2)_x - bu_{xx} = g(x, t), && (x, t) \in [0, 2\pi] \times (0, T], \\ &u(x, 0) = \sin(3x), && x \in [0, 2\pi] \end{aligned}$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Table 2 Errors and rates for Example 6.1 with $b = 1.2$, $T = 1$

	N	$\|e_{un}\|$	Rate	$\|e_u\|_c$	Rate	$\|e_{ur}\|$	Rate	$\|e_{u\ell}\|$	Rate
P^1	40	2.54E-04	–	1.36E-03	–	1.62E-03	–	5.06E-03	–
	80	3.17E-05	3.00	1.73E-04	2.97	2.17E-04	2.90	6.32E-04	3.00
	160	3.97E-06	2.99	2.18E-05	2.99	2.78E-05	2.96	8.09E-05	2.97
	320	4.97E-07	3.00	2.73E-06	3.00	3.51E-06	2.98	1.02E-05	2.99
P^2	20	1.28E-04	–	1.47E-04	–	8.28E-04	–	3.57E-03	–
	40	3.95E-06	5.01	4.51E-06	5.02	4.60E-05	4.17	2.20E-04	4.02
	80	1.20E-07	5.03	1.38E-07	5.03	2.84E-06	4.02	1.49E-05	3.88
	160	3.66E-09	5.04	4.24E-09	5.03	1.72E-07	4.04	9.50E-07	3.97
P^3	30	1.04E-07	–	1.15E-07	–	4.37E-06	–	3.25E-05	–
	40	1.32E-08	7.15	1.47E-08	7.15	1.01E-06	5.10	8.58E-06	4.62
	50	2.74E-09	7.05	3.04E-09	7.05	3.46E-07	4.78	2.81E-06	4.99
	60	7.56E-10	7.06	8.37E-10	7.06	1.40E-07	4.98	1.10E-06	5.13

	N	$\|e_{qn}\|$	Rate	$\|e_q\|_c$	Rate	$\|e_{qr}\|$	Rate	$\|e_{q\ell}\|$	Rate
P^1	40	2.67E-04	–	1.38E-03	–	1.36E-01	–	5.54E-03	–
	80	3.35E-05	2.99	1.77E-04	2.96	3.52E-02	1.96	6.92E-04	3.00
	160	4.20E-06	2.99	2.23E-05	2.98	8.82E-03	1.99	8.86E-05	2.96
	320	5.26E-07	3.00	2.80E-06	2.99	2.20E-03	2.00	1.11E-05	2.99
P^2	20	1.26E-05	–	9.42E-05	–	1.02E-01	–	3.91E-03	–
	40	4.70E-07	4.75	3.30E-06	4.83	1.25E-02	3.03	2.41E-04	4.02
	80	1.48E-08	4.98	1.04E-07	4.98	1.65E-03	2.92	1.63E-05	3.88
	160	4.63E-10	5.00	3.26E-09	5.00	2.07E-04	2.99	1.04E-06	3.97
P^3	30	1.67E-08	–	6.61E-08	–	2.18E-03	–	3.56E-05	–
	40	2.07E-09	7.25	9.94E-09	6.58	6.86E-04	4.03	9.40E-06	4.63
	50	4.26E-10	7.09	2.10E-09	6.97	2.97E-04	3.75	3.08E-06	4.99
	60	1.16E-10	7.13	5.86E-10	6.99	1.43E-04	4.02	1.21E-06	5.13

with the periodic boundary condition. The source term $g(x,t)$ is specially chosen such that the exact solution is

$$u(x,t) = e^{-bt}\sin(3x + t).$$

We list various errors and corresponding convergence rates when $b = 1.0, T = 1$ in Table 4. Superconvergence of order $2k+1$ for the numerical trace as well as cell averages, and $(k+2)$th order of the function value approximation at Radau points confirm the sharpness of Theorem 4.2 when the flow direction does change its sign. Moreover, the derivative approximation at Radau points achieves $(k+1)$th order as expected. In addition, superconvergence results for ξ_u and $P_h^- u - u_h$ in the L^2 norm are shown in Table 5, for $b = 0.1$, $b = 2.0$. For this problem, results of the generalized alternating fluxes with different weights θ with $b = 2.0$ are shown in Table 6, demonstrating that superconvergence results are still valid for the generalized numerical fluxes in Sect. 5.1.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Table 3 Errors and rates for Example 6.1 with $b = 2.0$, $T = 1$

	N	$\|\xi_u\|$	Rate	$\|P_h^- u - u_h\|$	Rate	$\|\xi_q\|$	Rate	$\|P_h^+ q - q_h\|$	Rate
P^1	40	1.64E-04	–	1.79E-03	–	1.18E-03	–	1.27E-03	–
	80	2.06E-05	2.99	2.29E-04	2.97	1.54E-04	2.93	1.64E-04	2.96
	160	2.60E-06	2.98	2.88E-05	2.99	1.97E-05	2.97	2.07E-05	2.98
	320	3.27E-07	2.99	3.61E-06	2.99	2.48E-06	2.98	2.61E-06	2.99
P^2	20	1.81E-04	–	8.63E-04	–	3.21E-05	-	5.72E-04	–
	40	5.97E-06	4.92	5.49E-05	3.97	1.06E-06	4.92	3.56E-05	4.00
	80	1.87E-07	5.00	3.45E-06	3.99	3.16E-08	5.06	2.24E-06	3.99
	160	5.77E-09	5.02	2.16E-07	3.99	9.68E-10	5.03	1.41E-07	3.99
P^3	30	1.46E-07	–	7.35E-06	–	1.51E-08	-	4.96E-06	–
	40	1.95E-08	6.98	1.76E-06	4.97	1.82E-09	7.34	1.18E-06	4.98
	50	4.12E-09	6.97	5.79E-07	4.98	3.65E-10	7.19	3.90E-07	4.98
	60	1.15E-09	7.00	2.33E-07	4.99	9.84E-11	7.19	1.57E-07	4.99

Table 4 Errors and rates for Example 6.2 with $b = 1.0$, $T = 1$

	N	$\|e_{un}\|$	Rate	$\|e_u\|_c$	Rate	$\|e_{ur}\|$	Rate	$\|e_{u\ell}\|$	Rate
P^1	40	2.00E-04	–	4.56E-04	–	7.32E-04	–	2.38E-03	–
	80	2.61E-05	2.94	5.79E-05	2.98	9.28E-05	2.98	5.07E-04	2.23
	160	3.30E-06	2.98	7.26E-06	2.99	1.16E-05	3.00	1.16E-04	2.13
	320	4.13E-07	3.00	9.09E-07	3.00	1.45E-06	3.00	2.75E-05	2.07
P^2	20	1.24E-05	–	1.37E-05	–	1.38E-04	–	6.27E-04	–
	40	4.51E-07	4.79	4.98E-07	4.79	8.12E-06	4.08	6.18E-05	3.34
	80	1.49E-08	4.92	1.64E-08	4.92	4.89E-07	4.05	6.59E-06	3.23
	160	4.77E-10	4.97	5.26E-10	4.97	2.99E-08	4.03	7.57E-07	3.12
P^3	30	3.70E-09	–	4.98E-09	–	4.52E-07	–	9.72E-06	–
	40	5.07E-10	6.91	6.77E-10	6.93	1.07E-07	5.02	3.23E-06	3.84
	50	1.06E-10	6.99	1.42E-10	6.99	3.48E-08	5.01	1.31E-06	4.04
	60	2.97E-11	7.01	3.96E-11	7.01	1.39E-08	5.02	6.15E-07	4.15

Example 6.3 To illustrate the case with different boundary conditions and long time behaviors, consider the following problem

$$u_t + (u^2/2)_x - bu_{xx} = g(x,t), \qquad (x,t) \in [0,1] \times (0,T],$$

$$u(x,0) = \cos(\pi(x-1)/2) - \frac{e^{(x-1)/b} - 1}{1 - e^{-1/b}}, \qquad x \in [0,1]$$

with mixed boundary conditions

$$u(0,t) = \cos(\pi t), \quad u_x(1,t) = \frac{\cos(\pi t)}{(e^{-1/b} - 1)b}.$$

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Table 5 Errors and rates for Example 6.2 with $b = 0.1$, $b = 2.0$, $T = 1$

	N	$b = 0.1$				$b = 2.0$			
		$\|\xi_u\|$	Rate	$\|P_h^- u - u_h\|$	Rate	$\|\xi_u\|$	Rate	$\|P_h^- u - u_h\|$	Rate
P^1	40	7.68E-03	–	1.16E-02	–	2.84E-04	–	4.98E-04	–
	80	9.69E-04	2.99	1.52E-03	2.93	3.68E-05	2.95	6.32E-05	2.98
	160	1.21E-04	3.01	1.93E-04	2.97	4.64E-06	2.99	7.92E-06	2.99
	320	1.50E-05	3.01	2.43E-05	2.99	5.82E-07	3.00	9.91E-07	3.00
P^2	20	1.53E-03	–	1.98E-03	–	1.82E-05	–	1.17E-04	–
	40	3.03E-05	5.66	1.18E-04	4.07	6.32E-07	4.85	7.26E-06	4.01
	80	7.50E-07	5.33	7.23E-06	4.03	2.04E-08	4.95	4.52E-07	4.01
	160	2.13E-08	5.13	4.47E-07	4.02	6.48E-10	4.98	2.82E-08	4.00
P^3	30	2.33E-06	–	1.35E-05	–	4.95E-09	–	5.83E-07	–
	40	3.07E-07	7.04	3.30E-06	4.91	6.83E-10	6.89	1.39E-07	4.99
	50	6.42E-08	7.02	1.10E-06	4.92	1.45E-10	6.96	4.55E-08	5.00
	60	1.78E-08	7.02	4.47E-07	4.94	4.04E-11	6.99	1.83E-08	5.00

Table 6 Errors and rates for Example 6.2 with generalized fluxes for $b = 2.0$, $T = 1$

	N	$\|e_{un}\|$	Rate	$\|e_u\|_c$	Rate	$\|e_{ur}\|$	Rate	$\|e_{ur}^\star\|$	Rate
	40	6.22E-05	–	1.01E-04	–	2.90E-04	–	3.00E-03	–
P^1	80	7.52E-06	3.05	1.19E-05	3.08	3.45E-05	3.07	7.15E-04	2.07
$\theta = 1.5$	160	9.32E-07	3.01	1.47E-06	3.02	4.25E-06	3.02	1.77E-04	2.02
	320	1.16E-07	3.00	1.82E-07	3.01	5.29E-07	3.01	4.41E-05	2.00
	20	6.89E-06	–	6.95E-06	–	1.40E-04	–	2.28E-03	–
P^2	40	1.71E-07	5.33	1.72E-07	5.34	8.42E-06	4.06	2.70E-04	3.07
$\theta = 0.8$	80	5.03E-09	5.08	5.06E-09	5.08	5.18E-07	4.02	3.32E-05	3.02
	160	1.56E-10	5.01	1.57E-10	5.01	3.22E-08	4.01	4.13E-06	3.00
	30	2.28E-09	–	2.34E-09	–	9.38E-07	–	3.85E-05	–
P^3	40	2.94E-10	7.13	3.01E-10	7.13	2.31E-07	4.86	1.26E-05	3.86
$\theta = 1.2$	50	6.05E-11	7.07	6.22E-11	7.07	7.56E-08	5.01	5.16E-06	4.02
	60	1.67E-11	7.04	1.72E-11	7.04	3.04E-08	4.99	2.49E-06	4.00

The source term $g(x, t)$ is specially chosen such that the exact solution is

$$u(x,t) = \left[\cos(\pi(x-1)/2) - \frac{e^{(x-1)/b} - 1}{1 - e^{-1/b}}\right]\cos(\pi t). \tag{6.1}$$

When $b = 100$, $T = 3$, the errors and their orders are presented in Table 7, from which we observe $(2k+1)$th order superconvergence for the cell averages and the numerical fluxes in the discrete L^2 norm. Superconvergence of order $k+2$ $(k+1)$ can be seen for the function (derivative) value approximations and interior right (left) Radau points. This example indicates that the superconvergence results are also sharp when mixed boundary conditions are adopted.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Table 7 Errors and rates for Example 6.3 with mixed boundary conditions, $b = 100$, $T = 3$

	N	$\|e_{un}\|$	Rate	$\|e_u\|_c$	Rate	$\|e_{ur}\|$	Rate	$\|e_{u\ell}\|$	Rate
P^1	25	1.89E-08	–	2.46E-06	–	3.06E-06	–	4.76E-06	–
	30	1.09E-08	3.01	1.42E-06	3.00	1.77E-06	3.00	3.03E-06	2.47
	35	6.85E-09	3.01	8.94E-07	3.00	1.12E-06	3.00	2.08E-06	2.43
	40	4.58E-09	3.01	5.99E-07	3.00	7.48E-07	3.00	1.51E-06	2.39
P^2	10	3.02E-11	–	4.73E-11	–	2.27E-07	–	5.08E-07	–
	15	3.89E-12	5.05	6.27E-12	4.98	4.51E-08	3.99	1.25E-07	3.45
	20	8.05E-13	5.48	1.58E-12	4.80	1.43E-08	3.99	4.77E-08	3.36
	25	2.28E-13	5.65	5.46E-13	4.75	5.86E-09	4.00	2.28E-08	3.31

Table 8 Errors and rates for Example 6.3 with Dirichlet boundary conditions, $b = 1.5$, $T = 100$

	N	$\|e_{un}\|$	Rate	$\|e_u\|_c$	Rate	$\|e_{ur}\|$	Rate	$\|e_{u\ell}\|$	Rate
P^1	25	4.58E-07	–	3.12E-06	–	3.35E-06	–	1.74E-04	–
	30	2.65E-07	3.00	1.81E-06	3.00	1.94E-06	2.99	1.17E-04	2.18
	35	1.67E-07	3.00	1.14E-06	3.00	1.23E-06	2.99	8.38E-05	2.16
	40	1.12E-07	3.00	7.63E-07	3.00	8.22E-07	2.99	6.29E-05	2.14
P^2	10	1.74E-09	–	5.32E-09	–	3.89E-06	–	1.61E-04	–
	15	2.67E-10	4.62	6.93E-10	5.03	7.79E-07	3.96	4.84E-05	2.96
	20	7.48E-11	4.42	1.59E-10	5.11	2.48E-07	3.98	2.06E-05	2.97
	25	2.50E-11	4.92	5.77E-11	4.55	1.02E-07	3.98	1.06E-05	2.98

To verify superconvergence results for Dirichlet boundary conditions with long time simulations, we consider Example 6.3 with the following Dirichlet boundary conditions

$$u(0,t) = \cos(\pi t), \qquad u(1,t) = \cos(\pi t).$$

When $b = 1.5$, $T = 100$, the results with $\kappa = 3.5/h$ in numerical fluxes (5.4) are shown in Table 8, demonstrating that the conclusions still hold for Dirichlet boundary conditions and long time simulations.

Example 6.4 To illustrate the time-dependent singularly perturbed problems with a stationary outflow boundary layer, we would like to consider a nonlinear problem

$$u_t + (e^u)_x - bu_{xx} = g(x,t), \qquad (x,t) \in [0,1] \times (0,T],$$

with Dirichlet boundary conditions and the same exact solution as (6.1). The initial solution and the source term $g(x,t)$ is determined by this exact solution. Note that when we take $b = 10^{-5}$, the solution (6.1) varies quickly with a large gradient and forms an outflow boundary layer near the outflow boundary point $x = 1$. When the Gauss–Radau projection $P_h^- u$ is used as the initial condition, we observe errors and the corresponding superconvergent rates in the local region [13]

$$(0, x_c) = (0, 1 - \lceil \ln N \rceil h),$$

where $\lceil \ln N \rceil$ denotes the minimal integer no less than $\ln N$. The results with $\kappa = 2/h$ in numerical fluxes (5.4) are shown in Table 9, from which we can see superconvergence

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Table 9 Local errors and rates for Example 6.4 with Dirichlet boundary conditions, $b = 10^{-5}$, $T = 2$

	N	$\|e_{un}\|$	Rate	$\|e_u\|_c$	Rate	$\|e_{ur}\|$	Rate	$\|e_{u\ell}\|$	Rate
P^1	20	5.75E-07	–	1.96E-06	–	6.32E-06	–	3.51E-04	–
	40	7.20E-08	3.00	2.45E-07	3.00	7.90E-07	3.00	8.75E-05	2.00
	80	9.24E-09	2.96	3.06E-08	3.00	9.86E-08	3.00	2.18E-05	2.00
	160	1.18E-09	2.96	3.82E-09	3.00	1.23E-08	3.00	5.45E-06	2.00
P^2	20	1.76E-09	–	2.71E-10	–	5.52E-08	–	4.45E-06	–
	40	1.74E-11	6.66	7.63E-12	5.15	3.53E-09	3.97	5.75E-07	2.95
	60	2.34E-12	4.94	1.35E-12	4.27	7.00E-10	3.99	1.72E-07	2.97
	80	3.29E-13	6.82	2.78E-13	5.48	2.22E-10	3.99	7.31E-08	2.98

property similar to the nonlinear hyperbolic equations [4] in the local region $(0, x_c)$. This is, both the cell averages error and numerical flux in the discrete L^2 norm converge at a rate of $2k+1$, and the LDG error (its derivative) is superconvergent at interior right (left) Radau points with an order of $k+2$ $(k+1)$.

7 Concluding Remarks

In this paper, we investigate superconvergence of the LDG method for one-dimensional nonlinear convection-diffusion equations. The main techniques are the construction of new projections and correction functions, allowing us to derive a supercloseness result between the LDG solution and an interpolation function. We have established $(2k+1)$th order superconvergence for the numerical flux and cell averages, as well as superconvergence at Radau points, even when the flow direction changes. The results are extended to generalized alternating fluxes and mixed boundary conditions. The sharpness of the theoretical results is verified by numerical experiments.

In further work, we will consider the degenerate nonlinear diffusion problems and multidimensional diffusion equations.

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Boffi, D., Brezzi, F., Fortin, M.: Mixed finite element methods and applications, Springer Series in Computational Mathematics, vol. 44. Springer, Heidelberg (2013)
2. Cao, W., Huang, Q.: Superconvergence of local discontinuous Galerkin methods for partial differential equations with higher order derivatives. J. Sci. Comput. **72**(2), 761–791 (2017). https://doi.org/10.1007/s10915-017-0377-z
3. Cao, W., Li, D., Yang, Y., Zhang, Z.: Superconvergence of discontinuous Galerkin methods based on upwind-biased fluxes for 1D linear hyperbolic equations. ESAIM Math. Model. Numer. Anal. **51**(2), 467–486 (2017). https://doi.org/10.1051/m2an/2016026

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

4. Cao, W., Shu, C.W., Yang, Y., Zhang, Z.: Superconvergence of discontinuous Galerkin method for scalar nonlinear hyperbolic equations. SIAM J. Numer. Anal. **56**(2), 732–765 (2018). https://doi.org/10.1137/17M1128605
5. Cao, W., Shu, C.W., Zhang, Z.: Superconvergence of discontinuous Galerkin methods for 1-D linear hyperbolic equations with degenerate variable coefficients. ESAIM Math. Model. Numer. Anal. **51**(6), 2213–2235 (2017). https://doi.org/10.1051/m2an/2017026
6. Cao, W., Zhang, Z.: Superconvergence of local discontinuous Galerkin methods for one-dimensional linear parabolic equations. Math. Comp. **85**(297), 63–84 (2016). https://doi.org/10.1090/mcom/2975
7. Cao, W., Zhang, Z., Zou, Q.: Superconvergence of discontinuous Galerkin methods for linear hyperbolic equations. SIAM J. Numer. Anal. **52**(5), 2555–2573 (2014). https://doi.org/10.1137/130946873
8. Castillo, P.: A superconvergence result for discontinuous Galerkin methods applied to elliptic problems. Comput. Methods Appl. Mech. Eng. **192**(41–42), 4675–4685 (2003). https://doi.org/10.1016/S0045-7825(03)00445-6
9. Castillo, P., Gómez, S.: On the conservation of fractional nonlinear Schrödinger equation's invariants by the local discontinuous Galerkin method. J. Sci. Comput. **77**(3), 1444–1467 (2018). https://doi.org/10.1007/s10915-018-0708-8
10. Celiker, F., Cockburn, B.: Superconvergence of the numerical traces of discontinuous Galerkin and hybridized methods for convection-diffusion problems in one space dimension. Math. Comp. **76**(257), 67–96 (2007). https://doi.org/10.1090/S0025-5718-06-01895-3
11. Cheng, Y., Meng, X., Zhang, Q.: Application of generalized Gauss-Radau projections for the local discontinuous Galerkin method for linear convection-diffusion equations. Math. Comp. **86**(305), 1233–1267 (2017). https://doi.org/10.1090/mcom/3141
12. Cheng, Y., Shu, C.W.: Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension. SIAM J. Numer. Anal. **47**(6), 4044–4072 (2010). https://doi.org/10.1137/090747701
13. Cheng, Y., Zhang, F., Zhang, Q.: Local analysis of local discontinuous Galerkin method for the time-dependent singularly perturbed problem. J. Sci. Comput. **63**(2), 452–477 (2015). https://doi.org/10.1007/s10915-014-9901-6
14. Cockburn, B., Guzmán, J., Wang, H.: Superconvergent discontinuous Galerkin methods for second-order elliptic problems. Math. Comp. **78**(265), 1–24 (2009). https://doi.org/10.1090/S0025-5718-08-02146-7
15. Cockburn, B., Kanschat, G., Schotzau, D.: A locally conservative LDG method for the incompressible Navier-Stokes equations. Math. Comp. **74**(251), 1067–1095 (2005). https://doi.org/10.1090/S0025-5718-04-01718-1
16. Cockburn, B., Luskin, M., Shu, C.W., Süli, E.: Enhanced accuracy by post-processing for finite element methods for hyperbolic equations. Math. Comp. **72**(242), 577–606 (2003). https://doi.org/10.1090/S0025-5718-02-01464-3
17. Cockburn, B., Shu, C.W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J. Numer. Anal. **35**(6), 2440–2463 (1998). https://doi.org/10.1137/S0036142997316712
18. Guo, W., Zhong, X., Qiu, J.M.: Superconvergence of discontinuous Galerkin and local discontinuous Galerkin methods: eigen-structure analysis based on Fourier approach. J. Comput. Phys. **235**, 458–485 (2013). https://doi.org/10.1016/j.jcp.2012.10.020
19. Ji, L., Xu, Y., Ryan, J.K.: Accuracy-enhancement of discontinuous Galerkin solutions for convection-diffusion equations in multiple-dimensions. Math. Comp. **81**(280), 1929–1950 (2012). https://doi.org/10.1090/S0025-5718-2012-02586-5
20. Meng, X., Ryan, J.K.: Divided difference estimates and accuracy enhancement of discontinuous Galerkin methods for nonlinear symmetric systems of hyperbolic conservation laws. IMA J. Numer. Anal. **38**(1), 125–155 (2018). https://doi.org/10.1093/imanum/drw072
21. Meng, X., Shu, C.W., Wu, B.: Optimal error estimates for discontinuous Galerkin methods based on upwind-biased fluxes for linear hyperbolic equations. Math. Comp. **85**(299), 1225–1261 (2016). https://doi.org/10.1090/mcom/3022
22. Meng, X., Shu, C.W., Zhang, Q., Wu, B.: Superconvergence of discontinuous Galerkin methods for scalar nonlinear conservation laws in one space dimension. SIAM J. Numer. Anal. **50**(5), 2336–2356 (2012). https://doi.org/10.1137/110857635
23. Yang, Y., Shu, C.W.: Analysis of sharp superconvergence of local discontinuous Galerkin method for one-dimensional linear parabolic equations. J. Comput. Math. **33**(3), 323–340 (2015). https://doi.org/10.4208/jcm.1502-m2014-0001
24. Zhang, Q., Shu, C.W.: Stability analysis and a priori error estimates of the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws. SIAM J. Numer. Anal. **48**(3), 1038–1063 (2010). https://doi.org/10.1137/090771363

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

SPI Journal: 10915 Article No.: 1446 TYPESET DISK LE CP Disp.:2021/3/8 Pages: 30 Layout: Small